Local central limit theorems in stochastic geometry

Mathew D. Penrose[1] and Yuval Peres[2]
University of Bath and Microsoft Research

August 16, 2011

Abstract

We give a general local central limit theorem for the sum of two independent random variables, one of which satisfies a central limit theorem while the other satisfies a local central limit theorem with the same order variance. We apply this result to various quantities arising in stochastic geometry, including: size of the largest component for percolation on a box; number of components, number of edges, or number of isolated points, for random geometric graphs; covered volume for germ-grain coverage models; number of accepted points for finite-input random sequential adsorption; sum of nearest-neighbour distances for a random sample from a continuous multidimensional distribution.

Key words and phrases: Local central limit theorem, stochastic geometry, percolation, random geometric graph, nearest neighbours.
AMS classifications: 60F05, 60D05, 60K35, 05C80

[1] Department of Mathematical Sciences, University of Bath, Bath BA1 7AY, United Kingdom: m.d.penrose@bath.ac.uk
[1] Partially supported by the Alexander von Humboldt Foundation through a Friedrich Wilhelm Bessel Research Award.
[2] Microsoft research, Redmond, WA USA: peres@microsoft.com

Contents

1 Introduction

A number of general central limit theorems (CLTs) have been proved recently for quantities arising in stochastic geometry subject to a certain local dependence. See [18, 19, 20, 21, 22] for some examples. The present work is concerned with *local* central limit theorems for such quantities. The local CLT for a binomial (n, p) variable says that for large n with p fixed, its probability mass function minus that of the corresponding normal variable rounded to the nearest integer, is uniformly $o(n^{-1/2})$. The classical local CLT provides similar results for sums of i.i.d. variables with an arbitrary distribution possessing a finite second moment. Here we are concerned with sums of variables with some weak dependence, in the sense that the summands can be thought of as contributions from spatial regions with only local interactions between different regions.

Among the examples for which we obtain local CLTs here are the following. In Section 3 we give local CLTs for the number of clusters in percolation on a large finite lattice box, and for the size of the largest open cluster for supercritical percolation on a large finite box, as the box size becomes large. In Sections 4 and 5 we consider continuum models, starting with random geometric graphs [18] for which we demonstrate local CLTs for the number of copies of a fixed subgraph (for example the number of edges) both in the thermodynamic limit (in which the mean degree is $\Theta(1)$) and in the sparse limit (in which the mean degree vanishes). For the thermodynamic limit we also derive local CLTs for the number of components of a given type (for example the number of isolated points), as an example of a more general local CLT for functionals which have finite range interactions or which are sums of functions determined by nearest neighbours (Theorem 5.1). This also yields local CLTs for quantities associated with a variety of other models, including germ-grain models and random sequential adsorption in the continuum.

We derive these local CLTs using the following idea which has been seen (in somewhat different form) in [8], in [4], and no doubt elsewhere. If the random variable of interest is known to satisfy a CLT, and can be decomposed (with high probability) as the sum of two independent parts, one of which satisfies a local CLT with the same order of variance growth, then one can find a local CLT for the original variable. Theorem 2.1 below formalises this idea. The statement of this result has no geometrical content and it could be of use elsewhere.

In the geometrical context, one can often use the geometrical structure to effect such a decomposition. Loosely speaking, in these examples one can represent a positive proportion of the spatial region under consideration as a union of disjoint boxes or balls, in such a way that with high probability a non-vanishing proportion of the boxes are 'good' in some sense, where the contributions to the variable of interest from a good box, given the configuration outside the box and given that it has the 'good' property, are i.i.d. Then the classical local CLT applies to the total contribution from good boxes, and one can represent the variable of interest as the sum of two independent contributions, one of which (namely the contribution from good boxes) satisfies a local CLT, and then apply Theorem 2.1. This technique is related to a method used by Avram and Bertsimas [1] to find lower bounds on the variance for certain quantities in stochastic geometry, although the examples considered here are mostly different from those considered in [1].

In any case, our results provide extra information on the CLT behaviour for variables for numerous geometrical and multivariate stochastic settings, which have arisen in a variety of applications (see the examples in Section 5).

2 A general local CLT

In the sequel we let ϕ denote the standard ($\mathcal{N}(0,1)$) normal density function, i.e. $\phi(x) = (2\pi)^{-1/2}\exp(-(1/2)x^2)$. Note that for $\sigma > 0$, the probability density function of the $\mathcal{N}(0,\sigma^2)$ distribution is then $\sigma^{-1}\phi(x/\sigma)$, $x \in \mathbb{R}$. Define the $\mathcal{N}(0,0)$ distribution to be that of a random variable that is identically zero.

We say a random variable X is *integrable* if $\mathbb{E}\,[|X|] < \infty$. We say X has a *lattice* distribution if there exists $h > 0$ such that $(X-a)/h \in \mathbb{Z}$ almost surely for some $a \in \mathbb{R}$. If X is lattice, then the largest such h is called the *span* of X, and here denoted h_X. If X is non-lattice, then we set $h_X := 0$. If X is degenerate, i.e. if $\mathrm{Var}[X] = 0$, then we set $h_X := +\infty$. As usual with local central limit theorems, we need to distinguish between the lattice and non-lattice cases. For real numbers $a \geq 0, b > 0$, we shall write $a|b$ to mean that either b is an integer multiple of a or $a = 0$. When $a = +\infty, b < \infty$ we shall say by convention that $a|b$ does not hold.

Theorem 2.1. *Let $V, V_1, V_2, V_3, \ldots$ be independent identically distributed random variables. Suppose for each $n \in \mathbb{N}$ that (Y_n, S_n, Z_n) is a triple of integrable random variables on the same sample space such that (i) Y_n and S_n are independent, with $S_n \stackrel{\mathcal{D}}{=} \sum_{j=1}^n V_j$; (ii) both $n^{-1/2}\mathbb{E}\,[|Z_n - (Y_n + S_n)|]$ and $n^{1/2}P[Z_n \neq Y_n + S_n]$ tend to zero as $n \to \infty$; and (iii) for some $\sigma \in [0,\infty)$,*

$$n^{-1/2}(Z_n - \mathbb{E}\,Z_n) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0,\sigma^2) \quad \text{as} \quad n \to \infty. \tag{2.1}$$

Then $\mathrm{Var}[V] \leq \sigma^2$ and if $b, c_1, c_2, c_3, \ldots$ are positive constants with $h_V|b$ and $c_n \sim n^{1/2}$ as $n \to \infty$, then

$$\sup_{u\in\mathbb{R}} \left\{ \left| c_n P[Z_n \in [u, u+b)] - \sigma^{-1} b \phi\left(\frac{u - \mathbb{E}\,Z_n}{c_n \sigma} \right) \right| \right\} \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Also,

$$n^{-1/2}(Y_n - \mathbb{E}\,Y_n) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma^2 - \mathrm{Var}[V]). \tag{2.3}$$

Remarks. The main case to consider is $c_n = n^{1/2}$. The more general formulation above is convenient in some applications, e.g., in the proof of Theorem 4.1. Theorem 2.1 is proved in Section 7. Our main interest is in the conclusion (2.2), but (2.3), which comes out for free from the proof, is also of interest.

3 Percolation

Most of our applications of Theorem 2.1 will be in the continuum, but we start with applications to percolation on the lattice. We consider *site percolation* with

parameter p, where each site (element) of $\mathbb{Z}^d$ is open with probability p and closed otherwise, independently of all the other sites. Given a finite set $B \subset \mathbb{Z}^d$, the *open clusters in* B are defined to be the components of the (random) graph with vertex set consisting of the open sites in B, and edges between each pair of open sites in B that are at unit Euclidean distance from each other. Let $\Lambda(B)$ denote the *number of open clusters* in B. Listing the open clusters in B as $\mathbf{C}_1, \ldots, \mathbf{C}_{\Lambda(B)}$, and denoting by $|\mathbf{C}_j|$ the order (i.e., the number of vertices) of the cluster $\mathbf{C}_j$, we denote by $L(B)$ the random variable $\max(|\mathbf{C}_1|, \ldots, |\mathbf{C}_{\Lambda(B)}|)$, and refer to this as the *size of the largest open cluster in* B. Given a growing sequence of regions $(B_n)_{n\geq 1}$ in $\mathbb{Z}^d$, we shall demonstrate local CLTs for the random variables $\Lambda(B_n)$ and $L(B_n)$, subject to some conditions on the sets B_n which are satisfied, for example, if they are cubes of side n. There should not be any difficulty adapting these results to bond percolation.

For $B \subset \mathbb{Z}^d$ let $|B|$ denote the number of elements of B. Let $|\partial B|$ denote the number of elements of $\mathbb{Z}^d \setminus B$ lying at unit Euclidean distance from some element of B. We say a sequence $(B_n)_{n\geq 1}$ of nonempty finite sets in $\mathbb{Z}^d$ has *vanishing relative boundary* if

$$\lim_{n\to\infty} |\partial B_n|/|B_n| = 0. \tag{3.1}$$

We write $\liminf(B_n)$ for $\cup_{n\geq 1} \cap_{m\geq n} B_m$.

Theorem 3.1. *Suppose $d \geq 2$ and $p \in (0,1)$. Then there exists $\sigma > 0$ such that if $(B_n)_{n\geq 1}$ is any sequence of nonempty finite subsets in $\mathbb{Z}^d$ with vanishing relative boundary and with $\liminf(B_n) = \mathbb{Z}^d$, then*

$$|B_n|^{-1/2}(\Lambda(B_n) - \mathbb{E}\,\Lambda(B_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \tag{3.2}$$

and

$$\sup_{j\in\mathbb{Z}} \left| |B_n|^{1/2} P[\Lambda(B_n) = j] - \sigma^{-1}\phi\left(\frac{j - \mathbb{E}\,\Lambda(B_n)}{\sigma |B_n|^{1/2}}\right)\right| \to 0. \tag{3.3}$$

For the size of the largest open cluster we consider a more restricted class of sequences $(B_n)_{n\geq 1}$. Let us say that $(B_n)_{n\geq 1}$ is a *cube-like sequence of lattice boxes* if each set B_n is of the form $\prod_{j=1}^d ([-a_{j,n}, b_{j,n}] \cap \mathbb{Z})$, where $a_{j,n} \in \mathbb{N}$ and $b_{j,n} \in \mathbb{N}$ for all j, n, and moreover

$$\liminf_{n\to\infty} \frac{\inf\{a_{1,n}, b_{1,n}, a_{2,n}, b_{2,n}, \ldots, a_{d,n}, b_{d,n}\}}{\sup\{a_{1,n}, b_{1,n}, a_{2,n}, b_{2,n}, \ldots, a_{d,n}, b_{d,n}\}} > 0 \tag{3.4}$$

which says, loosely speaking, that the sets B_n are not too far away from all being cubes.

Given $d \geq 2$, and $p \in (0,1)$, let $\theta_d(p)$ denote the percolation probability, that is, the probability that the graph with vertices consisting of all open sites in $\mathbb{Z}^d$ and edges between any two open sites that are unit Euclidean distance apart includes an infinite component containing the origin. Let $p_c(d)$ denote the critical value of p for site percolation in d dimensions, i.e., the infimum of all $p \in (0,1)$ such that $\theta_d(p) > 0$. It is well known that $p_c(d) \in (0,1)$ for all $d \geq 2$.

