FOURIER TRANSFORM FOR THE SPATIAL QUINCUNX LATTICE

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ABSTRACT

We derive a new, two-dimensional nonseparable signal transform for computing the spectrum of spatial signals residing on a finite quincunx lattice. The derivation uses the connection between transforms and polynomial algebras, which has long been known for the discrete Fourier transform (DFT), and was extended to other transforms in recent research. We also show that the new transform can be computed with $O(n^2 \log(n))$ operations, which puts it in the same complexity class as its separable counterparts.

1. INTRODUCTION

It is known that applying a two-dimensional discrete Fourier transform (DFT) to a finite, discrete two-dimensional signal assumes that the signal resides on a rectangular directed lattice with periodic boundary conditions (b.c.'s), which effectively makes the lattice a (directed) torus. Similarly, and less well-known, applying a two-dimensional discrete cosine or sine transform (DCT or DST) to the same signal assumes that it resides on an *undirected* lattice, shown in Fig. 1, with symmetric or antisymmetric b.c.'s (that depend on the DCT or DST chosen and are not shown) [1, 2]. Intuitively, "undirected" implies that the associated "Fourier transforms" (the DCTs and DSTs) are better suited for spatial signals as for time signals; this is confirmed by the ubiquitous use of DCTs in image processing.

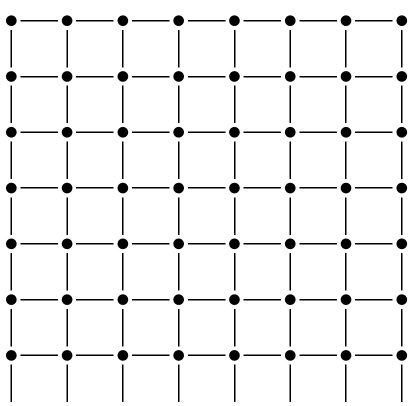

Fig. 1. Finite $n \times n$ lattice associated with the two-dimensional DCTs and DSTs. The boundary conditions are omitted and depend on the DCT or DST chosen. For a DCT, type 3, the associated polynomial algebra (explained in the paper) is $\mathcal{A} = \mathbb{C}[x,y]/\langle T_n(x), T_n(y)\rangle = \mathbb{C}[x]/\langle T_n(x)\rangle \otimes \mathbb{C}[y]/\langle T_n(y)\rangle$.

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In this paper we derive a Fourier transform for spatial signals residing on the quincunx lattice shown in Fig. 2. The transform is nonseparable, and, as expected, related to the DCTs. This connection also enables us to show that the transform can be computed in roughly $2n^2 \log_2(n)$ operations for a two-power n.

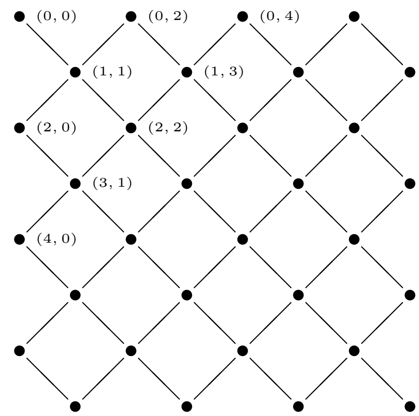


Fig. 2. The finite undirected quincunx lattice of size $n/2 \times n$; it is obtained from Fig. 1 by omitting every other point in the pattern shown. As derived in the paper, the associated polynomial algebra is nonseparable and is given by $\mathcal{A} = \mathbb{C}[u,v,w]/\langle T_{n/2}(u), T_{n/2}(v), 4w^2-(u+1)(v+1)\rangle$. We use $\mathcal{A}$ to derive the associated Fourier transform.

Our derivation uses the connection between transforms and polynomial algebras, which has long been known for the DFT, and recently been extended to the DCTs and DSTs [2, 3]. Using this connection, we already derived a Fourier transform for the *directed* quincunx lattice, which was closely related to the DFT [4], and a nonseparable transforms for the spatial hexagonal lattice in [5].

Signal processing on arbitrary two- and higher-dimensional lattices was developed by Mersereau et al. (e.g., [6]), but only for time signals, i.e., directed lattices. Specifically the quincunx lattice (also directed and infinite) has been studied in [7].

Organization. In Section 2 we provide the background on the relationship between polynomial algebras and signal transforms. In Section 3, we derive the Fourier transform for the spatial quincunx lattice from a polynomial algebra capturing the lattice's structure. Finally, we offer conclusions in Section 4.

