A bijective proof of Macdonald's reduced word formula

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We describe a bijective proof of Macdonald's reduced word identity using pipe dreams and Little's bumping algorithm. The proof extends to a principal specialization of the identity due to Fomin and Stanley. Our bijective tools also allow us to address a problem posed by Fomin and Kirillov from 1997, using work of Wachs, Lenart and Serrano-Stump.

Nous donnons une preuve bijective de l'identité des mots réduits de Macdonald, en utilisant des "pipe dreams" et l'algorithme "bumping" de Little. La preuve est étendue à une spécialisation principale de l'identité due à Fomin et Stanley. La méthode bijective nous permet aussi de traiter un problème posé par Fomin et Kirillov en 1997, en utilisant les travaux de Wachs, Lenart et Serrano-Stump.

Keywords: reduced words, Schubert polynomials, RC graphs, pipe dreams, bijective proof, algorithmic bijection

1 Introduction

Macdonald gave a remarkable formula connecting a weighted sum of reduced words for permutations with the number of terms in a Schubert polynomial $\mathfrak{S}_\pi(x_1,\ldots,x_n)$. For a permutation $\pi \in S_n$, let $\ell(\pi)$ be its inversion number and let $R(\pi)$ denote the set of its reduced words. (See Section 2 for definitions.)

Theorem 1.1 (Macdonald [Mac91, (6.11)]) *Given a permutation $\pi \in S_n$ with $\ell(\pi) = p$, one has*

$$\sum_{(a_1,a_2,\ldots,a_p)\in R(\pi)} a_1 \cdot a_2 \cdots a_p \ = p!\ \mathfrak{S}_\pi(1,\ldots,1). \tag{1.1}$$

For example, the permutation $[3,2,1] \in S_3$ has 2 reduced words, $R([3,2,1]) = \{(1,2,1),(2,1,2)\}$. The inversion number is $\ell([3,2,1]) = 3$, and the Schubert polynomial $\mathfrak{S}_\pi$ is the single term $x_1^2x_2$. We observe that Macdonald's formula holds: $1\cdot 2\cdot 1 + 2\cdot 1\cdot 2 = 3!\cdot 1$.

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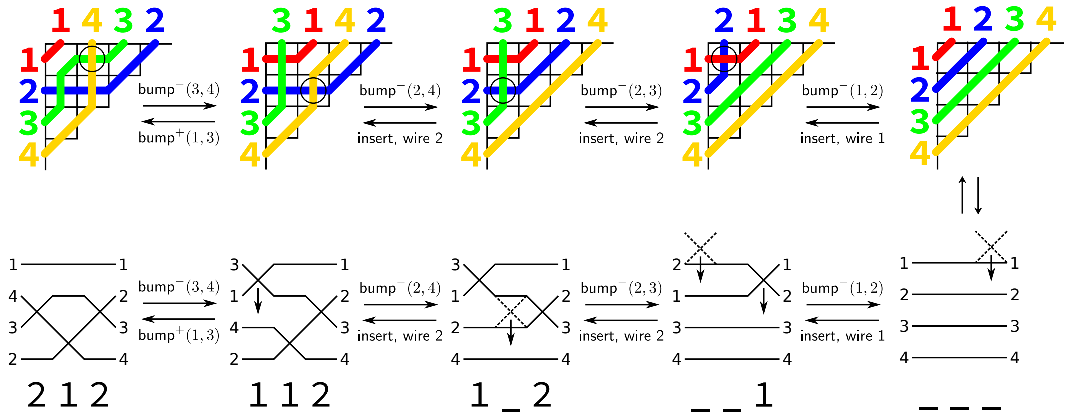


Fig. 1: An example of the bijection M for $w = [1,4,3,2]$ where the pair $(\mathbf{a},\mathbf{b})$ is mapped to $(\mathbf{c}, D)$ with $\mathbf{a} = (2,3,2)$, $\mathbf{b} = (2,1,2)$, $\mathbf{c} = (1,1,2)$, and D is the pipe dream in the top left corner of the picture. Its transition chain is $Y(D) = ((1,3),(2,0),(2,0),(1,0))$. Each vertical pair in the picture is also demonstrating the bijection for a different permutation; note the permutations on the wires agree on the vertical pairs.

In this extended abstract, we outline a bijective proof of Theorem 1.1. This approach has been sought for over 20 years. It has been listed as an open problem in both [FK97] and [R.P09]. Fomin and Sagan have stated that they have a bijective proof, but that it is unpublished due to its complicated nature; see [FK97]. Our proof builds on the work of the third author on a Markov Process on reduced words for the longest permutation [You14].