Theorem 3.2. *Suppose $d \geq 2$ and $p \in (p_c(d), 1)$. Then there exists $\sigma > 0$ such that if $(B_n)_{n\geq 1}$ is any cube-like sequence of lattice boxes $\mathbb{Z}^d$ with $\liminf(B_n) = \mathbb{Z}^d$, we have*

$$|B_n|^{-1/2}(L(B_n) - \mathbb{E}\, L(B_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \tag{3.5}$$

and

$$\sup_{j\in\mathbb{Z}} \left| |B_n|^{1/2} P[L(B_n) = j] - \sigma^{-1}\phi\left(\frac{j - \mathbb{E}\, L(B_n)}{\sigma |B_n|^{1/2}}\right)\right| \to 0. \tag{3.6}$$

Theorems 3.1 and 3.2 are proved in Section 8. Theorem 3.1 is the simplest of our applications of Theorem 2.1 and we give its proof with some extra detail for instructional purposes.

4 Random geometric graphs

For our results in this section and the next, on continuum stochastic geometry, let $X_1, X_2, \ldots$ be i.i.d. d-dimensional random vectors with common density f. Assume throughout that $f_{\max} := \sup_{x\in\mathbb{R}^d} f(x) < \infty$, and that f is almost everywhere continuous. Define the induced *binomial point processes*

$$\mathcal{X}_n := \mathcal{X}_n(f) := \{X_1, ..., X_n\}, \quad n \in \mathbb{N}. \tag{4.1}$$

In the special case where f is the density of the uniform distribution on the unit $[0,1]^d$ cube we write $f \equiv f_U$.

For locally finite $\mathcal{X} \subset \mathbb{R}^d$ and $r > 0$, let $\mathcal{G}(\mathcal{X}, r)$ denote the graph with vertex set $\mathcal{X}$ and with edges connecting each pair of vertices x, y in $\mathcal{X}$ with $|y - x| \leq r$; here $|\cdot|$ denotes the Euclidean norm though there should not be any difficulty extending our results to other norms. Sometimes $\mathcal{G}(\mathcal{X}, r)$ is called a *geometric graph* or *Gilbert graph*.

Let $(r_n)_{n\geq 1}$ be a sequence with $r_n \to 0$ as $n \to \infty$. Graphs of the type of $\mathcal{G}(\mathcal{X}_n, r_n)$ are the subject of the monograph [18]. Among the quantities of interest associated with $\mathcal{G}(\mathcal{X}_n, r_n)$ are the number of edges, the number of triangles, and so on; also the

number of isolated points, the number of isolated edges, and so on. CLTs for such quantities are given in Chapter 3 of [18] (see the notes therein for other references) for a large class of limiting regimes for r_n. Here we give some associated local CLTs.

Let $\kappa \in \mathbb{N}$ and let Γ be a fixed connected graph with κ vertices. We follow terminology in [18]. With $\sim$ denoting graph isomorphism, let G_n be the number of κ-subsets $\mathcal{Y}$ of $\mathcal{X}_n$ such that $\mathcal{G}(\mathcal{Y}, r_n) \sim \Gamma$ (i.e., the number of induced subgraphs of $\mathcal{G}(\mathcal{X}_n, r_n)$ that are isomorphic to Γ). Let G_n^* (denoted J_n in [18]) denote the number of *components* of $\mathcal{G}(\mathcal{X}_n, r_n)$ that are isomorphic to Γ. To avoid certain trivialities, assume that Γ is *feasible* in the sense of [18], i.e. that $\mathcal{G}(\mathcal{X}_\kappa, r)$ is isomorphic to Γ with strictly positive probability for some $r > 0$. When considering G_n, we shall also assume that $\kappa \geq 2$. We shall give local CLTs for G_n and G_n^*.

We assume existence of the limit

$$\rho := \lim_{n\to\infty} (nr_n^d) < \infty, \tag{4.2}$$

so that ρ could be zero. If $\rho > 0$ then we are taking the *thermodynamic limit*.

We also assume that

$$\tau_n^2 := n(nr_n^d)^{\kappa-1} \to \infty \quad \text{as} \quad n \to \infty. \tag{4.3}$$

Then (see Theorems 3.12 and 3.13 of [18]) there exists a constant $\sigma = \sigma(f, \Gamma, \rho) > 0$, given explicitly in terms of f, Γ and ρ in [18], such that

$$\lim_{n\to\infty} \tau_n^{-2} \mathrm{Var}(G_n) = \sigma^2; \tag{4.4}$$

$$\tau_n^{-1}(G_n - \mathbb{E}\, G_n) \xrightarrow{\mathcal{D}} N(0, \sigma^2). \tag{4.5}$$

We prove here an associated local central limit theorem for the case $f \equiv f_U$.

Theorem 4.1. *Suppose $f \equiv f_U$. Suppose $k \geq 2$, and suppose assumptions* (4.2) *and* (4.3) *hold. Then as $n \to \infty$,*

$$\sup_{j\in\mathbb{Z}} \left| \tau_n P[G_n = j] - \sigma^{-1}\phi\left(\frac{j - \mathbb{E}\, G_n}{\sigma\tau_n}\right) \right| \to 0. \tag{4.6}$$

We prove Theorem 4.1 in Section 9. It should be possible to obtain similar results for G_n^*, but we shall do so only for the thermodynamic limit with $\rho > 0$, as an example in the next section. In the next section we shall see that for the case with $\rho > 0$, it is possible to relax the assumption that $f \equiv f_U$ in Theorem 4.1; when $\rho = 0$, a similar extension to non-uniform densities should be possible, but we content ourselves here with the case $f \equiv f_U$ so as to provide one example where the simplicity and the appeal of the approach do not get buried.

5 General local CLTs in stochastic geometry

In this section we present some general local central limit theorems in stochastic geometry. We shall illustrate these by some examples in the next section.

For our general local CLTs in stochastic geometry, we consider *marked* point sets in $\mathbb{R}^d$. Let $\mathcal{M}$ be an arbitrary measurable space (the *mark space*), and let $\mathbb{P}_{\mathcal{M}}$ be a probability distribution on $\mathcal{M}$. Given $\mathbf{x} = (x, t) \in \mathbb{R}^d \times \mathcal{M}$ and given $y \in \mathbb{R}^d$, set $y + \mathbf{x} := (y + x, t)$. Given also $a \in \mathbb{R}$, set $a\mathbf{x} = (ax, t)$. We think of t as a mark attached to the point $x \in \mathbb{R}^d$ that is unaffected by translation or scalar multiplaction. Given $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}, y \in \mathbb{R}^d$, and $a \in (0, \infty)$, let $y + a\mathcal{X}^* := \{y + a\mathbf{x} : \mathbf{x} \in \mathcal{X}^*\}$. Let 0 denote the origin of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, and $r > 0$, let $B(x; r)$ denote the Euclidean ball $\{y \in \mathbb{R}^d : |y - x| \leq r\}$, and set $B^*(x; r) := B(x; r) \times \mathcal{M}$. Set $B(r) := B(0; r)$ and $B^*(r) := B^*(0; r)$. Given nonempty $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ and $\mathcal{Y}^* \subset \mathbb{R}^d \times \mathcal{M}$, write

$$D(\mathcal{X}^*, \mathcal{Y}^*) := \inf\{|x - y| : (x, t) \in \mathcal{X}^*, (y, u) \in \mathcal{Y}^* \text{ for some } t, u \in \mathcal{M}\}.$$

Let ω_d denote the volume of the d-dimensional unit ball $B(1)$.

Suppose $H(\mathcal{X}^*)$ is a measurable $\mathbb{R}$-valued function defined for all finite $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$. Suppose H is translation invariant, i.e. $H(y + \mathcal{X}^*) = H(\mathcal{X}^*)$ for all $y \in \mathbb{R}^d$ and all $\mathcal{X}^*$.

Throughout this section we consider the thermodynamic limit; let $r_n, n \geq 1$ be a sequence of constants such that (4.2) holds with $\rho > 0$. Define

$$H_n(\mathcal{X}^*) := H(r_n^{-1}\mathcal{X}^*). \tag{5.1}$$

Let the point process $\mathcal{X}_n := \{X_1, \ldots, X_n\}$ in $\mathbb{R}^d$ be as given in (4.1), with f as in Section 4 (so $f_{\max} < \infty$ and f is Lebesgue-almost everywhere continuous). Define the corresponding marked point processs (i.e., point process in $\mathbb{R}^d \times \mathcal{M}$) by

$$\mathcal{X}_n^* := \{(X_1, T_1), \ldots, (X_n, T_n)\},$$

where $(T_1, T_2, T_3, \ldots)$ is a sequence of independent $\mathcal{M}$-valued random variables with distribution $\mathbb{P}_{\mathcal{M}}$, independent of everything else. We are interested in local CLTs for $H_n(\mathcal{X}_n^*)$, for general functions H. We give two distinct types of condition on H, either of which is sufficient to obtain a local CLT.

We shall say that H has *finite range interactions* if there exists a constant $\tau \in (0, \infty)$ such that

$$H(\mathcal{X}^* \cup \mathcal{Y}^*) = H(\mathcal{X}^*) + H(\mathcal{Y}^*) \quad \text{whenever} \quad D(\mathcal{X}^*, \mathcal{Y}^*) > \tau. \tag{5.2}$$

In many examples it is natural to write $H(\mathcal{X}^*)$ as a sum. Suppose $\xi(\mathbf{x}; \mathcal{X}^*)$ is a measurable $\mathbb{R}$-valued function defined for all pairs $(\mathbf{x}, \mathcal{X}^*)$, where $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ is

finite and $\mathbf{x}$ is an element of $\mathcal{X}^*$. Suppose ξ is translation invariant, i.e. $\xi(y+\mathbf{x}; y+\mathcal{X}^*) = \xi(\mathbf{x}; \mathcal{X}^*)$ for all $y \in \mathbb{R}^d$ and all $\mathbf{x}, \mathcal{X}^*$. Then ξ induces a translation-invariant functional $H^{(\xi)}$ defined on finite point sets $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ by

$$H^{(\xi)}(\mathcal{X}^*) := \sum_{\mathbf{x} \in \mathcal{X}^*} \xi(\mathbf{x}; \mathcal{X}^*). \tag{5.3}$$

Given $r \in (0, \infty)$ we say ξ has range r if $\xi((x,t); \mathcal{X}^*) = \xi((x,t); \mathcal{X}^* \cap B_r^*(x))$ for all finite $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ and all $(x,t) \in \mathcal{X}^*$. It is easy to see that if ξ has range r for some (finite) r then $H^{(\xi)}$ has finite range interactions, although not all H with finite range interactions arise in this way.

Let $\kappa \in \mathbb{N}$. Given any set $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ with more than κ elements, and given $\mathbf{x} = (x,t) \in \mathcal{X}^*$, set $R_\kappa(\mathbf{x}; \mathcal{X}^*)$ to be the κ-nearest neighbour distance from x to $\mathcal{X}^*$, i.e. the smallest $r \geq 0$ such that $\mathcal{X}^* \cap B^*(x; r)$ has at least κ elements other than $\mathbf{x}$ itself. If $\mathcal{X}^*$ has κ or fewer elements, set $R_\kappa(\mathbf{x}; \mathcal{X}^*) := \infty$.

We say that ξ *depends only on the κ nearest neighbours* if for all $\mathbf{x}$ and $\mathcal{X}^*$, writing $\mathbf{x} = (x,t)$ we have

$$\xi(\mathbf{x}; \mathcal{X}^*) = \xi(\mathbf{x}; \mathcal{X}^* \cap B^*(x; R_\kappa(\mathbf{x}; \mathcal{X}))).$$

We give local CLTs for H under two alternative sets of conditions: either (i) when H has finite range interactions, or (ii) when H is induced, according to the definition (5.3), by a functional $\xi(\mathbf{x}; \mathcal{X}^*)$ which depends only on the κ nearest neighbours, for some fixed κ.