2. POLYNOMIAL ALGEBRAS AND TRANSFORMS

In this section, we provide the necessary background on polynomial algebras and their connection to signal transforms in one and two dimensions. But first, we recall the definition and some properties of Chebyshev polynomials, which are necessary to algebraically describe spatial transforms such as the DCTs and DSTs and our new transform.

Chebyshev polynomials. Chebyshev polynomials [8] in one variable are defined by the recursion

$$C_n(x) = 2xC_{n-1}(x) - C_{n-2}(x), \quad n \geq 2. \tag{1}$$

The exact form of C_n is determined by the initial conditions C_0 and C_1, which are chosen as polynomials of degree 0 and 1, respectively; (1) then implies that C_n is a polynomial of degree n for $n \geq 0$. The Chebyshev polynomials of the first kind, denoted with $C = T$ arise from

$$T_0(x) = 1,\ T_1(x) = x.$$

The (much lesser known) Chebyshev polynomials of the third kind $C = V$ arise from

$$V_0(x) = 1,\ V_1(x) = 2x - 1.$$

Running the recurrence (1) in the other direction yields, for given initial conditions, Chebyshev polynomials C_n for negative indices n, and thus for all $n \in \mathbb{Z}$.

We will use the following properties:

$$\text{closed form: } T_n = \cos n\theta, \quad x = \cos\theta, \tag{2}$$

$$V_n = \cos\frac{(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \quad x = \cos\theta, \tag{3}$$

$$\text{product: } T_k C_n = (C_{n+k} + C_{n-k})/2, \quad C = T, V, \tag{4}$$

$$\text{relation: } T_n + T_{n+1} = (x+1)V_n, \tag{5}$$

$$\text{decomposition: } T_{km} = T_k(T_m), \tag{6}$$

$$n \text{ zeros of } T_n: \alpha_k = \cos(k+1/2)\pi/n,\ 0 \leq k < n. \tag{7}$$

2.1. Polynomial Algebras (One Variable)

Polynomial algebra. An algebra $\mathcal{A}$ is a vector space that is also a ring, i.e., closed under multiplication and the distributivity law holds. Examples include the set of complex numbers $\mathcal{A} = \mathbb{C}$ and the set of polynomials $\mathcal{A} = \mathbb{C}[x]$ in one variable, or in several variables $\mathcal{A} = \mathbb{C}[x_1, \dots, x_k]$.

If $p(x) \in \mathbb{C}[x]$ is given, then the set of polynomials of degree less than $\deg(p)$ with addition and multiplication modulo p,

$$\mathcal{A} = \mathbb{C}[x]/p(x) = \{q(x) \mid \deg(q) < \deg(p)\},$$

is called a *polynomial algebra* (in one variable). As a vector space, $\mathcal{A}$ has dimension $\dim(\mathcal{A}) = \deg(p)$. Further, $\mathcal{A}$ is obviously generated by x, since every element of $\mathcal{A}$ is a polynomial in x.

Polynomial transform. Let $\mathcal{A} = \mathbb{C}[x]/p(x)$ and assume that p has pairwise distinct zeros $\alpha = (\alpha_0, \dots, \alpha_{n-1})$, where $n = \deg(p)$. Then the Chinese remainder theorem (CRT) establishes the (isomorphic) decomposition of $\mathcal{A}$ into its spectrum, which is a direct sum (or Cartesian product with elementwise operations) of one-dimensional polynomial algebras:

$$\begin{aligned} \Phi:\ \mathbb{C}[x]/p(x) &\to \mathbb{C}[x]/(x-\alpha_0) \oplus \ldots \oplus \mathbb{C}[x]/(x-\alpha_{n-1}), \\ q(x) &\mapsto (q(\alpha_0), \dots, q(\alpha_{n-1})). \end{aligned}$$

In particular, Φ is a linear mapping. Thus, if we choose a basis $b = (p_0, \dots, p_{n-1})$ in $\mathcal{A}$, then Φ is represented by the matrix

$$\mathcal{P}_{b,\alpha} = [p_\ell(\alpha_k)]_{0 \leq k,\ell < n},$$

which we call *polynomial transform* or Fourier transform for $\mathcal{A}$.

