AN EXTENSION OF A THEOREM OF HAMADA ON
THE CAUCHY PROBLEM WITH SINGULAR DATA

BY LESLIE LAMPORT
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Introduction. Hamada [1] proved the following result about the propagation of singularities in the Cauchy problem for an analytic linear partial differential operator. Assume that the initial data are analytic at the point 0 except for singularities along a submanifold $T$ of the initial surface containing 0. Let $K^{(1)}, \ldots, K^{(m)}$ be the characteristic surfaces of the operator emanating from $T$. Under the assumption that the $K^{(i)}$ have multiplicity one, he showed that the solution of the Cauchy problem is analytic at 0 except for logarithmic singularities along the $K^{(i)}$. We extend his result to the case where the $K^{(i)}$ have constant multiplicity.

1. Definitions and theorem. Let $C^{n+1}$ denote the set of $(n + 1)$-tuples $x = (x^0, \ldots, x^n)$ of complex numbers. Let $S$ be an $n$-dimensional analytic submanifold of $C^{n+1}$, and let $T$ be an $(n - 1)$-dimensional analytic submanifold of $S$. Since our results are local, we can assume $S = \{(0, x^1, \ldots, x^n) \in C^{n+1}\}$ and $T = \{(0, 0, x^2, \ldots, x^n) \in C^{n+1}\}$.

Let $D_i = \partial / \partial x^i, \mathbf{D} = (D_0, \ldots, D_n)$, and let $\mathbf{a} : x \to \mathbf{a}(x; D)$ be an analytic partial differential operator on a neighborhood of 0 in $C^{n+1}$. Let $h(x; \mathbf{D})$ be the principal part of $\mathbf{a}(x; D)$. We assume that $S$ is not a characteristic surface of $\mathbf{a}$ at 0, so $h(0; 1, 0, \ldots, 0) \neq 0$. Let $\mathbf{p} = (p_0, \ldots, p_n)$ be an $(n + 1)$-tuple of formal variables, so $h(x; \mathbf{p})$ is a homogeneous polynomial in $\mathbf{p}$ with analytic coefficients.

We say that the operator $\mathbf{a}$ has constant multiplicity at 0 in the direction of $T$ if we can factor $h$ as

$$h(x; \mathbf{p}) = [h_1(x; \mathbf{p})]^{k_1} \cdots [h_s(x; \mathbf{p})]^{k_s}$$

for all $x$ in a neighborhood of 0, where each $h_i(x; \mathbf{p})$ is a polynomial in $\mathbf{p}$ of degree $m_i$ with analytic coefficients, and the $\Sigma m_i$ roots of the polynomials $h_i(0; \tau, 1, 0, \ldots, 0)$ in $\tau$ are all distinct. If $s = k_1 = 1$, then $a$ is said to be of multiplicity one at 0 in the direction of $T$.

Assume now that $\mathbf{a}$ has constant multiplicity at 0 in the direction of $T$. It can be shown that we can find $m = \Sigma m_i$ analytic characteristic functions $\varphi^{(1)}, \ldots, \varphi^{(m)}$ of $h$ defined in a neighborhood $N$ of 0 satisfying:


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1. \( h(x; D \varphi^{(i)}(x)) = 0 \) for all \( x \in N \).
2. \( \varphi^{(i)}(0, x^1, \ldots, x^n) = x^1 \) for all \( (0, x^1, \ldots, x^n) \in N \cap S \).
3. For each \( y \in N \cap S \), the \( m \) numbers \( D_0 \varphi^{(i)}(y) \) are distinct.

Note that this implies that the numbers \( D_0 \varphi^{(i)}(y) \) are the distinct roots of the polynomials \( h(y; \tau, 1, 0, \ldots, 0) \) for each \( y \in N \cap S \). Let \( K^{(i)} = \{ x : \varphi^{(i)}(x) = 0 \} \), so each \( K^{(i)} \) is a characteristic surface of \( a \).

Using these notations, we now state our result.

**Theorem.** Let \( a, N, S, T, \varphi^{(i)} \) and \( K^{(i)} \) be as above. Let \( v \) be an analytic function on \( N \), and let \( w^j \) be an analytic function on \( N \cap (S - T) \) for \( j = 0, \ldots, r - 1 \), where \( r \) is the degree of the operator \( a \). Then there exists a neighborhood \( U \) of \( 0 \) such that the Cauchy problem

\[
\begin{align*}
(1) \quad a(x; D)u(x) &= v(x), \\
(D_0^{(i)}u(y) &= w^j(y), \quad \text{for } y \in S, j = 0, \ldots, r - 1,
\end{align*}
\]

has a solution \( u \) of the form

\[
u(x) = \sum_{i=1}^{m} F^{(i)}(x) + G^{(i)}(x) \log [\varphi^{(i)}(x)],
\]

where each \( F^{(i)} \) is analytic on \( U - K^{(i)} \) and each \( G^{(i)} \) is analytic on \( U \).