Given $K > 0$ and $n \in \mathbb{N}$, define point processes $\mathcal{U}_{n,K}$, and $\mathcal{Z}_n$ in $\mathbb{R}^d$, and point processes $\mathcal{U}_{n,K}^*$, and $\mathcal{Z}_n^*$ in $\mathbb{R}^d \times \mathcal{M}$, as follows. Let $\mathcal{U}_{n,K}$ denote the point process consisting of n independent uniform random points $U_{1,K}, \ldots, U_{n,K}$ in $B(K)$, and let $\mathcal{Z}_n$ be the point process consisting of n independent points $Z_1, \ldots, Z_n$ in $\mathbb{R}^d$, each with a d-dimensional standard normal distribution (any other positive continuous density on $\mathbb{R}^d$ would do just as well). The corresponding marked point processs are defined by

$$\begin{aligned} \mathcal{U}_{n,K}^* &:= \{(U_{1,K}, T_1), \ldots, (U_{n,K}, T_n)\}; \\ \mathcal{Z}_n^* &:= \{(Z_1, T_1), \ldots, (Z_n, T_n)\}. \end{aligned}$$

Define the limiting span

$$h(H) := \liminf_{n \to \infty} h_{H(\mathcal{Z}_n^*)}. \tag{5.4}$$

Theorem 5.1. *Suppose that either (i) H has finite range interactions and $h_{H(\mathcal{Z}_n^*)} < \infty$ for some $n \in \mathbb{N}$, or (ii) for some $\kappa \in \mathbb{N}$, H is induced by a functional $\xi(\mathbf{x}; \mathcal{X}^*)$ which depends only on the κ nearest neighbours, and $h_{H(\mathcal{Z}_n^*)} < \infty$ for some $n \in \mathbb{N}$*

with $n > \kappa$. Suppose also that $H_n(\mathcal{X}_n^*)$ and $H(\mathcal{U}_{n,K}^*)$ are integrable for all $n \in \mathbb{N}$ and $K > 0$. Finally suppose that

$$n^{-1/2}(H_n(\mathcal{X}_n^*) - \mathbb{E}\, H_n(\mathcal{X}_n^*)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty. \tag{5.5}$$

Then $\sigma > 0$ and $h(H) < \infty$ and for any $b \in (0, \infty)$, with $h(H)|b$,

$$\sup_{u \in \mathbb{R}} \left\{ \left| n^{1/2} P[H_n(\mathcal{X}_n^*) \in [u, u+b)] - \sigma^{-1} b \phi \left(\frac{u - \mathbb{E}\, H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma} \right) \right| \right\} \to 0 \quad \text{as } n \to \infty. \tag{5.6}$$

We prove Theorem 5.1 in Section 10. Analogues to this result and to Theorem 4.1 should also hold if one Poissonizes the number of points in the sample, but we do not give details.

The corresponding result for unmarked point sets in $\mathbb{R}^d$ goes as follows; we adapt our terminology to this case in an obvious manner.

Corollary 5.1. *Suppose $H(\mathcal{X})$ is $\mathbb{R}$-valued and defined for all finite $\mathcal{X} \subset \mathbb{R}^d$. Suppose H is translation invariant, and set $H_n(\mathcal{X}) := H(r_n^{-1}\mathcal{X})$. Suppose that either (i) H has finite range interactions and $h_{H(\mathcal{Z}_n)} < \infty$ for some $n \in \mathbb{N}$, or (ii) for some $\kappa \in \mathbb{N}$, H is induced by a functional $\xi(x; \mathcal{X})$ which depends only on the κ nearest neighbours, and $h_{H(\mathcal{Z}_n)} < \infty$ for some $n \in \mathbb{N}$ with $n > \kappa$. Suppose also that $H_n(\mathcal{X}_n)$ and $H(\mathcal{U}_{n,K})$ are integrable for all $n \in \mathbb{N}$ and $K > 0$. Finally suppose*

$$n^{-1/2}(H_n(\mathcal{X}_n) - \mathbb{E}\, H_n(\mathcal{X}_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty. \tag{5.7}$$

Then $\sigma > 0$ and $h(H) < \infty$ and for any $b \in (0, \infty)$, with $h(H)|b$,

$$\sup_{u \in \mathbb{R}} \left\{ \left| n^{1/2} P[H_n(\mathcal{X}_n) \in [u, u+b)] - \sigma^{-1} b \phi \left(\frac{u - \mathbb{E}\, H_n(\mathcal{X}_n)}{n^{1/2}\sigma} \right) \right| \right\} \to 0 \quad \text{as } n \to \infty. \tag{5.8}$$

Corollary 5.1 is easily obtained from Theorem 5.1 by taking $\mathcal{M}$ to have just a single element, denoted t_0 say, and identifying each element $(x, t_0) \in \mathbb{R}^d \times \mathcal{M}$ with the corresponding element x of $\mathbb{R}^d$.

To apply Theorem 5.1 in examples, we need to check condition (5.5). For some examples this is best done directly. However, if we strengthen the other hypotheses of Theorem 5.1, we can obtain (5.5) from known results and so do not need to include it as an extra hypothesis. The next three theorems illustrate this. As well (5.5), these results give us the associated variance convergence result

$$\lim_{n \to \infty} n^{-1} \mathrm{Var}[H_n(\mathcal{X}_n^*)] = \sigma^2. \tag{5.9}$$

In the next three theorems, we impose some extra assumptions besides those of Theorem 5.1. Writing $\mathrm{supp}(f)$ for the support of f, we shall assume that $\mathrm{supp}(f)$ is compact, and that also r_n satisfy

$$|r_n^{-d} - n| = O(n^{1/2}), \tag{5.10}$$

which implies (4.2) with $\rho = 1$. We also assume certain polynomial growth bounds; see (5.11), (5.13) and (5.14) below.

First consider the case where $H = H^{(\xi)}$ is induced by a functional $\xi(\mathbf{x}; \mathcal{X}^*)$ with finite range $r > 0$. For any set A, let $\mathrm{card}(A)$ denotes the number of elements of A.

Theorem 5.2. *Suppose $H = H^{(\xi)}$ is induced by a translation invariant functional $\xi(\mathbf{x}; \mathcal{X}^*)$ having finite range r and and satisfying for some $\gamma > 0$ the polynomial growth bound*

$$|\xi((x,t); \mathcal{X}^*)| \leq \gamma(\mathrm{card}(\mathcal{X}^* \cap B^*(x; r)))^\gamma \quad \forall \text{ finite } \mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}, \ \forall \ (x,t) \in \mathcal{X}^*. \tag{5.11}$$

Suppose $h_{H(\mathcal{Z}_n^*)} < \infty$ for some $n \in \mathbb{N}$, and suppose $\mathrm{supp}(f)$ is compact. Finally, suppose that (5.10) holds. Then there exists $\sigma \in (0, \infty)$ such that (5.5) and (5.9) hold, and $h(H) < \infty$ and (5.6) holds for all b with $h(H)|b$.

Now we turn to the general case of Condition (i) in Theorem 5.1, where H has finite range interactions but is not induced by a finite range ξ. For this case we shall borrow some concepts from continuum percolation. For $\lambda > 0$, let $\mathcal{H}_\lambda$ denote a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity λ. Let $\mathcal{H}_\lambda^*$ denote the same Poisson point process with each point given an independent $\mathcal{M}$-valued mark with the distribution $\mathbb{P}_\mathcal{M}$.

Let λ_c be the critical value for percolation in d dimensions, that is, the supremum of the set of all $\lambda > 0$ such that the component of the geometric (Gilbert) graph $G(\mathcal{H}_\lambda \cup \{0\}, 1)$ containing the origin is almost surely finite. It is known (see e.g. [18]) that $0 < \lambda_c < \infty$ when $d \geq 2$ and $\lambda_c = \infty$ when $d = 1$.

For nonempty $\mathcal{X} \subset \mathbb{R}^d$, write $\mathrm{diam}(\mathcal{X})$ for $\sup\{|x - y| : x, y \in \mathcal{X}\}$. For $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$, write $\mathrm{diam}(\mathcal{X}^*)$ for $\mathrm{diam}(\pi(\mathcal{X}^*))$, where π denotes the canonical projection from $\mathbb{R}^d \times \mathcal{M}$ onto $\mathbb{R}^d$.

Theorem 5.3. *Suppose $H(\mathcal{X}^*)$ is a measurable $\mathbb{R}$-valued function defined for all finite $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$, and is translation invariant. Suppose $\mathrm{supp}(f)$ is compact. Suppose for some $\tau > 0$ that the finite range interaction condition (5.2) holds, and suppose f and τ satisfy the subcriticality condition*

$$\tau^d f_{\max} < \lambda_c, \tag{5.12}$$

Assume $(r_n)_{n\geq 1}$ *satisfies* (5.10)*, and suppose also that* $h_{H(\mathcal{Z}_n^*)} < \infty$ *for some* $n \in \mathbb{N}$*, and that there exists a constant* $\gamma > 0$ *such that for all finite non-empty* $\mathcal{X}^* \subset \mathbb{R}^d$ *we have*

$$H(\mathcal{X}^*) \leq \gamma(\mathrm{diam}(\mathcal{X}^*) + \mathrm{card}(\mathcal{X}^*))^\gamma. \tag{5.13}$$

Then there exists $\sigma \in (0, \infty)$ *such that* (5.5) *and* (5.9) *hold, and* $h(H) < \infty$ *and if* $b \in (0, \infty)$ *with* $h(H)|b$*, then* (5.6) *holds.*

Now we turn to condition (ii) in Theorem 5.1. Following [24], we say that a closed region $A \subset \mathbb{R}^d$ is a *d-dimensional* C^1 *submanifold-with-boundary of* $\mathbb{R}^d$ if it has a differentiable boundary in the following sense: for every x in the boundary ∂A of A, there is an open $U \subset \mathbb{R}^d$, and a continuously differentiable injection g from U to $\mathbb{R}^d$, such that $0 \in U$ and $g(0) = x$ and $g(U \cap ([0, \infty) \times \mathbb{R}^{d-1})) = g(U) \cap A$.

Theorem 5.4. *Let* $\kappa \in \mathbb{N}$*. Suppose* $H = H^{(\xi)}$ *is induced by a* ξ *which depends only on the* κ *nearest neighbours, and for some* $\gamma \in (0, \infty)$ *suppose we have for all* $(\mathbf{x}, \mathcal{X}^*)$ *that*

$$|\xi(\mathbf{x}; \mathcal{X}^*)| \leq \gamma(1 + R_\kappa(\mathbf{x}, \mathcal{X}^*))^\gamma. \tag{5.14}$$

Suppose also that $\mathrm{supp}(f)$ *is either a compact convex region in* $\mathbb{R}^d$ *or a compact d-dimensional submanifold-with-boundary of* $\mathbb{R}^d$*, and suppose* f *is bounded away from zero on* $\mathrm{supp}(f)$*. Finally suppose that the sequence* $(r_n)_{n\geq 1}$ *satisfies* (5.10)*, and that* $h_{H(\mathcal{Z}_n^*)} < \infty$ *for some* $n \in \mathbb{N}$ *with* $n > \kappa$*. Then there exists* $\sigma \in (0, \infty)$ *such that* (5.5) *and* (5.9) *hold, and* $h(H) < \infty$ *and if* $b \in (0, \infty)$ *with* $h(H)|b$ *then* (5.6) *also holds.*

We prove Theorems 5.2, 5.3 and 5.4 in Section 11. In proving each of these results, we apply Theorem 5.1, and check the CLT condition (5.5) using a general CLT from [20], stated below as Theorem 11.1.