Examples. Let $\mathcal{A} = \mathbb{C}[x]/(x^n - 1)$ with chosen basis $b = (x^0, x^1, \dots, x^{n-1})$. Then $\mathcal{P}_{b,\alpha} = \mathrm{DFT}_n$ is the discrete Fourier transform. Letting the generator x of $\mathcal{A}$ operate on b, namely $x \cdot x^\ell = x^{\ell+1}$, yields a directed circle, which is the structure implicitly imposed on a signal, when the DFT is applied. In particular, $x^n - 1 = 0$ or $x^n = 1$ captures the cyclic b.c.

If $\mathcal{A} = \mathbb{C}[x]/T_n(x)$ with basis $b = (T_0, T_1, \dots, T_{n-1})$, then $\mathcal{P}_{b,\alpha} = \mathrm{DCT}\text{-}3_n$ is the DCT of type 3 [3, 2]. Letting x operate on b, using (4) with $k = 1$, yields $xT_\ell = (T_{\ell-1} + T_{\ell+1})/2$. Visually,

$$T_{\ell-1} \longleftarrow T_\ell \longrightarrow T_{\ell+1}$$

(the weights $1/2$ are omitted), which produces the line graph

$$T_0 \rightleftarrows T_1 \longleftrightarrow T_2 \cdots\cdots T_{n-3} \longleftrightarrow T_{n-2} \longleftrightarrow T_{n-1}$$

Since the line graph is undirected, it is the suitable structure for spatial signals. The left boundary has the b.c. $T_{-1} = T_1$, and the right boundary the b.c. $T_n = 0$, which implies (again from (4)) $T_{n+1} = -T_{n-1}$.

The same algebra $\mathcal{A} = \mathbb{C}[x]/T_n(x)$, but now with basis $b = (V_0, V_1, \dots, V_{n-1})$ yields $\mathcal{P}_{b,\alpha} = D_n \mathrm{DCT}\text{-}4_n$, with

$$D_n = \mathrm{diag}_{0 \leq k < n}\left(1/\cos\tfrac{(k+1/2)\pi}{2n}\right). \tag{8}$$

The polynomial algebras for all 16 DCTs/DST can be found in [3, 2], where they are used to concisely derive fast algorithms.

2.2. Polynomial Algebras (Two or more Variables)

Polynomial algebra. Similar to the one variable case, we can define polynomial algebras in more variables. We consider two variables as example and define a polynomial algebra as

$$\mathcal{A} = \mathbb{C}[x,y]/\langle p(x,y), q(x,y)\rangle. \tag{9}$$

Note that here we need to compute modulo two or more[1] polynomials to ensure that the dimension of $\mathcal{A}$ is finite.

Polynomial transform. To define a polynomial transform for the algebra in (9) we assume that the total degrees[2] of p and q are n and m, respectively, and that the equations $p(x,y) = q(x,y) = 0$ have precisely mn solutions $\alpha = (\alpha_k, \beta_k)_{0 \leq k < mn}$. The CRT now becomes the isomorphic decomposition

$$\Phi:\ \mathcal{A} \to \bigoplus_{0 \leq k < mn} \mathbb{C}[x,y]/\langle x - \alpha_k, y - \beta_k\rangle, \tag{10}$$

which also implies that $\dim(\mathcal{A}) = mn$. With respect to a basis $b = (p_\ell(x,y) | 0 \leq \ell < mn)$ of $\mathcal{A}$, the polynomial transform for $\mathcal{A}$ is given by the matrix $\mathcal{P}_{b,\alpha} = [p_\ell(\alpha_k, \beta_k)]_{0 \leq k,\ell < mn}$. As in the case of one variable, $\mathcal{P}_{b,\alpha}$ can be considered as a Fourier transform for $\mathcal{A}$.

The separable case. The case of a separable polynomial algebra plays an important role in signal processing. It is the special case in (9), where $p(x,y) = p(x)$ and $q(x,y) = q(y)$ depend only on x and y, respectively. In this case, we can choose as basis

[1] In contrast to the one-dimensional case, were one polynomial suffices, more than two polynomials may indeed be necessary [9].

[2] The total degree of $p(x,y)$ is defined as the maximum value of $i + j$, as i and j range over all summands cx^iy^j of $p(x,y)$.

in $\mathcal{A}$ the product[3] of bases b and c of $\mathbb{C}[x]/p(x)$ and $\mathbb{C}[y]/q(y)$, respectively. Further, the set of solutions of $p(x) = q(y) = 0$ is given by the Cartesian product $\alpha \times \beta$ of the zeros α of p and β of q. Mathematically, this shows that

$$\mathbb{C}[x, y]/\langle p(x), q(y)\rangle \cong \mathbb{C}[x]/p(x) \otimes \mathbb{C}[y]/q(y) \qquad (11)$$

is the *tensor product* of one-dimensional polynomial algebras. The associated polynomial transform is correspondingly the tensor or Kronecker product

$$\mathcal{P}_{b,\alpha} \otimes \mathcal{P}_{c,\beta},$$

where $A \otimes B = [a_{k,\ell} B]_{k,\ell}$ for $A = [a_{k,\ell}]$.