The Schubert polynomial $\mathfrak{S}_\pi$ can be expressed as a sum over *pipe dreams* (or *RC graphs*) corresponding to π, and its evaluation at $(1,\ldots,1)$ is simply the number of such pipe dreams. (See Section 2 for definitions, and [BJS93] for a proof.) Thus, the right side of (1.1) is the number of pairs $(\mathbf{c}, D)$, where $\mathbf{c} = (c_1,\ldots,c_p)$ is a vector with $1 \le c_i \le i$ for each i, and D is a pipe dream for π. The left side is the number of pairs $(\mathbf{a},\mathbf{b})$ where $\mathbf{a} \in R(\pi)$ and $\mathbf{b}$ is a vector satisfying $1 \le b_i \le a_i$ for each $i = 1,\ldots,p$. Our bijection is between pairs $(\mathbf{a},\mathbf{b})$ and $(\mathbf{c}, D)$ that satisfy these conditions.

The outline of the bijection is quite simple given some well-known properties of Schubert polynomials, together with the *bumping algorithm* for reduced words introduced by Little [Lit03]. These properties and objects will be defined in Section 2. We first give a modification of Little's bumping algorithm that applies to pipe dreams, and use it to give a bijective interpretation to Lascoux and Schützenberger's transition equation for Schubert polynomials. Essentially the same construction has been given by Buch [Knu12, p.11]. The key step is to recursively apply the corresponding transition map to D until it is the empty pipe dream and record the sequence of steps according to what happened at the end of each bump so that the process is reversible. We call this sequence a transition chain, and denote it by $Y(D)$.

Next we use Little's bumping algorithm to push crossings up in value in a wiring diagram according to the reverse of the transition chain. The vector $\mathbf{c} = (c_1,\ldots,c_p)$ tells us where to add a new crossing on the steps in the transition chain where we need to insert a new crossing; when adding the ith new crossing it should become the c_ith entry in the word. For example, because $\mathbf{c} = (1,1,2)$ was chosen, we see that reading from right to left along the second row of Figure 1, a crossing is added to the empty wiring diagram in the first column, the second time a crossing is added it again becomes the first column, and the third time a crossing is added it becomes the second column. The row of the added crossing is determined by placing its feet on the wire specified by the corresponding step of the transition chain. Each new crossing is immediately pushed up in value, initiating a Little bump. The result is a reduced wiring diagram for π corresponding to a reduced word $\mathbf{a} = (a_1, a_2, \ldots, a_p)$. If we bumping processes, we obtain a vector $\mathbf{b} = (b_1, \ldots, b_p)$ of the same length such that $a_i \geq b_i$ for all i, as required. See Figure 1 for an illustration of the algorithm. Each step is reversible.

A computer implementation of this bijection will be made available at `http://www.math.washington.edu/~billey/papers/macdonald/`.

1.1 Further results: q-analog

Our bijective proof extends to a q-analog of (1.1) that was conjectured by Macdonald and subsequently proved by Fomin and Stanley. To state this formula, let q be a formal variable. Define the q-analog of a positive integer k to be $[k] = [k]_q := (1 + q + q^2 + \cdots + q^{k-1})$. The q-analog of the factorial $k!$ is defined to be $[k]! = [k]_q! := [k][k-1]\cdots[1]$. (We use the blackboard bold symbol ! to distinguish it from the ordinary factorial of k.) For $\mathbf{a} = (a_1, a_2, \ldots, a_p) \in R(\pi)$, define the co-major index

$$\operatorname{comaj}(\mathbf{a}) := \sum_{\substack{1 \leq i \leq p: \\ a_i < a_{i+1}}} i.$$

Theorem 1.2 (Fomin and Stanley [FS94, Thm. 2.4]) *Given a permutation $\pi \in S_n$ with $\ell(\pi) = p$, one has*

$$\sum_{\mathbf{a}=(a_1,a_2,\ldots,a_p)\in R(\pi)} [a_1] \cdot [a_2] \cdots [a_p]\, q^{\operatorname{comaj}(\mathbf{a})} = [p]!\ \mathfrak{S}_\pi(1, q, q^2, \ldots, q^{n-1}). \tag{1.2}$$

Continuing with the example $\pi = [3, 2, 1]$, we observe that the q-analog formula indeed holds: $q[1] \cdot [2] \cdot [1] + q^2[2] \cdot [1] \cdot [2] = q(1+q) + q^2(1+q)^2 = q + 2q^2 + 2q^3 + q^4 = [3]! \cdot q = [3]! \cdot \mathfrak{S}_{[3,2,1]}(1, q, q^2)$.