The conclusion that $\sigma > 0$ in Theorems 5.1–5.4 and Corollary 5.1 is noteworthy because the result from [20] on its own does not guarantee this. Our approach to showing $\sigma > 0$ here is related to that given in [1] (and elsewhere) but is more generic. A different approach to providing generic variance lower bounds was used in [21] and [3] but is less well suited to the present setting.

6 Applications

This section contains discussion of some examples of concrete models in stochastic geometry, to which the general local central limit theorems presented in Section 5 are applicable. Further examples where the conditions for these general theorems can be verified are discussed in [20, 21, 22, 23].

6.1 Further quantities associated with random geometric graphs

Suppose the graph $\mathcal{G}(\mathcal{X}_n, r_n)$ is as in Section 4. We assume here that (4.2) holds with $\rho > 0$. Theorem 5.1 enables us to extend the case $\rho > 0$ of Theorem 4.1 to non-uniform f. It also yields local CLTs for some graph quantities not covered by Theorem 4.1; we now give some examples.

Number of components for $\mathcal{G}(\mathcal{X}_n, r_n)$. This quantity can be written in the form $H_n(\mathcal{X}_n)$, where $H(\mathcal{X})$ is the number of components of the geometric graph $\mathcal{G}(\mathcal{X}, 1)$ (which clearly has finite range interactions). In the the thermodynamic limit, this quantity satisfies the CLT (5.7) (see Theorem 13.26 of [18]). Therefore, Corollary 5.1 is applicable here and shows that it satisfies the local CLT (5.8).

Number of components for $\mathcal{G}(\mathcal{X}_n, r_n)$ *isomorphic to a given feasible graph* Γ. This quantity, denoted G_n^* in Section 4, can be written in the form $H_n(\mathcal{X}_n)$, with $H(\mathcal{X})$ the number of components of $G(\mathcal{X}, 1)$ isomorphic to Γ. Clearly, this H has finite range interactions since (5.2) holds for $\tau = 2$. Also, it satisfies (5.7) by Theorem 3.14 of [18]. Therefore we can apply Corollary 5.1 to deduce (5.8) in this case.

Independence number. The independence number of a finite graph is the maximal number k such that there exists a set of k vertices in the graph such that none of them are adjacent Clearly this quantity is the sum of the independence numbers of the graph's components, and therefore if for $\mathcal{X} \subset \mathbb{R}^d$ we set $H(\mathcal{X})$ to be the independence number of $\mathcal{G}(\mathcal{X}, \tau)$ (also known as the *off-line packing number* since it is the maximum number of balls of radius $\tau/2$ that can be packed centred at points of $\mathcal{X}$) then H satisfies the finite range interactions condition (5.2) with $r = 2$. Therefore we can apply Theorem 5.3 to derive a local CLT for the independence number of $\mathcal{G}(\mathcal{X}_n, r_n)$, as follows.

Theorem 6.1. . *Let* $\tau > 0$ *and suppose* (5.12) *holds. Suppose* r_n *is satisfies* (5.10). *Then if for* $\mathcal{X} \subset \mathbb{R}^d$ *we set* $H(\mathcal{X})$ *to be the independence number of* $\mathcal{G}(\mathcal{X}, \tau)$, *then there exists* $\sigma \in (0, \infty)$ *such that* (5.7) *holds, and if* $b \in \mathbb{N}$ *then* (5.8) *holds.*

6.2 Germ-grain models

Consider a coverage process in which each point X_i has an associated mark T_i, the T_i (defined for $i \geq 1$) being i.i.d. nonnegative random variables with a distribution having bounded support (i.e., with $P[T_i \leq K] = 1$ for some finite K). Define the random coverage process

$$\Xi_n := \cup_{i=1}^n B(r_n^{-1} X_i; T_i). \tag{6.1}$$

For U a finite union of convex sets in $\mathbb{R}^d$, let $|U|$ denote the volume of U (i.e. its Lebesgue measure) and let $|\partial U|$ denote the surface area of U (i.e. the $(d-1)$-dimensional Hausdorff measure of its boundary).

Theorem 6.2. *Under the above assumptions, if* (5.10) *holds then there exists* $\sigma > 0$ *and* $\tilde{\sigma} > 0$ *such that* $n^{-1/2}(|\Xi_n| - \mathbb{E}\,|\Xi_n|) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ *and* $n^{-1/2}(|\partial\Xi_n| - \mathbb{E}\,|\partial\Xi_n|) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{\sigma}^2)$*, and moreover for any* $b \in (0, \infty)$*,*

$$\sup_{u\in\mathbb{R}} \left\{ \left| n^{1/2} P[|\Xi_n| \in [u, u+b)] - \sigma^{-1} b \phi\left(\frac{u - \mathbb{E}\,|\Xi_n|}{n^{1/2}\sigma} \right) \right| \right\} \to 0 \quad \text{as } n \to \infty. \tag{6.2}$$

and

$$\sup_{u\in\mathbb{R}} \left\{ \left| n^{1/2} P[|\partial\Xi_n| \in [u, u+b)] - \tilde{\sigma}^{-1} b \phi\left(\frac{u - \mathbb{E}\,|\partial\Xi_n|}{n^{1/2}\tilde{\sigma}} \right) \right| \right\} \to 0 \quad \text{as } n \to \infty. \tag{6.3}$$

Proof. The volume $|\Xi_n|$ can be viewed as a functional $H_n(\mathcal{X}_n^*)$, where $H(\mathcal{X}) = H^{(\xi)}(\mathcal{X}^*)$ with $\xi((x,t); \mathcal{X}^*)$ given by the volume of that part of the ball centred at x with radius given by the associated mark t, which is not covered by any corresponding ball for some other point $x' \in \mathcal{X}$ with x' preceding x in the lexicographic ordering. Since we assume the support of the distribution of the T_i is bounded, this ξ has finite range $r = 2K$. Moreover, it satisfies the polynomial growth bound (5.11) so by Theorem 5.2 we get the CLT (5.5) and local CLT (5.6) for any $b > 0$ (in this example $h(H) = 0$). Thus we have (6.2).

Turning to the surface area $|\partial\Xi_n|$, this can also be viewed as a functional $H_n(\mathcal{X}_n)$ for a different $H = H^{(\xi)}$, this time taking $\xi(\mathbf{x}; \mathcal{X})$ to be the uncovered surface area of the ball at x, which again has range $r = 2K$ and satisfies (5.11). Hence by Theorem 5.2. we get the CLT (5.5) and local CLT (5.6) for any $b > 0$ for this choice of H (in this example, again $h(H) = 0$). Thus we have (6.3). □

Remark. The preceding argument still works if the independent balls of random radius in the preceding discussion are replaced by independent copies of a random compact shape that is almost surely contained in the ball $B(K)$ for some K (cf. Section 6.1 of [20]).

Other functionals for the germ-grain model. When $f \equiv f_U$, the scaled point process $r_n^{-1/d}\mathcal{X}_n$ can be viewed as a uniform point process in a window of side $r_n^{-1/d}$. CLTs for a large class of other functionals on germ-grain models in such a window are considered in [13], for the Poissonised point process with a Poisson distributed

number of points. Since the Poissonised version of Theorems 5.1 and 5.2 should also hold, it should be possible to derive local CLTs for many of the quantities considered in [13], at least in the case where the grains (i.e., the balls or other shapes attached to the random points) are of uniformly bounded diameter.

6.3 Random sequential adsorption (RSA).

RSA (on-line packing) is a model of irreversible deposition of particles onto an initially empty d-dimensional surface where particles of fixed finite size arrive sequentially at random locations in an initially empty region A of a d-dimensional space (typically $d = 1$ or $d = 2$), and each successive particle is accepted if it does not overlap any previously accepted particle. The region A is taken to be compact and convex. The locations of successive particles are independent and governed by some density f on A. In the present setting, we take the mark space $\mathcal{M}$ to be $[0, 1]$ with $\mathbb{P}_{\mathcal{M}}$ the uniform distribution. Each point $\mathbf{x} = (x, t)$ of $\mathcal{X}^*$ represents an incoming particle with arrival time t. The marks determine the order in which particles arrive, and two particles at $\mathbf{x} = (x, t)$ and $\mathbf{y} = (y, u)$ are said to overlap if $|x - y| \leq 1$. Let $H(\mathcal{X}^*)$ denote the number of accepted particles. This choice of H clearly has finite range interactions ((5.2) holds for $\tau = 2$).

Then $H_n(\mathcal{X}_n^*)$ represents the number of accepted particles for the re-scaled marked point process $r_n^{-1}\mathcal{X}_n^*$; note that the density f and hence the region A on which the particles are deposited, does not vary with n. At least for $r_n = n^{-1/d}$, the central limit theorem for $H_n(\mathcal{X}_n)$ is known to hold; see [22] for the case when $A = [0, 1]^d$ and $f \equiv f_U$ and [3] for the extension to the non-uniform case on arbitrary compact convex A (note that these results do not require the sub-criticality condition (5.12) to be satisfied). Thus, the H under consideration here satisfies the condition (5.5). Therefore we can apply Theorem 5.1 to obtain a local CLT for the number of accepted particles in this model.

Theorem 6.3. *Suppose f has compact convex support and is bounded away from zero and infinity on its support. Suppose $r_n = n^{-1/d}$, and suppose $Z_n = H_n(\mathcal{X}_n^*)$ is the number of accepted particles in the rescaled RSA model described above. In other words, suppose Z_n be the number of accepted particles when RSA is performed on $\mathcal{X}_n$ with distance parameter $r_n = n^{-1/d}$. Then there is a constant $\sigma \in (0, \infty)$ such that* (2.1) *holds and for $b = 1$ and $c = n^{1/2}$,* (2.2) *holds.*

It is likely that in the preceding result the condition $r_n = n^{-1/d}$ can be relaxed to (4.2) holding with $\rho > 0$. We have not checked the details.

In the *infinite input* version of RSA with range of interaction r, particles continue to arrive until the region A is saturated, and the total number of accepted particles is a random variable with its distribution determined by r. A central limit theorem

for the (random) total number of accepted particles (in the limit $r \to 0$) is known to hold, at least for $f \equiv f_U$; see [25]. It would be interesting to know if a corresponding local central limit theorem holds here as well.

6.4 Nearest neighbour functionals

Many functionals have arisen in the applied literature which can be expressed as sums of functionals of κ-nearest neighbours, for such problems as multidimensional goodness-of-fit tests [5, 2], multidimensional two-sample tests [14], entropy estimation of probability distributions [17], dimension estimation [16], and nonparametric regression [10]. Functionals considered include: sums of power-weighted nearest neighbour distances, sums of logarithmic functions of the nearest-neighbour distances, number of nearest-neighbours from the same sample in a two-sample problem, and others. Central limit theorems have been obtained explicitly for some of these examples [5, 14, 2] and in other cases they can often be derived from more general results [1, 20, 21, 7]. Thus, for many of these examples it should be possible to check the conditions of Theorem 5.1 (case (ii)).

We consider just one simple example where Theorem 5.4 is applicable. Suppose for some fixed $\alpha > 0$ that $H(\mathcal{X})$ is the sum of the α-power-weighted nearest neighbour distances in $\mathcal{X}$ (for $\alpha = 1$ this is known as the total length of the directed nearest neighbour graph on $\mathcal{X}$). That is, suppose $H(\mathcal{X}) = \mathcal{H}^{(\xi)}(\mathcal{X})$ with $\xi(x; \mathcal{X})$ given by $\min\{|y - x|^\alpha : y \in \mathcal{X} \setminus \{x\}\}$. Then $H_n(\mathcal{X}) = r_n^{-\alpha} H(\mathcal{X})$, and ξ clearly satisfies (5.14) for some γ, so provided f is supported by a compact convex region in $\mathbb{R}^d$ or by a compact d-dimensional submanifold-with-boundary of $\mathbb{R}^d$, and provided f is bounded away from zero on its support, Theorem 5.4 is applicable with $\kappa = 1$. Hence in this case there exists $\sigma \in (0, \infty)$ such that (5.5) and (for any $b \in (0, \infty)$) (5.6) are valid.