Example. The two-dimensional $\mathrm{DCT}\text{-}3_n \otimes \mathrm{DCT}\text{-}3_n$ is a polynomial transform for $\mathcal{A} = \mathbb{C}[x, y]/\langle T_n(x), T_n(y)\rangle$, with basis

$$b = (T_i(x)T_j(y) \mid 0 \le i, j < n). \qquad (12)$$

If we assume b to be arranged into an $n \times n$ lattice, then the operation of the generators x (horizontal) and y (vertical) yields the undirected structure in Fig. 1. The nodes in Fig. 1 can be considered as being indexed by the elements in b.

3. SPATIAL QUINCUNX LATTICE AND TRANSFORM

Using the theory of polynomial algebras, we construct the Fourier transform for the spatial quincunx lattice as follows. We start with a suitable (explained below) two-dimensional DCT and its polynomial algebra $\mathcal{A}$ for the lattice in Fig. 1 and identify the *subalgebra* $\mathcal{B} < \mathcal{A}$ corresponding to the quincunx lattice in Fig. 2. Then we put $\mathcal{B}$ into the form (9), and read off the spectrum and the transform using (10). Finally, we discuss the complexity of the transform.

Constructing the algebra. We start with a two-dimensional DCT, type 3 of size $n \times n$, and its associated polynomial algebra $\mathcal{A} = \mathbb{C}[x, y]/\langle T_n(x), T_n(y)\rangle$ and basis shown in (12). Further, we assume that n is even. Omitting the basis polynomials in (12) that do not reside on the quincunx lattice in Fig. 2 leaves exactly

$$b = (T_{i,j}(x, y) = T_i(x)T_j(y) \mid i + j \equiv 0 \bmod 2), \qquad (13)$$

which we assume to be ordered in row-major order. Obviously, b spans a subvector space $\mathcal{B}$ of $\mathcal{A}$ of dimension $n^2/2$. But is $\mathcal{B}$ also an algebra, i.e., closed under multiplication? The answer is yes because of the b.c.'s associated to this DCT; namely, $T_{-1} = T_1$ and $T_{n+1} = -T_{n-1}$ preserve the defining condition in (13). For example, products like $T_2(x) \cdot T_1(x)T_1(y) = (T_{-1}(x)T_1(y) + T_3(x)T_1(y))/2$ (using (4)) are again in $\mathcal{B}$.[4]

To bring $\mathcal{B}$ into the form (9), we need to find a set of generators, which then become our variables. Natural choices are

$$u = T_2(x) = 2x^2 - 1, \quad v = T_2(y) = 2y^2 - 1,$$

but their spanned algebra does not contain

$$w = T_1(x)T_1(y) = xy,$$

which thus we choose as the third generator. Since we have three polynomials u, v, w in two variables, they cannot be independent.

[3]The product of two lists $(c_1, \dots, c_n)$ and $(d_1, \dots, d_m)$ is the list of all products $(c_1d_1, c_1d_2, \dots)$ of length mn.

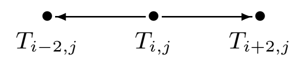
[4]In fact, using this argument, one can show that exactly 4 of the 16 DCTs/DSTs can be used as a starting point, namely the DCTs/DSTs of type 1 and 3. This yields 4 types of quincunx transforms. We consider only the DCT, type 3, since it is the best-known among those and due to lack of space.

Indeed, they satisfy the polynomial relation $4w^2 - (u+1)(v+1) = 0$. Further, from (6), $T_{n/2}(u) = T_n(x) = 0$ in $\mathcal{A}$. Analogously, $T_{n/2}(v) = 0$. Thus, we obtain the polynomial algebra

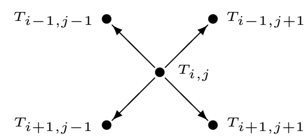
$$\mathcal{B} = \mathbb{C}[u, v, w]/\langle T_{n/2}(u), T_{n/2}(v), 4w^2 - (u+1)(v+1)\rangle. \qquad (14)$$

Using Hilbert's method [9, ch. 9], we confirm that $\dim(\mathcal{B}) = n^2/2$, i.e., $\mathcal{B}$ is indeed the *entire* algebra spanned by b in (13). Note that this algebra is not separable.

