FOURIER TRANSFORM FOR THE SPATIAL QUINCU NX LATTICE

Markus Püschel
Dept. of Electrical and Computer Engineering
Carnegie Mellon University
Pittsburgh, U.S.A.

Martin Rötteler
Dept. of Combinatorics and Optimization
University of Waterloo
Waterloo, Canada

Abstract

We derive a new, two-dimensional nonseparable signal transform for computing the spectrum of spatial signals residing on a finite quincunx lattice. The derivation uses the connection between transforms and polynomial algebras, which has long been known for the discrete Fourier transform (DFT), and was extended to other transforms in recent research. We also show that the new transform can be computed with $O(n^2 \log(n))$ operations, which puts it in the same complexity class as its separable counterparts.

1. INTRODUCTION

It is known that applying a two-dimensional discrete Fourier transform (DFT) to a finite, discrete two-dimensional signal assumes that the signal resides on a rectangular directed lattice with periodic boundary conditions (b.c.’s), which effectively makes the lattice a (directed) torus. Similarly, and less well-known, applying a two-dimensional discrete cosine or sine transform (DCT or DST) to the same signal assumes that it resides on an undirected lattice, shown in Fig. 1, with symmetric or antisymmetric b.c.’s (that depend on the DCT or DST chosen and are not shown) [1, 2].

Intuitively, “undirected” implies that the associated “Fourier transforms” (the DCTs and DSTs) are better suited for spatial signals as for time signals; this is confirmed by the ubiquitous use of DCTs for time signals, i.e., directed lattices. Specifically the quincunx lattice (also directed and infinite) has been studied in [7].

Our derivation uses the connection between transforms and polynomial algebras, which has long been known for the DFT, and recently been extended to the DCTs and DSTs [2, 3]. Using this connection, we already derived a Fourier transform for the directed quincunx lattice, which was closely related to the DFT [4], and a nonseparable transforms for the spatial hexagonal lattice in [5].

Signal processing on arbitrary two- and higher-dimensional lattices was developed by Mersereau et al. (e.g., [6]), but only for time signals, i.e., directed lattices. Specifically the quincunx lattice (also directed and infinite) has been studied in [7].

Organization. In Section 2 we provide the background on the relationship between polynomial algebras and signal transforms. In Section 3, we derive the Fourier transform for the spatial quincunx lattice from a polynomial algebra capturing the lattice’s structure. Finally, we offer conclusions in Section 4.

2. POLYNOMIAL ALGEBRAS AND TRANSFORMS

In this section, we provide the necessary background on polynomial algebras and their connection to signal transforms in one and two dimensions. But first, we recall the definition and some
properties of Chebyshev polynomials, which are necessary to algebraically describe spatial transforms such as the DCTs and DSTs and our new transform.

**Chebyshev polynomials.** Chebyshev polynomials \([T_n(x)]\) in one variable are defined by the recursion

\[ C_n(x) = 2x C_{n-1}(x) - C_{n-2}(x), \quad n \geq 2. \tag{1} \]

The exact form of \(C_n\) is determined by the initial conditions \(C_0\) and \(C_1\), which are chosen as polynomials of degree 0 and 1, respectively; (1) then implies that \(C_n\) is a polynomial of degree \(n\) for \(n \geq 0\). The Chebyshev polynomials of the first kind, denoted with \(C = T\) arise from

\[ T_0(x) = 1, \quad T_1(x) = x. \]

The (much less known) Chebyshev polynomials of the third kind \(V\) arise from

\[ V_0(x) = 1, \quad V_1(x) = 2x - 1. \]

Running the recurrence (1) in the other direction yields, for given initial conditions, Chebyshev polynomials \(C_n\) for negative indices \(n\), and thus for all \(n \in Z\).