7 Proof of Theorem 2.1

Let $V, V_1, V_2, V_3, \ldots$ be independent identically distributed random variables. Define $\sigma_V := \sqrt{\mathrm{Var}(V)} \in [0, \infty]$. In the case $\sigma_V = 0$, Theorem 2.1 is trivial, so from now on in this section, we assume $\sigma_V > 0$. Let $b, c_1, c_2, c_3, \ldots$ be positive constants with $h_V | b$ and $c_n \sim n^{1/2}$ as $n \to \infty$.

We prove Theorem 2.1 first in the special case where $Z_n = S_n$, then in the case where $Z_n = Y_n + S_n$, and then in full generality. Before starting we recall a fact about characteristic functions.

Lemma 7.1. *If $\sigma_V = \infty$ then for all $t \in \mathbb{R}$, as $n \to \infty$*

$$\mathbb{E}\left[\exp\left(itn^{-1/2}\sum_{j=1}^{n}(V_j - \mathbb{E}[V])\right)\right] \to 0.$$

Proof. See for example Section 3, and in particular the final display, of [26]. □

Lemma 7.2. *Suppose $S_n \stackrel{\mathcal{D}}{=} \sum_{j=1}^{n} V_j$ and $\sigma_V < \infty$. Then as $n \to \infty$,*

$$\sup_{u\in\mathbb{R}}\left\{\left|c_n P[S_n \in [u, u+b)] - \sigma^{-1} b\phi\left(\frac{u - \mathbb{E}\,S_n}{c_n\sigma_V}\right)\right|\right\} \to 0 \tag{7.1}$$

Proof. First consider the special case with $c_n = n^{1/2}$. In this case, (7.1) holds by the classical local central limit theorem for sums of i.i.d. non-lattice variables with finite second moment in the case where $h_V = 0$ (see page 232 of [6], or Theorem 2.5.4 of [9]), and by the local central limit theorem for sums of i.i.d. lattice variables in the case where $h_V > 0$ and $b/h_V \in \mathbb{Z}$ (see Theorem XV.5.3 of [11], or Theorem 2.5.2 of [9]).

To extend this to the general case with $c_n \sim n^{1/2}$, observe first that by the special case considered above, $n^{1/2}P[S_n \in [u, u+b)]$ remains bounded uniformly in u and n, and hence

$$\sup_{u\in\mathbb{R}}\{|(n^{1/2} - c_n)P[S_n \in [u, u+b)]|\} = \sup_{u\in\mathbb{R}}\left\{n^{1/2}\left|1 - \frac{c_n}{n^{1/2}}\right| P[S_n \in [u, u+b)]\right\} \to 0. \tag{7.2}$$

Also, for any $K > 1$,

$$\begin{aligned}\sup_{|x|\le Kn^{1/2}}\left\{\left|\phi\left(\frac{x}{n^{1/2}}\right) - \phi\left(\frac{x}{c_n}\right)\right|\right\} &\le (2\pi e)^{-1/2}\sup_{|x|\le Kn^{1/2}}\left\{\left|\left(\frac{x}{n^{1/2}}\right) - \left(\frac{x}{c_n}\right)\right|\right\} \\ &\le (2\pi e)^{-1/2}\left(\frac{Kn^{1/2}}{n^{1/2}}\right)\left|1 - \frac{n^{1/2}}{c_n}\right| \to 0. \end{aligned} \tag{7.3}$$

Also, for large enough n,

$$\sup_{|x|\ge Kn^{1/2}} \max\left(\phi\left(\frac{x}{c_n}\right), \phi\left(\frac{x}{n^{1/2}}\right)\right) \le \phi(K-1)$$

and since K is arbitrarily large, combined with (7.3), this shows that

$$\sup_{x\in\mathbb{R}}\left\{\left|\phi\left(\frac{x}{n^{1/2}}\right) - \phi\left(\frac{x}{c_n}\right)\right|\right\} \to 0.$$

Combined with (7.2), this shows that we can deduce (7.1) for general c_n satisfying $c_n \sim n^{1/2}$ from the special case with $c_n = n^{1/2}$ which was established earlier. □

Lemma 7.3. *Theorem 2.1 holds in the special case where $Z_n = Y_n + S_n$.*

Proof. Assume, along with the hypotheses of Theorem 2.1, that $Z_n = Y_n + S_n$. Considering characteristic functions, by (2.1) we have for $t \in \mathbb{R}$ that

$$\mathbb{E}\left[\exp\left(itn^{-1/2}(Y_n - \mathbb{E}\,Y_n)\right)\right]\mathbb{E}\left[\exp\left(itn^{-1/2}(S_n - \mathbb{E}\,S_n)\right)\right] \to \exp(-\frac{1}{2}t\sigma^2). \tag{7.4}$$

If $\sigma_V = \infty$ then by Lemma 7.1, the second factor in the left hand side of (7.4) tends to zero, giving a contradiction. Hence we may assume $\sigma_V < \infty$ from now on.

By the Central Limit Theorem,

$$n^{-1/2}(S_n - \mathbb{E}\,S_n) \xrightarrow{\mathcal{D}} N(0, \sigma_V^2). \tag{7.5}$$

By (7.4) and (7.5), $\sigma_V^2 \le \sigma^2$ and setting $\sigma_Y^2 := \sigma^2 - \sigma_V^2 \ge 0$, we have that $n^{-1/2}(Y_n - \mathbb{E}\,Y_n)$ is asymptotically $\mathcal{N}(0, \sigma_Y^2)$. Hence,

$$c_n^{-1}(Y_n - \mathbb{E}\,Y_n) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_Y^2). \tag{7.6}$$

That is, (2.3) holds.

Let $u \in \mathbb{R}$ and set

$$t := t(u, n) := c_n^{-1}(u - \mathbb{E}\,Z_n). \tag{7.7}$$

Assume that $Z_n = Y_n + S_n$. By independence of Y_n and S_n,

$$P[Z_n \in [u, u+b)] = P[c_n^{-1}(Z_n - \mathbb{E}\,[Z_n]) \in c_n^{-1}[u - \mathbb{E}\,Z_n, u + b - \mathbb{E}\,Z_n)]$$
$$= \int_{-\infty}^{\infty} P\left[\frac{Y_n - \mathbb{E}\,Y_n}{c_n} \in dx\right] P\left[\frac{S_n - \mathbb{E}\,S_n}{c_n} \in c_n^{-1}[u - \mathbb{E}\,Z_n, u + b - \mathbb{E}\,Z_n) - x\right]$$

so that

$$c_n P[Z_n \in [u, u+b)] = \int_{-\infty}^{\infty} P\left[\frac{Y_n - \mathbb{E}\,Y_n}{c_n} \in dx\right] \times \left(c_n P\left[S_n - \mathbb{E}\,S_n \in [u - \mathbb{E}\,Z_n - xc_n, u - \mathbb{E}\,Z_n - xc_n + b)\right]\right)$$
$$= \int_{-\infty}^{\infty} P\left[\frac{Y_n - \mathbb{E}\,Y_n}{c_n} \in dx\right] \left(c_n P\left[S_n - \mathbb{E}\,S_n \in [(t-x)c_n, (t-x)c_n + b)\right]\right).$$

By Lemma 7.2,

$$c_n P\left[S_n - \mathbb{E}\,S_n \in [yc_n, yc_n + b)\right] = \frac{b}{\sigma_V}\phi\left(\frac{y}{\sigma_V}\right) + g_n(y)$$

where

$$\sup_{y\in\mathbb{R}} |g_n(y)| \to 0 \qquad \text{as} \quad n \to \infty. \tag{7.8}$$

Hence,

$$c_n P[Z_n \in [u, u+b)] = \mathbb{E}\left[\frac{b}{\sigma_V}\phi\left(\frac{t - c_n^{-1}(Y_n - \mathbb{E}\,Y_n)}{\sigma_V}\right) + g_n\left(t - c_n^{-1}(Y_n - \mathbb{E}\,Y_n)\right)\right],$$

so by (7.8), to prove (2.2), it suffices to prove

$$\sup_{u\in\mathbb{R}}\left\{\left|\mathbb{E}\left[\sigma_V^{-1}\phi\left(\frac{t(u,n) - c_n^{-1}(Y_n - \mathbb{E}\,Y_n)}{\sigma_V}\right)\right] - \sigma^{-1}\phi\left(\frac{u - \mathbb{E}\,Z_n}{c_n\sigma}\right)\right|\right\} \to 0. \tag{7.9}$$

Suppose this fails. Then there is a strictly increasing sequence of natural numbers $(n(m), m \geq 1)$ and a sequence of real numbers $(u_m, m \geq 1)$ such that with $t_m := t(u_m, n(m))$, we have

$$\liminf_{m\to\infty}\left|\mathbb{E}\left[\sigma_V^{-1}\phi\left(\frac{t_m - c_{n(m)}^{-1}(Y_{n(m)} - \mathbb{E}\,Y_{n(m)})}{\sigma_V}\right)\right] - \sigma^{-1}\phi\left(\frac{u_m - \mathbb{E}\,Z_{n(m)}}{c_{n(m)}\sigma}\right)\right| > 0. \tag{7.10}$$

By taking a subsequence if necessary, we may assume without loss of generality, either that $t_m \to t$ for some $t \in \mathbb{R}$, or that or that $|t_m| \to \infty$ as $m \to \infty$. Consider first the latter case. If $|t_m| \to \infty$ as $m \to \infty$, then by (7.6),

$$P[|t_m - c_{n(m)}^{-1}(Y_{n(m)} - \mathbb{E}\,Y_{n(m)})| \leq |t_m|/2] \leq P[|c_{n(m)}^{-1}(Y_{n(m)} - \mathbb{E}\,Y_{n(m)})| \geq |t_m|/2] \to 0,$$

and hence

$$\mathbb{E}\left[\sigma_V^{-1}\phi\left(\frac{t_m - c_{n(m)}^{-1}(Y_{n(m)} - \mathbb{E}\,Y_{n(m)})}{\sigma_V}\right)\right] \to 0.$$

Since $c_{n(m)}^{-1}(u_m - \mathbb{E}\,Z_{n(m)})$ is equal to t_m by (7.7), we also have under this assumption that $\sigma^{-1}\phi\left(\frac{u_m - \mathbb{E}\,Z_{n(m)}}{c_{n(m)}\sigma}\right)$ tends to zero, and thus we obtain a contradiction of (7.10).