It is interesting to observe the operation of u, v, w on the quincunx lattice, i.e., the elements $T_{i,j} \in b$ in (13). Namely, using (4), $uT_{i,j} = (T_{i-2,j} + T_{i+2,j})/2$, which is visualized as ($1/2$ omitted)



Hence u acts on the quincunx lattice horizontally. Similarly, v acts vertically. The action of w, using again (4), is given by $wT_{i,j} = (T_{i-1,j-1} + T_{i-1,j+1} + T_{i+1,j-1} + T_{i+1,j+1})/4$, which is visualized as (the weights $1/4$ are omitted)



To compute the Fourier transform for $\mathcal{B}$, we need to 1) express the basis b in (13) in u, v, w, and 2) compute the spectrum of $\mathcal{B}$, i.e., the solutions of

$$T_{n/2}(u) = T_{n/2}(v) = 4w^2 - (u+1)(v+1) = 0. \qquad (15)$$

Basis. Let $T_{i,j}(x, y) \in b$ in (13). Then either both i, j are even or both are odd. If $i = 2i'$ and $j = 2j'$, then, using (6), $T_{i,j}(x, y) = T_{i'}(u)T_{j'}(v)$. If $i = 2i' + 1$, $j = 2j' + 1$, then

$$\begin{aligned} 4wT_{i,j}(x, y) &= (T_{i'}(u) + T_{i'+1}(u))(T_{j'}(v) + T_{j'+1}(v)) \\ &= (u+1)(v+1)V_{i'}(u)V_{j'}(v) \\ &= 4w^2 V_{i'}(u)V_{j'}(v) \end{aligned}$$

using (5) and (14). It follows that $T_{i,j}(x, y) = wV_{i'}(u)V_{j'}(v)$.

In summary, scanning the lattice in Fig. 2 in row major order, we get for rows with even index $i = 2i'$, $0 \le i' < n/2$,

$$T_{i'}(u)T_0(v), T_{i'}(u)T_1(v), \dots, T_{i'}(u)T_{n/2-1}(v), \qquad (16)$$

and for rows with odd index $i = 2i' + 1$, $0 \le i' < n/2$,

$$wV_{i'}(u)V_0(v), wV_{i'}(u)V_1(v), \dots, wV_{i'}(u)V_{n/2-1}(v). \qquad (17)$$

The basis b in (13) is the concatenation of these lists.

The spectrum of $\mathcal{B}$. The spectrum of $\mathcal{B}$ is determined by the solution of (15). Using (7), we get the $n^2/2$ solutions

$$\begin{aligned} u_k &= \cos\tfrac{k+1/2}{n/2}\pi, \quad 0 \le k < n/2, \\ v_\ell &= \cos\tfrac{\ell+1/2}{n/2}\pi, \quad 0 \le \ell < n/2, \\ w_{k,\ell,\pm} &= \pm\tfrac{1}{2}\sqrt{(1 + u_k)(1 + v_\ell)}. \end{aligned} \qquad (18)$$

Using $(1 + \cos(x)) = (\cos(\pi/4 + x/2) + \sin(\pi/4 + x/2))^2$, the latter can be simplified to

$$w_{k,\ell,\pm} = \pm\tfrac{1}{2}(\cos\gamma_k + \sin\gamma_k)(\cos\gamma_\ell + \sin\gamma_\ell), \qquad (19)$$

Table 1. The discrete quincunx transform DQT for computing the spectrum of a signal given on an $n/2 \times n$ spatial quincunx lattice.

$$\begin{aligned}\mathrm{DQT}_{n/2\times n} &= \mathrm{L}^{n^2/2}_{n^2/4}\begin{bmatrix}\text{DCT-3}_{n/2}\otimes\text{DCT-3}_{n/2} & D_{n^2/4}(\text{DCT-4}_{n/2}\otimes\text{DCT-4}_{n/2})\\ \text{DCT-3}_{n/2}\otimes\text{DCT-3}_{n/2} & -D_{n^2/4}(\text{DCT-4}_{n/2}\otimes\text{DCT-4}_{n/2})\end{bmatrix}(\mathrm{L}^n_2\otimes\mathrm{I}_{n/2})\\ &= \mathrm{L}^{n^2/2}_{n^2/4}(\mathrm{DFT}_2\otimes\mathrm{I}_{n^2/4})\left((\text{DCT-3}_{n/2}\otimes\text{DCT-3}_{n/2})\oplus D_{n^2/4}(\text{DCT-4}_{n/2}\otimes\text{DCT-4}_{n/2})\right)(\mathrm{L}^n_2\otimes\mathrm{I}_{n/2})\end{aligned}$$