Our proof of Theorem 1.2 is omitted from this abstract due to space constraints. Our approach is to define statistics on $(\mathbf{a}, \mathbf{b})$ pairs and on (c, D) pairs, and interpret the left and right sides of Theorem 1.2 as a statement about the corresponding generating functions. We then show that our map M preserves these statistics.

1.2 Further results: Fomin-Kirillov identity

In 1997, Fomin and Kirillov published an extension to Theorem 1.2 in the case where π is a dominant permutation whose Rothe diagram is the partition λ (see Section 2 for definitions of these terms). To state it we need the following definitions:

Let $rpp^\lambda(x)$ be the set of *weak reverse plane partitions* whose entries are all in the range $[0, x]$ for $x \in \mathbb{N}$. This is the set of x-bounded fillings of λ with rows and columns weakly increasing to the right and down. Given a weak reverse plane partition R, let $|R|$ be the sum of its entries. Let

$$[rpp^\lambda(x)]_q = \sum_{R \in rpp^\lambda(x)} q^{|R|}.$$

Finally, for $\pi = [\pi_1, \ldots, \pi_n]$, let $1^x \times \pi = [1, 2, \ldots, x, \pi_1 + x, \pi_2 + x, \ldots, \pi_n + x]$.

Theorem 1.3 *[FK97, Thm. 3.1] For any partition $\lambda \vdash p$, we have the following identity for all $x \in \mathbb{N}$*

$$\sum_{(a_1,a_2,\ldots,a_p)\in R(\sigma_\lambda)} q^{\mathrm{comaj}(a_1,a_2,\ldots,a_p)}[x+a_1]\cdot[x+a_2]\cdots[x+a_p] \tag{1.3}$$

$$= [p]_q!\ \mathfrak{S}_{1^x\times\sigma_\lambda}(1,q,q^2,\ldots,q^{x+p}) \tag{1.4}$$

$$= [p]_q!\ q^{b(\lambda)}\ [rpp^\lambda(x)]_q \tag{1.5}$$

where $b(\lambda) = \sum_i (i-1)\lambda_i$.

Fomin-Kirillov proved this non-bijectively, using known results from Schubert calculus, and asked for a bijective proof. Using our results, together with results of Lenart [Len04], and Serrano and Stump [SS11, SS12], we are able to provide a bijective proof. We omit this proof from the abstract. See also [BJS93, Woo04] for connections to plane partitions.

2 Background

2.1 Permutations

We recall some basic notation and definitions relating to permutations which are standard in the theory of Schubert polynomials. We refer the reader to [LS82, Mac91, Man01] for references.

For $\pi \in S_n$, an *inversion of* π is an ordered pair (i,j), with $1 \leq i < j \leq n$, such that $\pi(i) > \pi(j)$. The *length* $\ell(\pi)$ is the number of inversions of π. We write t_{ij} for the transposition which swaps i and j, and we write $s_i = t_{i,i+1}$ $(1 \leq i \leq n-1)$. The s_i are called *simple transpositions*; they generate S_n as a Coxeter group.

We will often write a permutation π in one-line notation, $[\pi(1), \pi(2), \ldots, \pi(n)]$. If $\pi(j) > \pi(j+1)$, then we say that π has a *descent* at j.

An alternate notation for a permutation π is its *Lehmer code*, or simply *code*, which is n-tuple

$$(L(\pi)_1, L(\pi)_2, \ldots, L(\pi)_n)$$

where $L(\pi)_i$ denotes the number of inversions (i,j) with the first coordinate fixed. The permutation π is said to be *dominant* if its code is a weakly decreasing sequence.

2.2 Reduced words

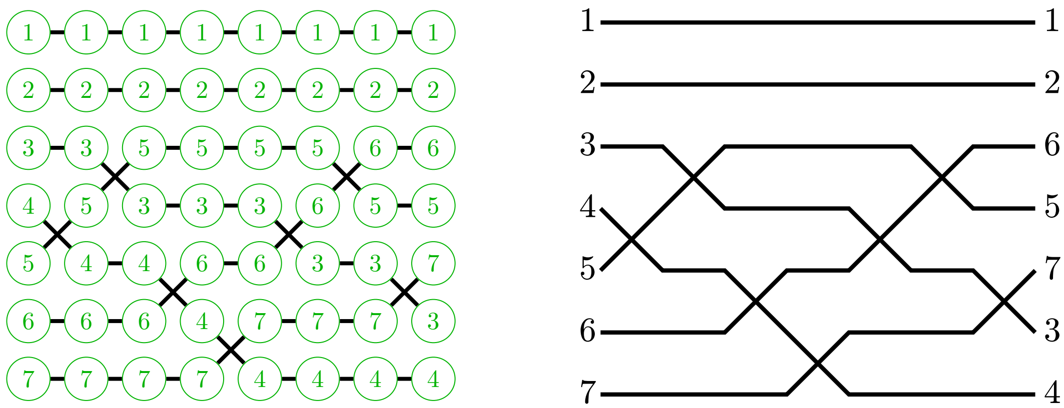
Let $\pi \in S_n$ be a permutation. A *word for* π is a k-tuple of numbers $\mathbf{a} = (a_1, \ldots, a_k)$ $(1 \leq a_i < n)$ such that