Hamada proved this result when \( a \) has multiplicity one. In this case, if each \( w^j \) has at most a polar singularity along \( T \), then each \( F^{(i)} \) has at most a polar singularity along \( K^{(i)} \). This is false in the general case, as is shown by the solution

\[
u(t, y) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k + 1)!} \frac{t^{2k+1}}{y^{k+1}}
\]

of the two-dimensional Cauchy problem

\[
\frac{\partial^2 u}{\partial t^2}(t, y) - \frac{\partial u}{\partial y}(t, y) = 0, \quad u(0, y) = 0, \quad \frac{\partial u}{\partial t}(0, y) = \frac{1}{y}.
\]

**2. Method of proof.** The problem is easily reduced to solving the Cauchy problem (1) with each \( w^j \equiv 0 \) and \( v \) analytic on \( N - K^{(1)} \). It can be shown that we may also assume that \( h(x; p) = h_1(x; p) \cdots h_s(x; p) \), where each \( h_i \) has multiplicity one in the direction of \( T \) and has \( \varphi^{(1)}, \ldots, \varphi^{(m)} \) as characteristic functions (so \( r = ms \)).

Let the functions \( f_k \) be the ones defined by Hamada satisfying \( df_k/\partial t = f_{k-1} \), for all integers \( k \), and \( f_0(t) = \log t \). The first step is to show that there exists a neighborhood \( V \) of \( 0 \) such that if \( v \) is of the form

\[
v(x) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} v^{(i)}_k(x) f_{k-1} [\varphi^{(i)}(x)],
\]

with each \( v^{(i)}_k \) analytic on \( V \), then the Cauchy problem
\( h(x;D)u(x) = v(x), \quad (D_0)^ju(y) = 0, \quad \text{for } y \in S, j = 0, \ldots, m - 1, \)

has a formal series solution of the form

\[
u(x) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} u_k^{(i)}(x)f_{k-l}^{m-1} \varphi^{(i)}(x),\]

with each \( u_k^{(i)} \) analytic on \( V \). Moreover, bounds are obtained for the partial derivatives of the \( u_k^{(i)} \) in terms of those of the \( v_k^{(i)} \). This procedure is similar to the one used by Hamada.

Employing this result \( s \) times shows that with \( v \) given by (2), the Cauchy problem

\[ h_1(x;D) \cdots h_r(x;D)u(x) = v(x), \quad (D_0)^ju(y) = 0, \quad \text{for } y \in S, j = 0, \ldots, r - 1, \]

has a formal solution

\[
u(x) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} u_k^{(i)}(x)f_{k-l+r-s}[\varphi^{(i)}(x)]\]

with the \( u_k^{(i)} \) analytic on \( V \). Again, bounds are obtained on the \( u_k^{(i)} \).

Now we write \( a(x;D) = h_1(x;D) \cdots h_r(x;D) + b(x;D) \), where the degree of \( b \) is less than \( r \). Using the above results, we solve the sequence of Cauchy problems

\[ h_1(x;D) \cdots h_r(x;D)u(x) = \begin{cases} v(x) & \text{if } q = 0, \\ -b(x;D)u_{q-1}(x) & \text{if } q > 0. \end{cases} \]

\[(D_0)^ju(y) = 0, \quad \text{for } y \in S, j = 0, \ldots, r - 1, \]

to get

\[ qu(x) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} qu_k^{(i)}(x)f_{k-l-q(s-1)}[\varphi^{(i)}(x)] \]

with each \( qu_k^{(i)} \) analytic on \( V \). Then

\[ \sum_{q=0}^{\infty} qu(x) \]

is easily seen to be a formal solution of (1) (with \( w^j = 0 \)).

Now assume \( v(x) = v_1(x)f_{-1}[\varphi^{(1)}(x)] \), with \( v_1 \) analytic on \( V \), and let the corresponding solution (4) be \( u(x) = \sum_{i=1}^{m} u^{(i)}(x) \). Using the bounds on the \( qu_k^{(i)} \), we can find a neighborhood \( W \) of \( 0 \) and demonstrate the absolute convergence of the sums (3) and (4) to prove that \( u^{(i)} \) is analytic on \( W - K^{(i)} \). Furthermore, we obtain a bound on \( u_k^{(i)} \) in terms of a bound on \( v_1 \).

Finally, we can write \( v(x) = \sum_{i=1}^{\infty} v_i(x)f_{-i}[\varphi^{(i)}(x)] \) (plus an analytic term which is handled by the Cauchy-Kowalewski theorem). It can be shown that there is a neighborhood \( U \) of \( 0 \) such that the sums \( u_k^{(i)}(x) = \sum_{i=1}^{\infty} u_i^{(i)}(x) \)
are absolutely convergent on $U - K^{(i)}$. It is then easily seen that the solution $u(x) = \sum_{i=1}^{m} w^{(i)}(x)$ has the desired form.

3. **Further generalizations.** It is evident from the proof that the theorem remains valid if $v$ has a singularity along any of the hypersurfaces $K^{(i)}$. The theorem is also true if $v$ has a singularity on any hypersurface $K$ containing $T$ which is not tangent to $S$ or to any $K^{(i)}$ at $0$.

By using different choices for the functions $f_k$, the result can be extended to the case where the $w^j$ are $p$-valued analytic functions on $N \cap (S - T)$—i.e., multiple-valued functions finitely ramified about $T$—and $v$ is a $p$-valued analytic function on $N - K^{(i)}$ or $N - K$. In this case, the $F^{(i)}$ become $p$-valued analytic functions on $U - K^{(i)}$. This result was also obtained by Wagschal [2] when $a$ has multiplicity one.

**REFERENCES**


Massachusetts Computer Associates, Lakeside Office Park, Wakefield, Massachusetts 01880