In the case where $t_m \to t$ for some finite t, we have by (7.6) that $t_m - c_{n(m)}^{-1}(Y_{n(m)} - \mathbb{E}\,Y_{n(m)})$ converges in distribution to $t - W_1$, where $W_1 \sim \mathcal{N}(0, \sigma_Y^2)$. Hence as $m \to \infty$,

$$\mathbb{E}\left[\sigma_V^{-1}\phi\left(\frac{t_m - c_{n(m)}^{-1}(Y_{n(m)} - \mathbb{E}\,Y_{n(m)})}{\sigma_V}\right)\right] \to \sigma_V^{-1}\mathbb{E}\,\phi((t - W_1)/\sigma_V) = \mathbb{E}\,f_{W_2}(t - W_1),$$

where $W_2 \sim N(0, \sigma_V^2)$, with probability density function $f_{W_2}(x) := \sigma_V^{-1}\phi(x/\sigma_V)$. If we assume W_1, W_2 are independent, then $\mathbb{E}\, f_{W_2}(t - W_1)$ is the convolution formula for the probability density function of $W_1 + W_2$, which is $\mathcal{N}(0, \sigma^2)$, so that

$$\mathbb{E}\, f_{W_2}(t - W_1) = f_{W_1+W_2}(t) = \sigma^{-1}\phi(t/\sigma).$$

On the other hand, since $c_{n(m)}^{-1}(u_m - \mathbb{E}\, Z_{n(m)})$ is equal (by (7.7)) to t_m which we assume converges to t, we also have that

$$\sigma^{-1}\phi\left(\frac{u_m - \mathbb{E}\, Z_{n(m)}}{c_{n(m)}\sigma}\right) \to \sigma^{-1}\phi\left(\frac{t}{\sigma}\right),$$

and therefore we obtain a contradiction of (7.10) in this case too.

Thus (7.10) fails, and therefore (7.9) holds. Hence, (2.2) holds in the case with $Z_n = Y_n + S_n$. □

Proof of Theorem 2.1. Set $Z'_n := Y_n + S_n$. By the integrability assumptions, Z'_n is integrable. By (2.1) and the assumption that $n^{-1/2}\mathbb{E}\,[|Z_n - Z'_n|] \to 0$ as $n \to \infty$,

$$n^{-1/2}(Z'_n - \mathbb{E}\, Z'_n) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2) \quad \text{as} \quad n \to \infty. \tag{7.11}$$

Let $b > 0$ with $h_V | b$. By Lemma 7.3, $\sigma^2 \geq \mathrm{Var} V$ and (2.3) holds and

$$\sup_{u\in\mathbb{R}} \left\{ \left| c_n P[Z'_n \in [u, u+b)] - \sigma^{-1} b\phi\left(\frac{u - \mathbb{E}\, Z'_n}{c_n\sigma}\right) \right| \right\} \to 0 \qquad \text{as} \quad n \to \infty.$$

Hence, by the assumption $n^{1/2}P[Z_n \neq Z'_n] \to 0$,

$$\sup_{u\in\mathbb{R}} \left\{ \left| c_n P[Z_n \in [u, u+b)] - \sigma^{-1} b\phi\left(\frac{u - \mathbb{E}\, Z'_n}{c_n\sigma}\right) \right| \right\} \to 0 \qquad \text{as} \quad n \to \infty,$$

and since the assumption $n^{-1/2}\mathbb{E}\,[|Z_n - Z'_n|] \to 0$ implies that $c_n^{-1}(\mathbb{E}\,[Z_n] - \mathbb{E}\,[Z'_n]) \to 0$ as $n \to \infty$, and ϕ is uniformly continuous on $\mathbb{R}$, we can then deduce (2.2). □

8 Proof of theorems for percolation

We shall repeatedly use the following Chernoff-type tail bounds for the binomial and Poisson distributions For $a > 0$ set $\varphi(a) := 1 - a + a\log a$. Then $\varphi(1) = 0$ and $\varphi(a) > 0$ for $a \in (0, \infty) \setminus \{1\}$.

Lemma 8.1. *If X is a binomial or Poisson distributed random variable with $\mathbb{E}\,[X] = \mu > 0$. Then we have for all $x > 0$ that*

$$P[X \geq x] \leq \exp(-\mu\varphi(x/\mu)), \qquad x \geq \mu; \tag{8.1}$$

$$P[X \leq x] \leq \exp(-\mu\varphi(x/\mu)), \qquad x \leq \mu. \tag{8.2}$$

Proof. See e.g. Lemmas 1.1 and 1.2 of [18]. □

Proof of Theorem 3.1. Let $(B_n)_{n\geq 1}$ be a sequence of nonempty finite subsets in $\mathbb{Z}^d$ with vanishing relative boundary. The first conclusion (3.2) follows from Theorem 3.1 of [19], so it remains to prove (3.3).

For $x \in \mathbb{Z}^d$ let $\|x\|_\infty$ denote the ℓ_∞-norm of x, i.e., the maximum absolute value of its coordinates. Let B_n^o be the set of points x in B_n such that all $y \in \mathbb{Z}^d$ with $\|y - x\|_\infty \leq 1$ are also in B_n. Since $|B_n \setminus B_n^o|/|\partial B_n|$ is bounded by a constant depending only on d, the vanishing relative boundary condition (3.1) implies $|B_n^o|/|B_n| \to 1$ as $n \to \infty$.

Hence, by the pigeonhole principle, for all large enough n we can choose a set of points $x_{n,1}, x_{n,2}, \ldots, x_{n,\lfloor 5^{-d}|B_n|/2\rfloor}$ in B_n^o such that $\|x_{n,j} - x_{n,k}\|_\infty \geq 3$ for each distinct j, k in $\{1, 2, \ldots, \lfloor 5^{-d}|B_n|/2\rfloor\}$ (let these points be chosen by some arbitrary deterministic rule).

For $1 \leq j \leq \lfloor 5^{-d}|B_n|/2\rfloor$, let $I_{n,j}$ be the indicator of the event that each vertex $y \in \mathbb{Z}^d$ with $\|y - x_{n,j}\|_\infty = 1$ is closed, and list the j for which $I_{n,j} = 1$, in increasing order, as $J(n, 1) \ldots, J(n, N_n)$, where $N_n := \sum_{j=1}^{\lfloor 5^{-d}|B_n|/2\rfloor} I_{n,j}$. Let $I'_{n,j}$ be the indicator of the event that the vertex $x_{n,j}$ is itself open. Then N_n is binomially distributed with parameter $(1-p)^{3^d-1}$, so by Lemma 8.1,

$$\limsup_{n\to\infty} |B_n|^{-1} \log P[N_n < 5^{-d}(1-p)^{3^d-1}|B_n|/4] < 0. \tag{8.3}$$

Set $b_n := \lfloor 5^{-d}(1-p)^{3^d-1}|B_n|/4\rfloor$. Let $V_1, V_2, \ldots$ be a sequence of independent Bernoulli variables with parameter p, independent of everything else. Recalling that $\Lambda(B)$ denotes the number of open clusters in B, set

$$S'_n := \sum_{j=1}^{\min(b_n, N_n)} I'_{n,J(n,j)}; \qquad Y_n := \Lambda(B_n) - S'_n,$$

and

$$S_n := S'_n + \sum_{j=1}^{(b_n - N_n)^+} V_j,$$

where $x^+ := \max(x, 0)$ as usual, and the sum $\sum_{i=1}^0$ is taken to be zero.

In this case, the 'good boxes' discussed in Section 1 are the unit ℓ_∞-neighbourhoods of the sites $x_{n,J(n,1)}, x_{n,J(n,2)}, \ldots x_{n,J(n,\min(b_n,N_n))}$. If $x_{n,j}$ is at the centre of a good box, it is (if open) isolated from other open sites, so that Y_n is simply the number of open clusters in B_n if one ignores all sites $x_{n,J(n,j)}$ $(1 \leq j \leq \min(b_n, N_n))$. Hence Y_n does not affect the open/closed status of these sites.

Thus S_n has the $\mathrm{Bin}(b_n, p)$ distribution and its distribution, given Y_n, is unaffected by the value of Y_n so S_n is independent of Y_n. Also,

$$\Lambda(B_n) - (Y_n + S_n) = S'_n - S_n = -\sum_{j=1}^{(b_n - N_n)^+} V_j$$

so that by (8.3), both $|B_n|^{1/2} P[\Lambda(B_n) \neq Y_n + S_n]$ and $|B_n|^{-1/2} \mathbb{E}\,[|\Lambda(B_n) - (Y_n + S_n)|]$ tend to zero as $n \to \infty$. Combined with (3.2) this shows that Theorem 2.1 is applicable, with $h_V = 1$, and that result shows that (3.3) holds. □

In the proof of Theorem 3.2, and again later on, we shall use the following.

Lemma 8.2. *Suppose $\xi_1, \ldots, \xi_m$ are independent identically distributed random elements of some measurable space $(E, \mathcal{E})$. Suppose $m \in \mathbb{N}$ and $\psi : E^m \to \mathbb{R}$ is measurable and suppose for some finite K that for $j = 1, \ldots, m$,*

$$K \geq \sup_{(x_1, \ldots, x_m, x'_j) \in E^{m+1}} |\psi(x_1, \ldots, x_j, \ldots, x_m) - \psi(x_1, \ldots, x'_j, \ldots, x_m)|.$$

Set $Y = \psi(\xi_1, \ldots, \xi_m)$. Then for any $t > 0$,

$$P[|Y - \mathbb{E}\, Y| \geq t] \leq 2 \exp(-t^2/(2mK^2)).$$

Proof. The argument is similar to e.g. the proof of Theorem 3.15 of [18]; we include it for completeness. For $1 \leq i \leq m$ let $\mathcal{F}_i$ be the σ-algebra generated by $\xi_1, \ldots, \xi_i$, and let $\mathcal{F}_0$ be the trivial σ-algebra. Then $Y - \mathbb{E}\,[Y] = \sum_{i=1}^m D_i$ with $D_i := \mathbb{E}\,[Y|\mathcal{F}_i] - \mathbb{E}\,[Y|\mathcal{F}_{i-1}]$, the ith martingale difference. Then with ξ'_i independent of $\xi_1, \ldots, \xi_m$ with the same distribution as them, we have

$$D_i = \mathbb{E}\,[\psi(\xi_1, \ldots, \xi_i, \ldots \xi_m) - \xi(\xi_1, \ldots, \xi'_i, \ldots, \xi_m)|\mathcal{F}_i]$$

so that $|D_i| \leq K$ almost surely and hence by Azuma's inequality (see e.g. [18]) we have the result.

Proof of Theorem 3.2. Assume $d \geq 2$ and $p > p_c(d)$. Let $(B_n)_{n \geq 1}$ be a cube-like sequence of lattice boxes in $\mathbb{Z}^d$. For finite nonempty $A \subset \mathbb{Z}^d$ we define the *diameter* of A, written $\mathrm{diam}(A)$, to be $\max\{\|x - y\|_\infty : x \in A, y \in A\}$.

Set $\gamma_n := \lceil \mathrm{diam}(B_n)^{1/(4d)} \rceil$. Let B_n^{in} be the set of points x in B_n such that all $y \in \mathbb{Z}^d$ with $\|y - x\|_\infty \leq \gamma_n$ are also in B_n. Then we claim that $|B_n^{\mathrm{in}}|/|B_n| \to 1$ as $n \to \infty$. Indeed, writing $B_n = \prod_{j=1}^d ([-a_{j,n}, b_{j,n}] \cap \mathbb{Z})$, from the cube-like condition (3.4) we have for $1 \leq j \leq d$ that $\gamma_n = o(a_{j,n} + b_{j,n})$ as $n \to \infty$, and therefore

$$|B_n^{\mathrm{in}}| = \prod_{j=1}^d (b_{j,n} + a_{j,n} - 2\gamma_n) = (1 + o(1)) \prod_{j=1}^d (a_{j,n} + b_{j,n}),$$

justifying the claim.

By the preceding claim, and the pigeonhole principle, for all large enough n there is a deterministic set of points $x_{n,1}, x_{n,2}, \ldots, x_{n,\lfloor 5^{-d}|B_n|/2\rfloor}$ in $B_n^{\rm in}$ such that $\|x_{n,j} - x_{n,k}\|_\infty \geq 3$ for each distinct j, k in $\{1, 2, \ldots, \lfloor 5^{-d}|B_n|/2\rfloor\}$.