$$s_{a_1} s_{a_2} \ldots s_{a_k} = \pi.$$

Throughout this paper we will identify S_n with its image under the standard embedding $\iota : S_n \hookrightarrow S_{n+1}$; observe that this map sends Coxeter generators to Coxeter generators, so a word for π is also a word for $\iota(\pi)$. If $k = \ell(\pi)$, then we say that $\mathbf{a}$ is *reduced*. The reduced words are precisely the minimum-length ways of representing π in terms of the simple transpositions. For instance, the permutation $321 \in S_3$ has two reduced words: 121 and 212.

Write $R(\pi)$ for the set of all reduced words of the permutation π. The set $R(\pi)$ has received much study, largely due to interest in Bott-Samelson varieties and Schubert calculus. Its size has an interpretation in terms of counting standard tableaux and the Stanley symmetric functions [LS82, Lit03, Sta84, EG87].

Fig. 2: The wiring diagram for the reduced word $(4, 3, 5, 6, 4, 3, 5) \in R([1, 2, 6, 5, 7, 3, 4])$ showing the intermediate permutations on the left and a simplified version of the same wiring diagram on the right. The crossings in positions 2 and 6 are both in row 3.



Define the *wiring diagram* for a word $\mathbf{a}$ as follows. First, for $0 \le t \le k$, define $\pi_t \in S_n$ by

$$\pi_t = s_{a_1} s_{a_1} \cdots s_{a_t}.$$

In particular, when $t = 0$ the product on the right is empty, so that π_0 is the identity permutation. Also, $\pi_k = \pi$. The ith *wire* of $\mathbf{a}$ is defined to be the piecewise linear path joining the points $(\pi_t(i), t)$ for $0 \le t \le k$. We will consistently use "matrix coordinates" to describe wiring diagrams, so that (i, j) refers to row i (numbered from the top of the diagram) and column j (numbered from the left). The *wiring diagram* is the union of these n wires. See Figure 2 for an example.

For all $t \ge 1$, observe that between columns $t - 1$ and t in the wiring diagram, precisely two wires i and j intersect. This configuration is called a *crossing*. One can identify a crossing by its *position* t. When the word $\mathbf{a}$ is reduced, the minimality of the length of $\mathbf{a}$ ensures that any two wires cross at most once. In this case, we can also identify a crossing by the unordered pair $\{i, j\}$ of wires which are involved. In a wiring diagram, we call a_t the *row* of the crossing at position t.

2.3 Little's bumping algorithm

Little's algorithm [Lit03] is a map on reduced words. It was introduced to study the decomposition of Stanley symmetric functions into Schur functions in a bijective way. Later, Little's algorithm was found to be related to the RSK [Lit05] and Edelman-Greene [HY14] algorithms; it has been extended to signed permutations [BHRY14] and affine permutations [LS05]. We describe here a variant of the algorithm, which is the key building block for our bijective proofs. Our exposition follows that of [You14].

Definition 2.1 *Let* $\mathbf{a} = (a_1, \ldots, a_k)$ *be a word. Define the* push up, push down *and* deletion *of* $\mathbf{a}$ *at* *position t, respectively, to be*

$$\begin{aligned}\mathscr{P}_t^- a &= (a_1, \ldots, a_{t-1}, a_t - 1, a_{t+1}, \ldots, a_k),\\ \mathscr{P}_t^+ a &= (a_1, \ldots, a_{t-1}, a_t + 1, a_{t+1}, \ldots, a_k),\\ \mathscr{D}_t a &= (a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_k).\end{aligned}$$

In [You14], the notation $\mathscr{P}^\uparrow$ was used to represent $\mathscr{P}^-$, and $\mathscr{P}^\downarrow$ was used to represent $\mathscr{P}^+$.

Definition 2.2 *Let* $\mathbf{a}$ *be a word. If* $\mathscr{D}_t a$ *is reduced, then we say that* $\mathbf{a}$ *is* nearly reduced at t.