We will use the following properties:

- **Closed form:**
  \[ V_n(x) = \cos(n \theta), \quad x = \cos \theta, \tag{2} \]

- **Cosine form:**
  \[ V_n(x) = \cos\left(\frac{n+1}{2} \theta\right), \quad x = \cos \theta, \tag{3} \]

- **Product:**
  \[ T_0 C_n = \left( C_{n+k} + C_{n-k} \right)/2, \quad C = T, \tag{4} \]

- **Relation:**
  \[ T_n + T_{n+1} = (x + 1) V_n, \tag{5} \]

- **Decomposition:**
  \[ T_{kn} = T_k(T_{nm}), \quad n \text{ zeros of } T_n : \alpha_k = \cos(k + 1/2) \pi/n, \quad 0 \leq k < n. \tag{7} \]

### 2.1. Polynomial Algebras (One Variable)

**Polynomial algebra.** An algebra \(A\) is a vector space that is also a ring, i.e., closed under multiplication and the distributivity law holds. Examples include the set of complex numbers \(\mathbb{C}\) and the set of polynomials \(\mathbb{A} = \mathbb{C}[x]\) in one variable, or in several variables \(A = \mathbb{C}[x_1, \ldots, x_k]\).

If \(p(x) \in \mathbb{C}[x]\) is given, then the set of polynomials of degree less than \(\deg(p)\) with addition and multiplication modulo \(p\),

\[ A = \mathbb{C}[x]/p(x) = \{ q(x) \mid \deg(q) < \deg(p) \}, \]

is called a polynomial algebra (in one variable). As a vector space, \(A\) has dimension \(\dim(A) = \deg(p)\). Further, \(A\) is obviously generated by \(x\), since every element of \(A\) is a polynomial in \(x\).

**Polynomial transform.** Let \(A = \mathbb{C}[x]/p(x)\) and assume that \(p\) has pairwise distinct zeros \(\alpha_0, \ldots, \alpha_{n-1}\), where \(n = \deg(p)\). Then the Chinese remainder theorem (CRT) establishes the (isomorphic) decomposition of \(A\) into its spectrum, which is a direct sum (or Cartesian product with elementwise operations) of one-dimensional polynomial algebras:

\[ \Phi : \mathbb{C}[x]/p(x) \to \bigoplus_{0 \leq k < m} \mathbb{C}[x]/(x - \alpha_k), \quad q(x) \mapsto \{ q(\alpha_0), \ldots, q(\alpha_{n-1}) \}. \tag{10} \]

In particular, \(\Phi\) is a linear mapping. Thus, if we choose a basis \(b = (p_0, \ldots, p_{n-1})\) in \(A\), then \(\Phi\) is represented by the matrix

\[ P_{b,\alpha} = \{ p_k(\alpha_k) \}_{0 \leq k, \ell < n}, \]

which we call polynomial transform or Fourier transform for \(A\).

**Examples.** Let \(A = \mathbb{C}[x]/(x^2n - 1)\) with chosen basis \(b = (x, x^2, \ldots, x^{n-1})\). Then \(P_{b,\alpha} = \text{DFT}_{n}\), is the discrete Fourier transform. Letting the generator \(x\) of \(A\) operate on \(b\), namely \(x \cdot x^\ell = x^{\ell+1}\), yields a directed circle, which is the structure implicitly imposed on a signal, when the DFT is applied. In particular, \(x^{n-1} = 1 = 0 \Rightarrow n = 1\) captures the cyclic b.c.

If \(A = \mathbb{C}[x]/T_n(x)\) with basis \(b = (T_0, T_1, \ldots, T_{n-1})\), then \(P_{b,\alpha} = \text{DCT}-3\) is the DCT of type 3 \([3, 2]\). Letting \(x\) operate on \(b\), using (4) with \(k = 1\), yields \(x T_k = (T_{k-1} + T_{k+1})/2\). Visually,

\[ x T_{n-1} \quad x T_1 \quad x T_2 \quad x T_{n-3} \quad x T_{n-2} \quad x T_{n-1} \]

(1)

Since the line graph is undirected, it is the suitable structure for spatial signals. The left boundary has the b.c. \(T_{-1} = T_1\), and the right boundary the b.c. \(T_n = 0\), which implies (again from (4)) \(T_{n+1} = -T_{n-1}\).

The same algebra \(A = \mathbb{C}[x]/T_n(x)\), but now with basis \(b = (V_0, V_1, \ldots, V_{n-1})\) yields \(P_{b,\alpha} = D_n\text{DCT-4}_n\), with

\[ D_n = \text{diag}_{0 \leq k < n}(1/(\cos k + 1/2 \pi /n)). \]

The polynomial algebras for all 16 DCTs/DST can be found in \([3, 2]\), where they are used to concisely derive fast algorithms.