For $1 \leq j \leq \lfloor 5^{-d}|B_n|/2\rfloor$, let $I_{n,j}$ be the indicator of the event that (i) each vertex $y \in \mathbb{Z}^d$ with $\|y - x_{n,j}\|_\infty = 1$ is open, and (ii) the open cluster in B_n containing all $y \in \mathbb{Z}^d$ with $\|y - x_{n,j}\|_\infty = 1$ has diameter at least γ_n.

Set $m(n) := \lfloor 5^{-d} p^{3^d-1} \theta_d(p) |B_n|/8\rfloor$, with $\theta_d(p)$ denoting the percolation probability. List the j for which $I_{n,j} = 1$ as $J(n,1), \ldots, J(n,N_n)$, with $N_n := \sum_{j=1}^{\lfloor 5^{-d}|B_n|/2\rfloor} I_{n,j}$. Then we have for n large that

$$\mathbb{E}\,[N_n] \geq \lfloor 5^{-d}|B_n|/2\rfloor p^{3^d-1}\theta_d(p) \geq 2m(n).$$

Changing the open/closed status of a single site z in B_n can change the value of $I_{n,j}$ only for those j for which $\|x_{n,j} - z\|_\infty \leq \gamma_n$, and the number of such j is at most $(2\gamma_n + 1)^d$. Moreover, for n large

$$(2\gamma_n + 1)^d \leq (2(\mathrm{diam} B_n)^{1/(4d)} + 3)^d \leq 3^d (\mathrm{diam} B_n)^{1/4} \leq 3^d |B_n|^{1/4}$$

so that the total change in N_n due to changing the status of a single site z is at most $3^d|B_n|^{1/4}$. So by Lemma 8.2,

$$P[N_n \leq m(n)] \leq P[|N_n - \mathbb{E}\,N_n| \geq m(n)] \leq 2\exp\left(-\frac{m(n)^2}{2|B_n|(3^d|B_n|^{1/4})^2}\right)$$

and hence

$$\limsup_{n\to\infty} |B_n|^{-1/2} \log P[N_n \leq m(n)] < 0. \tag{8.4}$$

Let $V_1, V_2, \ldots$ be a sequence of independent Bernoulli variables with parameter p, independent of everything else. For $1 \leq j \leq \lfloor 5^{-d}|B_n|/2\rfloor$, let $I'_{n,j}$ be the indicator of the event that the vertex $x_{n,j}$ is open. Set

$$S'_n := \sum_{j=1}^{\min(m(n),N_n)} I'_{n,J(n,j)}; \qquad S_n := S'_n + \sum_{j=1}^{(m(n)-N_n)^+} V_j.$$

Let Y_n be the size of the largest open cluster in B_n if the status of $x_{i,n}$ is set to 'closed' for the first $\min(m(n), N_n)$ values of j for which $I_{n,j} = 1$.

Then S_n has the $\mathrm{Bin}(m(n), p)$ distribution and we assert that its distribution, given Y_n, is unaffected by the value of Y_n so S_n is independent of Y_n. Indeed, Y_n is obtained without sampling the status of the sites $x_{n,j}$ for the first $\min(m(n), N_n)$ values of j for which $I_{n,j} = 1$.

To explain this further, consider algorithmically sampling the open/closed status of sites in B_n as follows. First sample the status of sites outside $\cup_j \{x_{n,j}\}$. Then sample the status of those $x_{n,j}$ for which the ℓ_∞-neighbouring sites are not all open (for these sites, $I_{n,j}$ must be zero). At this stage, it remains to sample the status of sites $x_{n,j}$ for which the ℓ_∞-neighbouring sites are all open, and for these sites one can tell, without revealing the value of $x_{n,j}$, whether or not $I_{n,j} = 1$ (and in particular one can determine the value of N_n). At the next step sample the status of all $x_{n,i}$ except for the first $\min(N_n, m(n))$ values of i which have $I_{n,j} = 1$. At this point, the value of Y_n is determined. However, the value of S_n is determined by the status of the remaining unsampled sites together with some extra Bernoulli variables in the case where $N_n < m(n)$, so its distribution is independent of the value of Y_n as asserted.

Next, we establish that $L(B_n) = Y_n + S_n$ with high probability. One way in which this could fail would be if $N_n < m(n)$, but we know from (8.4) that this has small probability. Also, we claim that with high probability, all sites $x_{n,j}$ for which $I_{n,j} = 1$ have all their neighbouring sites as part of the largest open cluster, regardless of the status of $x_{n,i}$. To see this, let A_n be the event that (i) there is a unique open cluster for B_n that crosses B_n in all directions (in the sense of [19]) and (ii) all other clusters in B_n have diameter less than γ_n. Then we claim that $P[A_n^c]$ decays exponentially in γ_n in the sense that

$$\limsup_{n\to\infty} (\mathrm{diam} B_n)^{1/(4d)} \log P[A_n^c] < 0. \tag{8.5}$$

The proof of (8.5) proceeds as in proof of Lemma 3.4 of [19]; we include a sketch of this argument here for completeness.

First suppose $d = 2$. For a given rectangle of dimensions $(\gamma_n/3) \times \gamma_n$, the probability that it fails to have an open crossing the long way decays exponentially in γ_n (see Lemma 3.1 of [19]). Consider the family of all rectangles of dimensions $(\gamma_n/3) \times \gamma_n$ or of dimensions $\gamma_n \times (\gamma_n/3)$, with all corners in $(\gamma_n/3)\mathbb{Z}^2$, having non-empty intersection with B_n. The number of such rectangles is $O(\mathrm{diam}(B_n)^{d-1/2})$. By the preceding probability estimate, all rectangles in this family have an open crossing the long way, except on an event of probablity decaying exponentially in γ_n. However, if all these rectangles have an open crossing the long way, then event A_n occurs and we have justified (8.5) for $d = 2$.

For $d \geq 3$, by the well known result of Grimmett and Marstrand [12], there exists a finite K such that there is an infinite open cluster in the slab $[0, K] \times \mathbb{R}^{d-1}$ with strictly positive probability. By dividing B_n into slabs of thickness K we see for $1 \leq i \leq d$ that the probabilty that there is no open crossing of B_n in the i-direction decays exponentially in $\mathrm{diam}(B_n)$. Moreover, for $i \neq j$, by a similar slab argument (consider successive slabs of thickness K in the i direction), the probability that

there is an open cluster in B_n that crosses B_n in the i direction but not the j direction decays exponentially in $\mathrm{diam}(B_n)$. Similarly the probability that there are two or more disjoint open clusters in B_n which cross in the i direction decays exponentially in n. Finally by a further slab argument, the probability that there is an open cluster which has diameter at least γ_n/d in the i direction but fails to cross the whole of B_n in the j direction, decreases exponentially in γ_n. This justifies (8.5) for $d \geq 3$.

Note that the occurrence or otherwise of A_n is unaffected by the open/closed status of those $x_{n,i}$ for which $I_{n,j} = 1$. Also, for large enough n, on event A_n, whatever status we give to these $x_{n,j}$, the unique crossing cluster is the largest one because it has at least $\mathrm{diam}(B_n)$ elements while all other clusters have at most $O(\mathrm{diam}(B_n)^{1/4})$ elements.

If $N_n \geq m(n)$ and event A_n occurs, then for each $j \leq m(n)$, the site $x_{n,J(n,j)}$ is in the largest open cluster if and only if it is open, since if it is open then it is in an open cluster of diameter at least γ_n. This shows that if $N_n \geq m(n)$ and event A_n occurs, we do indeed have $L(B_n) = Y_n + S_n$. Together with the previous probability estimates (8.4) and (8.5), this shows that $|B_n|^{1/2} P[L(B_n) \neq Y_n + S_n] \to 0$ as $n \to \infty$. Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left[|L(B_n) - (Y_n + S_n)|\right] &= \mathbb{E}\left[|L(B_n) - (Y_n + S_n)| \mathbf{1}_{\{N_n < m(n)\} \cup A_n^c}\right] \\
&\leq (P[N_n < m(n)] + P[A_n^c])^{1/2} (\mathbb{E}\left[(L(B_n) - (Y_n + S_n))^2\right])^{1/2} \\
&\leq (P[N_n < m(n)] + P[A_n^c])^{1/2} (|B_n| + m(n)) \to 0.
\end{aligned}$$

By Theorem 3.2 of [19], the first conclusion (3.5) holds, and by the preceding discussion, we can then apply Theorem 2.1 with $h_V = 1$, to derive the second conclusion (3.6). □

9 Proof of Theorem 4.1

We are now in the setting of Section 4. Assume $f \equiv f_U$, and fix a feasible connected graph Γ with κ vertices ($2 \leq \kappa < \infty$). Assume also that the sequence $(r_n)_{n \geq 1}$ is given and satisifies (4.2) and (4.3). Then $P[\mathcal{G}(\mathcal{X}_\kappa, 1/(\kappa+3)) \sim \Gamma] \in (0,1)$. Let $Q_{n,1}, Q_{n,2}, \ldots, Q_{n,m(n)}$ be disjoint cubes of side $(\kappa+5)r_n$, contained in the unit cube, with $m(n) \sim ((\kappa+5)r_n)^{-d}$ as $n \to \infty$. For $1 \leq j \leq m(n)$, let $I_{n,j}$ be the indicator of the event that $\mathcal{X}_n \cap Q_{n,j}$ consists of exactly κ points, all of them at a Euclidean distance greater than r_n from the boundary of $Q_{n,j}$. List the indices $j \leq m(n)$ such that $I_{n,j} = 1$, in increasing order, as $J_{n,1}, \ldots, J_{n,N_n}$, with $N_n := \sum_{j=1}^{m(n)} I_{n,j}$. Then

$$\mathbb{E}\left[N_n\right] = m(n)((\kappa+3)/(\kappa+5))^{d\kappa} P[\mathrm{Bin}(n, ((\kappa+5)r_n)^d) = \kappa], \tag{9.1}$$

and hence as $n \to \infty$, since nr_n^d is bounded by our assumption (4.2),

$$\mathbb{E}\left[N_n\right] \sim \kappa!^{-1}(\kappa+3)^{d\kappa}(\kappa+5)^{-d} n^{\kappa} r_n^{d(\kappa-1)} \exp(-n(\kappa+5)^d r_n^d). \tag{9.2}$$

Recalling from (4.3) that $\tau_n := \sqrt{n(nr_n^d)^{\kappa-1}}$, we can rewrite (9.2) as

$$\mathbb{E}\left[N_n\right] \sim \kappa!^{-1}(\kappa+3)^{d\kappa}(\kappa+5)^{-d} \tau_n^2 \exp(-n(\kappa+5)^d r_n^d) \tag{9.3}$$

as $n \to \infty$. Moreover, for the Poissonised version of this model where the number of points is Poisson distributed with mean n, we have the same asymptotics for the quantity corresponding to N_n (the binomial probability in (9.1) is asymptotic to the corresponding Poisson probability). Set α to be one-quarter of the coefficient of τ_n^2 in (9.3), if the exponential factor is replaced by its smallest value in the sequence, i.e. set

$$\alpha := (4\kappa!)^{-1}(\kappa+3)^{d\kappa}(\kappa+5)^{-d} \inf_n \exp(-n(\kappa+5)^d r_n^d). \tag{9.4}$$

Then $\alpha > 0$ by our assumption (4.2) on r_n.