The term "nearly reduced" was coined by Lam et al [LLM$^+$14, Chapter 3], who uses "t-marked nearly reduced". Words that are nearly reduced at t may or may not also be reduced; however, every reduced word $\mathbf{a}$ is nearly reduced at some index t. For instance, any reduced word $\mathbf{a}$ of length k is nearly reduced at 1 and also at k.

In order to define our variant of Little's bumping map, we need the following lemma, which to our knowledge first appeared in [Lit03, Lemma 4], and was later generalized to arbitrary Coxeter systems in [LS05, Lemma 21].

Lemma 2.3 *If* $\mathbf{a}$ *is not reduced, but is nearly reduced at* t*, then* $\mathbf{a}$ *is nearly reduced at exactly one other position* $t' \neq t$.

Definition 2.4 *In the situation of Lemma 2.3, we say that* t' forms a defect with t *in* $\mathbf{a}$*, and write* $\mathrm{Defect}_t(\mathbf{a}) = t'$.

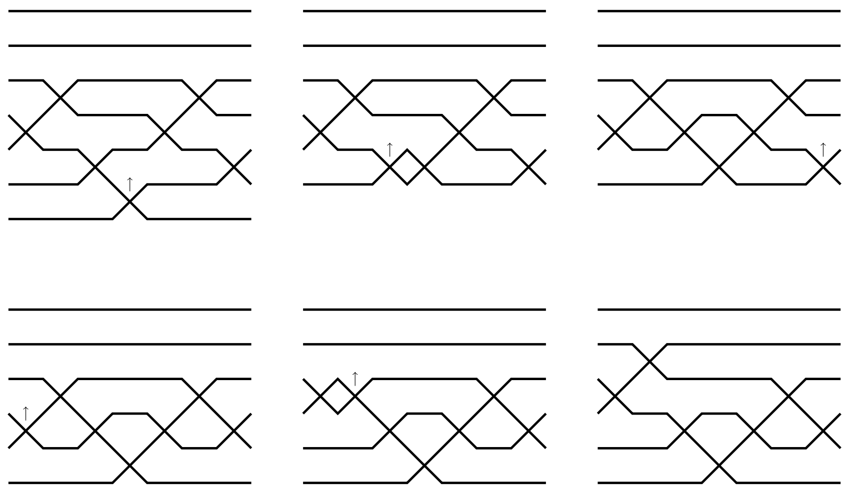
Note that in Definition 2.4, in the wiring diagram of $\mathbf{a}$, the two wires crossing in position t cross in exactly one other position t'.

Definition 2.5 *A word* $\mathbf{b}$ *is a* bounded word *for another word* $\mathbf{a}$ *if the words have the same length and* $1 \leq b_i \leq a_i$ *for all* i*. A* bounded pair *(for a permutation* π*) is an ordered pair* $(\mathbf{a}, \mathbf{b})$ *such that* $\mathbf{a}$ *is a reduced word (for* π*) and* $\mathbf{b}$ *is a bounded word for* $\mathbf{a}$*. Let* $\mathrm{BoundedPairs}(\pi)$ *be the set of all bounded pairs for* π.

Algorithm 2.6 (Bounded Bumping Algorithm) *The following is a modification of Little's generalized bumping algorithm, defined in [Lit03]. See Figure 3 for an example.*
Input: $(\mathbf{a}', \mathbf{b}', t_0, d)$*, where* $\mathbf{a}'$ *is a word that is nearly reduced at* t_0*,* $\mathbf{b}'$ *is a bounded word for* $\mathbf{a}'$*, and* $d \in \{-, +\}$ *is a direction.*
Output: $(\mathbf{a}, \mathbf{b}, i, t)$*, where* $\mathbf{a}$ *is a reduced word,* $\mathbf{b}$ *is a bounded word for* $\mathbf{a}$*,* i *is a wire number, and* t *is a position.*

1. *Initialize* $\mathbf{a} \leftarrow \mathbf{a}', \mathbf{b} \leftarrow \mathbf{b}', t \leftarrow t_0$.
2. $\mathbf{a} \leftarrow \mathscr{P}_t^d \mathbf{a}$*,* $\mathbf{b} \leftarrow \mathscr{P}_t^d \mathbf{b}$.
3. *If* $b_t = 0$*, let* i *be the larger-numbered of the two wires swapped by the crossing at position* t *of* $\mathbf{a}$*. Return* $(\mathscr{D}_t \mathbf{a}, \mathscr{D}_t \mathbf{b}, i, t)$ *and stop.*
4. *If* $\mathbf{a}$ *is reduced, return* $(\mathbf{a}, \mathbf{b}, i, 0)$ *and stop.*
5. $t \leftarrow \mathrm{Defect}_t(\mathbf{a})$*; Go to step 2.*

Fig. 3: An example of the Little bump algorithm with inputs $(\mathbf{a}, \mathbf{b}, 4, -)$, with $\mathbf{a} = (4, 3, 5, 6, 4, 3, 5)$ and $\mathbf{b} = \mathbf{a}$. The arrows indicate which crossing will move up in the next step.