### 2.2. Polynomial Algebras (Two or more Variables)

**Polynomial algebra.** Similar to the one variable case, we can define polynomial algebras in more variables. We consider two variables as example and define a polynomial algebra as

\[ A = \mathbb{C}[x, y]/(p(x, y), q(x, y)). \]

Note that here we need to compute modulo two or more\(^1\) polynomials to ensure that the dimension of \(A\) is finite.

**Polynomial transform.** To define a polynomial transform for the algebra in (9) we assume that the total degrees\(^2\) of \(p\) and \(q\) are \(n\) and \(m\), respectively, and that the equations \(p(x, y) = q(x, y) = 0\) have precisely \(mn\) solutions \(\alpha = (\alpha_k, \beta_k)_{0 \leq k < m,n}\). The CRT now becomes the isomorphic decomposition

\[ \Phi : A \to \bigoplus_{0 \leq k < m,n} \mathbb{C}[x, y]/(x - \alpha_k, y - \beta_k), \]

which also implies that \(\dim(A) = mn\). With respect to a basis \(b = (p(x, y)|0 \leq \ell < m, n)\) of \(A\), the polynomial transform for \(A\) is given by the matrix \(P_{b,\alpha} = [p_k(\alpha_k, \beta_k)]_{0 \leq k, \ell < m, n}\). As in the case of one variable, \(P_{b,\alpha}\) can be considered as a Fourier transform for \(A\).

**The separable case.** The case of a separable polynomial algebra plays an important role in signal processing. It is the special case in (9), where \(p(x, y) = p(x)\) and \(q(x, y) = q(y)\) depend only on \(x\) and \(y\), respectively. In this case, we can choose as basis

\(^1\)In contrast to the one-dimensional case, were one polynomial suffices, more than two polynomials may indeed be necessary \([9]\).

\(^2\) The total degree of \(p(x, y)\) is defined as the maximum value of \(i + j\), as \(i\) and \(j\) range over all summands \(cx^iy^j\) of \(p(x, y)\).
in \( \mathcal{A} \) the product\(^3\) of bases \( b \) and \( c \) of \( \mathbb{C}[x]/p(x) \) and \( \mathbb{C}[y]/q(y) \), respectively. Further, the set of solutions of \( p(x) = q(y) = 0 \) is given by the Cartesian product \( \alpha \times \beta \) of the zeros \( \alpha \) of \( p \) and \( \beta \) of \( q \). Mathematically, this shows that

\[
\mathbb{C}[x,y]/(p(x), q(y)) \cong \mathbb{C}[x]/p(x) \otimes \mathbb{C}[y]/q(y)
\]

(11)

is the tensor product of one-dimensional polynomial algebras. The associated transform is correspondingly the tensor or Kronecker product

\[
P_{\alpha,\beta} \otimes P_{\gamma,\delta},
\]

where \( \mathbb{A} \cup \mathbb{B} = \{a_{i,j} b_{i,j} \} \) for \( A = \{a_{i,j} \} \).

**Example.** The two-dimensional DCT-3 \( \mathcal{A} \otimes \mathcal{DCT} \) is a polynomial transform for \( \mathcal{A} = \mathbb{C}[x,y]/(T_n(x), T_n(y)) \), with basis

\[
b = (T_i(x)T_j(y) \mid 0 \leq i, j < n).
\]

(12)

If we assume \( b \) to be arranged into an \( n \times n \) lattice, then the operation of the generators \( x \) (horizontal) and \( y \) (vertical) yields the undirected structure in Fig. 1. The nodes in Fig. 1 can be considered as being indexed by the elements in \( b \).

3. **Spatial Quincunx Lattice and Transform**

Using the theory of polynomial algebras, we construct the Fourier transform for the spatial quincunx lattice as follows. We start with a suitable (explained below) two-dimensional DCT and its polynomial algebra \( \mathcal{A} \) for the lattice in Fig. 1 and identify the subalgebra \( \mathcal{B} \subset \mathcal{A} \) corresponding to the quincunx lattice in Fig. 2. Then we put \( \mathcal{B} \) into the form (9), and read off the spectrum and the transform using (10). Finally, we discuss the complexity of the transform.