with

$$\gamma_m = \frac{(m+1/2)\pi}{n} + \frac{\pi}{4}. \tag{20}$$

Transform. To determine the Fourier transform for the spatial quincunx lattice in Fig. 2, we evaluate the $n^2/2$ basis polynomials (given by concatenating alternately (16) and (17) for $0 \le i' < n/2$)) at the zeros in (18). We order the zeros lexicographically indexed as $(k, \ell, \pm)$, where $\pm \in \{+, -\}$ runs fastest, and $0 \le k, \ell < n/2$.

Evaluating all the even indexed rows (16) at all zeros indexed $(k, \ell, +)$ yields precisely a $\text{DCT-3}_{n/2} \otimes \text{DCT-3}_{n/2}$. The same is true for $(k, \ell, -)$ since (16) does not depend on w.

Evaluating the odd indexed rows (17) at all zeros indexed $(k, \ell, +)$ yields $D_{n^2/4}(\text{DCT-4}_{n/2} \otimes \text{DCT-4}_{n/2})$, where $D_{n^2/4}$ is a scaling diagonal arising from (8) and w in (17) evaluated at (19):

$$D_{n^2/4} = \mathrm{diag}_{0\le k,\ell<n/2}\, \frac{(\cos\gamma_k + \sin\gamma_k)(\cos\gamma_\ell + \sin\gamma_\ell)}{2\cos\frac{k+1/2}{n}\pi\cos\frac{\ell+1/2}{n}\pi},$$

where γ_n is defined in (20). Evaluating (17) at all zeros indexed $(k, \ell, -)$ yields $-D_{n^2/4}(\text{DCT-4}_{n/2} \otimes \text{DCT-4}_{n/2})$.

Definition 1 (Discrete quincunx transform) The discrete quincunx transform (DQT) is defined for an $n/2 \times n$ signal given in row major order on the spatial quincunx lattice in Fig. 2. The exact form is given in the first equation in Table 1.

The stride permutation, or perfect shuffle, L^m_2 interleaves the rows and columns to achieve the desired order of basis polynomials (alternating (16) and (17)); $\mathrm{L}^m_{m/2}$ is the inverse of L^m_2. The definition is

$$\mathrm{L}^m_2 : \ i(\tfrac{m}{2}) + j \mapsto 2j + i, \quad 0 \le i < 2,\ 0 \le j < \tfrac{m}{2}.$$

Fast algorithm and complexity. A straightforward decomposition of the $\mathrm{DQT}_{n/2\times n}$ yields the form in the second equation in Table 1, where

$$A \oplus B = \begin{bmatrix} A & \\ & B\end{bmatrix}.$$

This implies that the DQT can be computed using fast DCT-3 and DCT-4 algorithms. We analyze the complexity for a two-power n. In this case, DCT-3_n can be computed using $2n\log_2(n) - n + 1$ operations, and $D'\text{DCT-4}_n$, independent[5] of the diagonal matrix D', using $2n\log_2(n) + n$ operations [10]. Straightforward computation now yields the following lemma.

Lemma 1 (Complexity of the DQT) If n is a two-power, then the $\mathrm{DQT}_{n/2\times n}$ can be computed using

$$2n^2\log_2(n) + n^2/2 - n$$

operations.

[5] This is because there are best known algorithms, which have n multiplications in the end, which allows for fusion of a post-scaling.

Note that this bound assumes that the two-dimensional DCTs are computed using the row-column method, which is known to be suboptimal.

Lemma 1 shows that the DQT is with $O(n^2\log(n))$ in the same complexity class as its separable counterparts.

4. CONCLUSIONS

We derived a new spatial signal transform, the DQT, for computing the spectrum of a signal given on a finite quincunx lattice. The construction uses the connection between signal transforms and polynomial algebras, and is part of a larger effort to work towards an algebraic theory of signal processing (see [11]). The algebra also yields important properties of the DQT, such as the appropriate notion of filtering and diagonalization properties, which will be derived in a longer version of this paper.

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