Observe that $\mathbf{a}$ need not be a reduced word in order for Algorithm 2.6 to be well-defined. Rather, $\mathbf{a}$ need only be nearly reduced at t_0. Note also that the algorithm only halts in step 3 when $d = -$.

There are two significant differences between Little's map θ_r in [Lit03] and the map defined by Algorithm 2.6, other than the indexing. One difference is that θ_r shifts the entire word down (by applying $\prod_t \mathscr{P}_t^+$, in our terminology) if a crossing is pushed onto the zero line, whereas Algorithm 2.6 does not; our step 3 prevents this situation from occurring. The second difference is that Little's map acts only on the word and not its bounded word; Little's map does not have an equivalent of step 3.

Since the algorithm for Little's θ_r terminates [Lit03], so does Algorithm 2.6. Similarly, if Algorithm 2.6 is applied to $(\mathbf{a}', \mathbf{b}')$ and terminates during step 4 returning $(\mathbf{a}, \mathbf{b}, i, 0)$, then $\mathrm{comaj}(a) = \mathrm{comaj}(a')$; this is essentially because Little's map θ_r also preserves the comajor index (and indeed the descent set). If it terminates during step 3, the algorithm has a complicated effect on $\mathrm{comaj}\, a$, whose description we omit from this abstract; it is needed for our proof of Theorem 1.2.

2.4 Pipe Dreams and Schubert Polynomials

Schubert polynomials $\mathfrak{S}_\pi$ for $\pi \in S_n$ are a generalization of Schur polynomials that were invented by Lascoux and Schützenberger in the early 1980s [LS82]. They have been widely used over the past 30 years in research. An excellent summary of the early work on these polynomials appears in Macdonald's "Notes on Schubert polynomials" [Mac91]; see also the book by Manivel for a more recent treatment [Man01].

A *pipe dream* D is a finite subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$. We will usually draw a pipe dream as a modified wiring diagram as follows. Place a + at every point $(i, j) \in D$; place a pair of elbows at every point $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \setminus D$. This creates wires connecting points on the left side of the diagram to points on

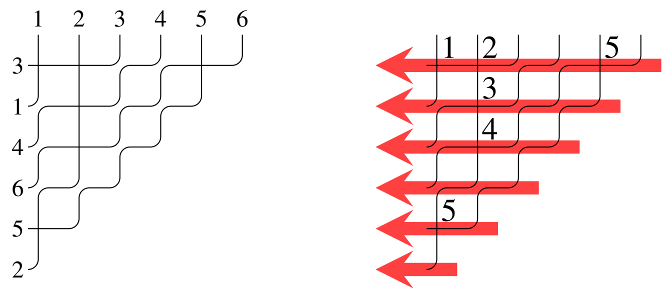


Fig. 4: A reduced pipe dream for $w = [3, 1, 4, 6, 5, 2]$. The corresponding reduced word is $\mathbf{a} = (5, 2, 1, 3, 4, 5)$ and $x^D = x_1^3 x_2 x_3 x_5$.

the top. If the wires are numbered $1, 2, 3, \ldots$ across the top of the diagram, then the corresponding wires reading down the left side of the diagram form a finitely supported permutation π of the positive integers called *the permutation of* D; again, we identify S_n with its image under the inclusion $S_n \hookrightarrow S_{\mathbb{N}}$, and thus the finiteness of D means that $\pi \in S_n$ for some n. We only need to draw a finite number of wires in a triangular array to represent a pipe dream since for all large enough wires there are no crossings. See Figure 4 for an example.

Following the terminology for reduced words, we say that D is *reduced* if π is the permutation of D, and $\ell(\pi) = |D|$. We write $\mathcal{RP}(\pi)$ for the set of all reduced pipe dreams for π. We say that the *weight* of a pipe dream D is

$$x^D = \prod_{(i,j)\in D} x_i$$

where $x_1, x_2, \ldots$ are formal variables.