**Constructing the Algebra.** We start with a two-dimensional DCT; type 3 of size \( n \times n \), and its associated polynomial algebra

\[
\mathcal{A} = \mathbb{C}[x,y]/(T_n(x), T_n(y))
\]

and basis shown in (12). Further, we assume that \( n \) is even. Omitting the basis polynomials in (12) that do not reside on the quincunx lattice in Fig. 2 leaves exactly

\[
b = (T_{i,j}(x,y) = T_i(x)T_j(y) \mid i + j \equiv 0 \mod 2),
\]

(13)

which we assume to be ordered in row-major order. Obviously, \( b \) spans a subvector space \( \mathcal{A} \) of \( \mathcal{A} \) of dimension \( n^2/2 \). But is \( \mathcal{B} \) also an algebra, i.e., closed under multiplication? The answer is yes because of the b.c.’s associated to this DCT; namely, \( T_{n-1} = T_1 \) and \( T_{n-1} = -T_{n-1} \) preserve the defining condition in (13). For example, products like \( T_3(x) \cdot T_1(x)T_1(y) = (T_3(x)T_1(y) + T_3(x)T_1(y))/2 \) (using (4)) are again in \( \mathcal{B} \).

To bring \( \mathcal{B} \) into the form (9), we need to find a set of generators, which then become our variables. Natural choices are

\[
u = T_2(x) = 2x^2 - 1, \quad v = T_2(y) = 2y^2 - 1,
\]

but their spanned algebra does not contain

\[
w = T_1(x)T_1(y) = xy,
\]

which thus we choose as the third generator. Since we have three polynomials \( u, v, w \) in two variables, they cannot be independent.

\(^{3}\)The product of two lists \( (c_1, \ldots, c_n) \) and \( (d_1, \ldots, d_m) \) is the list of all products \( (c_1 d_1, c_1 d_2, \ldots) \) of length \( mn \).

\(^{4}\)In fact, using this argument, one can show that exactly 4 of the 16 DCTs/DSTs can be used as a starting point, namely the DCTs/DSTs of type 1 and 3. This yields 4 types of quincunx transforms. We consider only the DCT, type 3, since it is the best-known among those and due to lack of space.

Indeed, they satisfy the polynomial relation

\[
4w^2 - (u+1)(v+1) = 0.
\]

Further, from (6), \( T_{n/2}(u) = T_n(x) = 0 \) in \( \mathcal{A} \). Analogously, \( T_{n/2}(v) = 0 \). Thus, we obtain the polynomial algebra

\[
\mathcal{B} = \mathbb{C}[u,v,w]/(T_{n/2}(u), T_{n/2}(v), 4w^2 - (u+1)(v+1)).
\]

(14)

Using Hilbert’s method [9, ch. 9], we confirm that \( \dim(\mathcal{B}) = n^2/2 \), i.e., \( \mathcal{B} \) is indeed the entire algebra spanned by \( b \) in (13). Note that this algebra is not separable.

It is interesting to observe the operation of \( u, v, w \) on the quincunx lattice, i.e., the elements \( T_{i,j} \) \( \in \mathcal{B} \). For example, products like

\[
u T_{i,j} = (T_{i-1,j+1} + T_{i+1,j-1} + T_{i+1,j+1} + T_{i-1,j-1})/2
\]

which we assume to be ordered in row-major order. Obviously, \( b \) spans a subvector space \( \mathcal{A} \) of \( \mathcal{A} \) of dimension \( n^2/2 \). But is \( \mathcal{B} \) also an algebra, i.e., closed under multiplication? The answer is yes because of the b.c.’s associated to this DCT; namely, \( T_{n-1} = T_1 \) and \( T_{n-1} = -T_{n-1} \) preserve the defining condition in (13). For example, products like \( T_3(x) \cdot T_1(x)T_1(y) = (T_3(x)T_1(y) + T_3(x)T_1(y))/2 \) (using (4)) are again in \( \mathcal{B} \).