Definition 2.7 *The* Schubert polynomial *of* $\pi \in S_n$ *is defined to be*

$$\mathfrak{S}_\pi = \sum_{D \in \mathcal{RP}(\pi)} x^D.$$

For example, the top line of Figure 1 shows pipe dreams for 5 different permutations. The pipe dream in the middle of the figure for $[3, 1, 2, 4]$ is unique so $\mathfrak{S}_{[3,1,2,4]} = x_1 x_2$. The pipe dream on the left for $[1, 4, 3, 2]$ is not the only one. There are 5 pipe dreams for $w = [1, 4, 3, 2]$ in total and

$$\mathfrak{S}_{[1,4,3,2]} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_2^2.$$

There are many other equivalent definitions of Schubert polynomials [LS82, BJS93, FS94, BB93]. Note that pipe dreams are also called *pseudo-line* arrangements and *RC-graphs* in the literature. See [KM05, Kog00] for other geometric and algebraic interpretations of individual pipe dreams.

We call the elements of $D \in \mathcal{RP}(\pi)$ *crossings* or *occupied positions*, and the elements of $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \setminus D$ *unoccupied positions*. Each crossing involves two wires, which are said to *enter the crossing horizontally* and *vertically*.

As explained in [BB93], a reduced pipe dream D is precisely a reduced word together with some extra information about where to place the crossings. More precisely, if D is a reduced pipe dream, and we replace each crossing $(i, j) \in D$ with the value $(j + i - 1)$ and then read the values along the rows of D from right to left, and then read the rows top to bottom (see Figure 4, right picture), we get a reduced word for the permutation of D.

This map from pipe dreams to reduced words is not bijective; to obtain a bijection, we need to introduce the notion of *compatible sequence* from [BB93]. The *compatible sequence* is read from D by reading the row numbers of the crossings in the same order (reading the values along the rows of D right to left, and then reading the rows top to bottom). Note the compatible sequence contains the same information as the monomial x^D. The compatible sequences for the reduced word $(a_1, a_2, \ldots, a_p)$ may be characterized as the set of sequences $(i_1, i_2, \ldots, i_p)$ which have the following three properties:

1. $i_1 \leq i_2 \leq \ldots \leq i_p$,
2. $a_j < a_{j+1}$ implies $i_j < i_{j+1}$,
3. $i_j \leq a_j$ for all j.

For example, Figure 4 shows the pipe dream corresponding with reduced word $(5, 2, 1, 3, 4, 5)$ and compatible sequence $(1, 1, 1, 2, 3, 5)$.

Finally, the *bounded word associated to* D is read from D by reading the *column* numbers of the crossings, again in the same order. Observe that D can be reconstructed from its associated reduced word and bounded word.

3 Transition equation

To describe the bijection M from cD-pairs to bounded pairs, we will first consider a sequence of pipe dreams starting with D and moving toward the empty pipe dream. This is done using Algorithm 2.6, following steps from the transition equation due to Lascoux-Schützenberger.

Theorem 3.1 (Lascoux and Schützenberger, Transition Equation) *[Mac91, 4.16] For all permutations π with $\ell(\pi) > 0$, the Schubert polynomial $\mathfrak{S}_\pi$ is determined by the recurrence*

$$\mathfrak{S}_\pi = x_r \mathfrak{S}_\nu + \sum_{\substack{i<r \\ l(\pi)=l(\nu t_{ir})}} \mathfrak{S}_{\nu t_{ir}} \tag{3.1}$$

where r is the last descent of π, s is the largest index such that $\pi_s < \pi_r$, $\nu = \pi t_{rs}$. The base case of the recurrence is $\mathfrak{S}_{id} = 1$.

Proof: (sketch) Algorithm 2.6 can be used to prove this theorem bijectively. Indeed, in this case, our algorithm coincides with an algorithm of Buch, described first in [Knu12, p.11].

Suppose that D' is a pipe dream associated to π. Let $(\mathbf{a}', \mathbf{b}')$ be the reduced word and bounded word associated to the pipe dream D'. Let π be the permutation associated to D', and let r, s be as in Theorem 3.1. By construction, $\ell(\nu) = \ell(pi) - 1$ since the inversions of ν are exactly the same as the inversions of π except for the pair (r, s). Therefore, if t' is the position of the r, s-wire crossing in the wiring diagram for $\mathbf{a}$, then $\mathbf{a}$ is nearly reduced in position t'. This is a key observation used in prior work as well [Lit03, LS82, Gar02]. Apply Algorithm 2.6 with arguments $(\mathbf{a}', \mathbf{b}', t', -)$, with output $(\mathbf{a}, \mathbf{b}, i, t)$. Since $\mathbf{b}$ is encoding the column number, Algorithm 2.6 returns a pair $(\mathbf{a}, \mathbf{b})$ which is associated to a new pipe

Tab. 1: An example of the M bijection starting with reduced word $(4,3,5,6,4,3,5)$ and compatible sequence $(1,1,2,3,3,3,4)$ for the permutation $\pi = [1,2,6,5,7,3,4]$. Here the code vector $\mathbf{c}$ is chosen to be $(1,1,1,3,2,1,3)$. Thus, $M(\mathbf{c}, D) = (5435645, 1423535) \in BoundedPairs(\pi)$.

rc pair	transition chain	$\mathbf{d}$	$\mathbf{a}, \mathbf{b}$	$perm$
$(4356435, 1123334)$	(25)	0	$(5435645, 1423535)$	[1265734]
$(3245434, 1123334)$	(40)	3	$(5324534, 1312424)$	[146523]
$(323543, 112333)$	(14)	0	$(523423, 121313)$	[146325]
$(312543, 112333)$	(30)	1	$(513423, 111313)$	[246135]
$(31243, 11233)$	(30)	2	$(13423, 11313)$	[24513]
$(3123, 1123)$	(30)	3	$(1323, 1213)$	[2431]
$(312, 112)$	(12)	0	$(132, 122)$	[2413]
$(212, 112)$	(20)	1	$(121, 111)$	[321]
$(21, 11)$	(10)	1	$(21, 11)$	[312]
$(1, 1)$	(10)	1	$(1, 1)$	[213]
$(\emptyset, \emptyset)$			$(\emptyset, \emptyset)$	[123]

dream D. One checks that D is one of the pipe dreams enumerated by the right side of the transition equation, using properties of Little's bumping map [Lit03]. □

4 Bijective proof of Macdonald formula - sketch

Our bijection M involves translating $(\mathbf{a}, \mathbf{b})$ pairs (both general ones, and the specific ones associated to pipe dreams D) into objects called *transition chains*, by iteratively applying Algorithm 2.6at the crossings specified by the transition equation.

Definition 4.1 *Given $(\mathbf{a}', \mathbf{b}')$ a bounded pair, define the associated* transition chain $Y(\mathbf{a}', \mathbf{b}')$ *recursively as follows. If $\mathbf{a}$ is the empty word, then $Y(\mathbf{a}', \mathbf{b}') = ()$, the empty list. Otherwise let π be the permutation associated to $\mathbf{a}$, let r, s be as in Theorem 3.1, and let t' be the position of the (r,s) crossing. Apply Algorithm 2.6 with arguments $(\mathbf{a}', \mathbf{b}', t', -)$ obtaining $(\mathbf{a}, \mathbf{b}, i, t)$. Recursively compute*

$$Y(\mathbf{a}, \mathbf{b}) = ((i_{q-1}, r_{q-1}), \ldots, (i_2, r_2), (i_1, r_1)).$$

Then define

$$Y(\mathbf{a}', \mathbf{b}') = ((i, r), (i_{q-1}, r_{q-1}), \ldots, (i_2, r_2), (i_1, r_1)).$$

One can prove inductively that $\mathcal{RP}(\pi)$ is in bijection with the set of all transition chains for π, using the fact that the map in Algorithm 2.6, with the parameters specified in the definition of the transition chain, is a bijective realization of the transition equation.

Similarly, one can map a pair $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}$ is a reduced word for the permutation π and $\mathbf{b}$ is a bounded sequence for $\mathbf{a}$, to the transition chain $Y(\mathbf{a}, \mathbf{b})$. However, this map is not bijective. To make it bijective, while computing the transition chain, let $\mathbf{c}(\mathbf{a}, \mathbf{b}) = (c_0, \ldots, c_k)$ be the sequence of nonzero values of t returned by Algorithm 2.6. $\mathbf{c}$ is the code of a permutation in S_k, where k is the length of $\mathbf{a}$. One can show that the map $(\mathbf{a}, \mathbf{b}) \mapsto (c, Y(\mathbf{a}, \mathbf{b}))$ is a bijection, by showing that the pairs $(\mathbf{a}, \mathbf{b})$ themselves satisfy a variant of the transition equation for bounded pairs. Unfortunately we need to omit the details of this argument.

Thus, the desired bijection which proves Macdonald's reduced word identity is the composition of the first transition chain bijection with the inverse of the second.

Definition 4.2 *Given (c, D) a CD pair, compute $(\mathbf{a}', \mathbf{b}')$ associated to D, and then compute $Y(\mathbf{a}', \mathbf{b}')$. Then, compute a new bounded pair $(\mathbf{a}, \mathbf{b})$ from $Y(\mathbf{a}', \mathbf{b}')$, inserting the crossings at positions indicated by the code vector c. Define $M(c, D)$ to be $(\mathbf{a}, \mathbf{b})$.*

An example of this bijection appears in Table 1. Full details will appear in the full version of this extended abstract.

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