To bring \( \mathcal{B} \) into the form (9), we need to find a set of generators, which then become our variables. Natural choices are

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\[^3\]The product of two lists \((c_1, \ldots, c_n)\) and \((d_1, \ldots, d_m)\) is the list of all products \((c_1 d_1, c_1 d_2, \ldots)\) of length \(mn\).

\[^4\]In fact, using this argument, one can show that exactly 4 of the 16 DCTs/DSTs can be used as a starting point, namely the DCTs/DSTs of type 1 and 3. This yields 4 types of quincunx transforms. We consider only the DCT, type 3, since it is the best-known among those and due to lack of space.
The discrete quincunx transform DQT for computing the spectrum of a signal given on an \( n/2 \times n \) spatial quincunx lattice.

\[
\text{DQT}_{n/2 \times n} = L_{n/2}^2 \left[ \text{DCT-3}_{n/2} \otimes \text{DCT-3}_{n/2} \right] = L_{n/2}^2 \left( \text{DCT-4}_{n/2} \otimes \text{DCT-4}_{n/2} \right)
\]

with
\[
\gamma_m = \frac{(m + 1/2)\pi}{n} + \frac{\pi}{4}.
\]

**Transform.** To determine the Fourier transform for the spatial quincunx lattice in Fig. 2, we evaluate the \( n^2/2 \) basis polynomials (given by concatenating alternately (16) and (17) for \( 0 \leq y' < n/2 \)) at the zeros in (18). We order the zeros lexicographically indexed as \((k, \ell, \pm)\), where \( \pm \in \{+, -\} \) runs fastest, and \( 0 \leq k, \ell < n/2 \).

Evaluating all even indexed rows (16) at all zeros indexed \((k, \ell, +)\) yields a DCT-3\(n/2 \otimes \text{DCT-3}_{n/2}\). The same is true for \((k, \ell, -)\) since (16) does not depend on \(w\).

Evaluating the odd indexed rows (17) at all zeros indexed \((k, \ell, +)\) yields \(D_{n/2}^2 (\text{DCT-4}_{n/2} \otimes \text{DCT-4}_{n/2})\), where \(D_{n/2}^2\) is a scaling diagonal arising from (8) and \(w\) in (17) evaluated at (19):

\[
D_{n/2}^2 = \text{diag}_{0 \leq k, \ell < n/2} \left( \cos \gamma_k + \sin \gamma_k \right) \left( \cos \gamma_\ell + \sin \gamma_\ell \right)
\]

where \(\gamma_k\) is defined in (20). Evaluating (17) at all zeros indexed \((k, \ell, -)\) yields \(-D_{n/2}^2 (\text{DCT-4}_{n/2} \otimes \text{DCT-4}_{n/2})\).

**Definition 1 (Discrete quincunx transform)** The discrete quincunx transform DQT is defined for an \( n/2 \times n \) signal given in row major order on the spatial quincunx lattice in Fig. 2. The exact form is given in the first equation in Table 1.

The stride permutation, or perfect shuffle, \(L_{n/2}^2\) interleaves the rows and columns to achieve the desired order of basis polynomials (alternating (16) and (17)); \(L_{n/2}^2\) is the inverse of \(L_{n/2}^2\). The definition is

\[
L_{n/2}^2 : i \left( \frac{n}{2} \right) + j \rightarrow 2j + i, \quad 0 \leq i < 2, \quad 0 \leq j < \frac{n}{2}.
\]

**Fast algorithm and complexity.** A straightforward decomposition of the DQT given in the second equation in Table 1, where

\[
A \otimes B = \begin{bmatrix} A & \ast \\ B & \ast \end{bmatrix}.
\]

This implies that the DQT can be computed using fast DCT-3 and DCT-4 algorithms. We analyze the complexity for a two-power \(n\). In this case, DCT-3\(n\) can be computed using \(2n \log_2(n) - n + 1\) operations, and \(D'\) DCT-4\(n\), independent of the diagonal matrix \(D'\), using \(2n \log_2(n) + n\) operations [10]. Straightforward computation now yields the following lemma.

**Lemma 1 (Complexity of the DQT)** If \(n\) is a two-power, then the DQT \(n \times n\) can be computed using

\[
2n^2 \log_2(n) + n^2/2 - n
\]

operations.

Note that this bound assumes that the two-dimensional DCTs are computed using the row-column method, which is known to be suboptimal.

Lemma 1 shows that the DQT is with \(O(n^2 \log(n))\) in the same complexity class as its separable counterparts.

**4. CONCLUSIONS**

We derived a new spatial signal transform, the DQT, for computing the spectrum of a signal given on a finite quincunx lattice. The construction uses the connection between signal transforms and polynomial algebras, and is part of a larger effort to work towards an algebraic theory of signal processing (see [11]). The algebra also yields important properties of the DQT, such as the appropriate notion of filtering and diagonalization properties, which will be derived in a longer version of this paper.

**5. REFERENCES**