Lemma 9.1. *It is the case that*

$$\limsup_{n\to\infty} \tau_n^{-2} \log P\left[N_n < \alpha\tau_n^2\right] < 0.$$

Proof. Let $\delta > 0$ (to be chosen later). Let M_n be Poisson distributed with parameter $(1-\delta)n$, independent of the sequence of random d-vectors $X_1, X_2, \ldots$. Define the Poisson point process

$$\mathcal{P}_{n(1-\delta)} := \{X_1, \ldots, X_{M_n}\}.$$

Let N_n' be defined in the same manner as N_n but in terms of $\mathcal{P}_{n(1-\delta)}$ rather than $\mathcal{X}_n$. That is, set

$$N_n' := \sum_{j=1}^{m(n)} I_{n,j}'$$

with $I_{n,j}'$ denoting the indicator of the event that $\mathcal{P}_{n(1-\delta)} \cap Q_{n,j}$ consists of exactly κ points, all at distance greater than r_n from the boundary of $Q_{n,j}$. List the indices $j \le M_n$ such that $I_{n,j}' = 1$ as $J_{n,1}', \ldots, J_{n,N_n'}'$.

Since (9.3) holds in the Poisson setting too, using the definition of τ_n we have as $n \to \infty$ that

$$\mathbb{E}\left[N_n'\right] \sim \kappa!^{-1}(\kappa+3)^{d\kappa}(\kappa+5)^{-d}(1-\delta)^{\kappa} \tau_n^2 \exp(-n(1-\delta)(\kappa+5)^d r_n^d). \tag{9.5}$$

By (9.3) and (9.5), we can and do choose $\delta > 0$ to be small enough so that $\mathbb{E}\, N'_n > (3/4)\mathbb{E}\, N_n$ for large n.

By (9.3) and (9.4) we have for large n that $2\alpha\tau_n^2 \leq (5/8)\mathbb{E}\, N_n$. Also, N'_n is binomially distributed, and hence by Lemma 8.1, $P[N'_n < 2\alpha\tau_n^2]$ decays exponentially in τ_n^2.

By Lemma 8.1, except on an event of probability decaying exponentially in n, the value of M_n lies between $n(1-2\delta)$ and n. If this happens, the discrepancy between N_n and N'_n is due to the addition of at most an extra $2\delta n$ points to $\mathcal{P}_{n(1-\delta)}$. If also $N'_n \geq 2\alpha\tau_n^2$ then to have $N_n < \alpha\tau_n^2$, at least $\alpha\tau_n^2$ of the added points must land in the union of the first $\lceil 2\alpha\tau_n^2 \rceil$ cubes contributing to N'_n.

To spell out the preceding argument in more detail, let $1 \leq j \leq m(n)$. If $M_n < n$ and $I'_{n,j} = 1$ and $X_k \notin Q_{n,j}$ for $M_n < k \leq n$, then $I_{n,j} = 1$, since in this case $\mathcal{X}_n \cap Q_{n,j} = \mathcal{P}_{n(1-\delta)} \cap Q_{n,j}$. Therefore if $M_n < n$ and $N'_n \geq 2\alpha\tau_n^2$ and

$$\sum_{k=M_n+1}^{n} \mathbf{1}\{X_k \in \cup_{j=1}^{\lceil 2\alpha\tau_n^2 \rceil} Q_{n,J'_{n,j}}\} < \alpha\tau_n^2,$$

then $\mathcal{X}_n \cap Q_{n,J'_{n,j}} \neq \mathcal{P}_{n(1-\delta)} \cap Q_{n,J'_{n,j}}$ for at most $\lfloor \alpha\tau_n \rfloor$ values of $j \in [1, 2\alpha\tau_n^2]$, and hence

$$N_n \geq \sum_{j=1}^{\lceil 2\alpha\tau_n^2 \rceil} I_{n,J'_{n,j}} \geq \lceil 2\alpha\tau_n^2 \rceil - \alpha\tau_n^2 \geq \alpha\tau_n^2.$$

Hence, if $n(1-2\delta) < M_n < n$ and $N'_n \geq 2\alpha\tau_n^2$ and $\sum_{k=M_n+1}^{M_n+\lceil 2\delta n \rceil} \mathbf{1}\{X_k \in \cup_{j=1}^{\lceil 2\alpha\tau_n^2 \rceil} Q_{n,J'_{n,j}}\} < \alpha\tau_n$, then $N_n \geq \alpha\tau_n^2$. Hence

$$\begin{aligned} &P[N_n < \alpha\tau_n^2 | N'_n \geq 2\alpha\tau_n^2, n - 2\delta n < M_n < n] \\ &\leq P[\mathrm{Bin}(\lceil 2\delta n \rceil, \lceil 2\alpha\tau_n^2 \rceil ((\kappa+5)r_n)^d) > \alpha\tau_n^2]. \end{aligned}$$

Since nr_n^d is assumed bounded, we can choose δ small enough so that the expectation of the binomial variable in the last line is less than $(\alpha/2)\tau_n^2$, and then appeal once more to Lemma 8.1 to see that the above conditional probability decays exponentially in τ_n^2. Combining all these probability estimates give the desired result. □

Proof of Theorem 4.1. Set $p := P[\mathcal{G}(\mathcal{X}_\kappa, 1/(\kappa+3)) \sim \Gamma]$. Let $V_1, V_2, \ldots$ be a sequence of independent Bernoulli variables with parameter p, independent of $\mathcal{X}_n$. Let

$$S'_n := \sum_{j=1}^{\min(\lfloor \alpha\tau_n^2 \rfloor, N_n)} \mathbf{1}\{\mathcal{G}(\mathcal{X}_n \cap Q_{n,J(n,j)}; r_n) \sim \Gamma\}; \qquad Y_n := G_n - S'_n,$$

and

$$S_n := S'_n + \sum_{j=1}^{(\lfloor \alpha\tau_n^2 \rfloor - N_n)^+} V_j,$$

where $x^+ := \max(x, 0)$ as usual, and the sum $\sum_{j=1}^{0}$ is taken to be zero.

For each j, given that $I_{n,j} = 1$, the distribution of the contribution to G_n from points in $Q_{n,j}$ is Bernoulli with parameter $P[\mathcal{G}((\kappa+3)r_n\mathcal{X}_\kappa, r_n) \sim \Gamma]$, which is p. Hence S_n is binomial $\mathrm{Bin}(\lfloor \alpha\tau_n^2 \rfloor, p)$. Moreover, the conditional distribution of S_n, given the value of Y_n, does not depend on the value of Y_n, and therefore S_n is independent of Y_n. By (4.5),

$$\lfloor \alpha\tau_n^2 \rfloor^{-1/2}(G_n - \mathbb{E}\, G_n) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \alpha^{-1}\sigma^2).$$

Moreover,

$$\mathbb{E}\left[|G_n - (Y_n + S_n)|\right] = \mathbb{E}\left[\sum_{j=1}^{(\lfloor \alpha\tau_n^2 \rfloor - N_n)^+} V_j\right] \leq p\lfloor \alpha\tau_n^2 \rfloor P[N_n < \alpha\tau_n^2]$$

so that by Lemma 9.1, both $\tau_n P[G_n \neq Y_n + S_n]$ and $\tau_n^{-1}\mathbb{E}\left[|G_n - Y_n - S_n|\right]$ tend to zero as $n \to \infty$. Hence, Theorem 2.1 (with $h_V = 1$) is applicable, with $\lfloor \alpha\tau_n^2 \rfloor$ playing the role of n in that result and $\alpha^{1/2}\tau_n$ playing the role of c_n, yielding

$$\sup_{k\in\mathbb{Z}} \left\{ \left| \alpha^{1/2}\tau_n P[G_n = k] - \alpha^{1/2}\sigma^{-1}\phi\left(\frac{k - \mathbb{E}\, G_n}{(\alpha^{1/2}\tau_n)\alpha^{-1/2}\sigma} \right) \right| \right\} \to 0,$$

as $n \to \infty$. Multiplying through by $\alpha^{-1/2}$ yields (4.6). □

10 Proof of Theorem 5.1

Recall the definition of h_X (the span of X) from Section 2.

Lemma 10.1. *If X and Y are independent random variables then $h_{X+Y} | h_X$.*

Proof. If $h_{X+Y} = 0$ there is nothing to prove. Otherwise, set $h = h_{X+Y}$. Then, considering characteristic functions, observe that

$$1 = |\mathbb{E}\, \exp(2\pi i(X+Y)/h)| = |\mathbb{E}\, \exp(2\pi i X/h)| \times |\mathbb{E}\, \exp(2\pi i Y/h)|$$

so that $|\mathbb{E}\, \exp(2\pi i X/h)| = 1$ and hence $h | h_X$. □

We are in the setup of Section 5. Recall that the point process $\mathcal{Z}_n$ consists of n normally distributed marked points in $\mathbb{R}^d$, while $\mathcal{U}_{n,K}$ consists of n uniformly distributed marked points in $B(K)$. Set $h_{n,K} := h_{H(\mathcal{U}^*_{n,K})}$. Set $h_n := h_{H(\mathcal{Z}^*_n)}$, and recall from (5.4) that $h(H) := \liminf_{n\to\infty} h_n$.

Lemma 10.2. *Suppose either (i) H has finite range interactions and $h_{H(\mathcal{Z}^*_n)} < \infty$ for some n, or (ii) $H = H^{(\xi)}$ is induced by a κ-nearest neighbour functional $\xi(\mathbf{x}; \mathcal{X}^*)$, and $h_{H(\mathcal{Z}^*_n)} < \infty$ for some $n > \kappa$. Then $h(H) < \infty$, and if $h(H) > 0$, there exists $\mu \in \mathbb{N}$ and $K > 0$ such that $h_{\mu,K} = h(H)$. If $h(H) = 0$, then for any $\varepsilon > 0$ there exists $\mu \in \mathbb{N}$ and $K > 0$ such that $h_{\mu,K} < \varepsilon$. In case (ii), we can take μ such that additionally $\mu \geq \kappa + 1$.*

Proof. The support of the distribution of $H(\mathcal{U}^*_{n,K})$ is increasing with K, so $h_{n,K'} | h_{n,K}$ for $K' \geq K$. Hence, there exists a limit $h_{n,\infty}$ such that

$$h_{n,\infty} = \lim_{K\to\infty} h_{n,K} \tag{10.1}$$

and also we have the implication

$$h_{n,\infty} > 0 \implies \exists K : h_{n,K} = h_{n,\infty}. \tag{10.2}$$

Also, for all K the support of the distribution of $H(\mathcal{U}^*_{n,K})$ is contained in the support of $H(\mathcal{Z}^*_n)$, so that

$$h_n = h_{H(\mathcal{Z}^*_n)} \leq h_{n,K}, \quad \forall K, \tag{10.3}$$

and hence $h_n \leq h_{n,\infty}$ for all n. We assert that in fact

$$h_{n,\infty} = h_n. \tag{10.4}$$

This is clear when $h_{n,\infty} = 0$. When $h_{n,\infty} > 0$, there exists a countable set S with span $h_{n,\infty}$ such that $P[H(\mathcal{U}^*_{n,K}) \in S] = 1$ for all K. But then it is easily deduced that $P[H(\mathcal{Z}^*_n) \in S] = 1$, so that $h_n \geq h_{n,\infty}$, and combined with (10.3) this gives (10.4).

We shall show in both cases (i) and (ii) that h_n tends to a finite limit; that is, for both cases we shall show that

$$h(H) = \lim_{n\to\infty} h_n = \lim_{n\to\infty} h_{n,\infty} < \infty. \tag{10.5}$$

Also, we show in both cases that

$$h(H) > 0 \implies \exists n_0 \in \mathbb{N} : h_n = h(H) \ \ \forall n \geq n_0. \tag{10.6}$$

If $h(H) > 0$, the desired conclusion follows from (10.6), (10.4) and (10.2). If $h(H) = 0$, the desired conclusion follows from (10.5) and (10.1).

Consider the case (i), where H has finite range interactions. In this case, we shall show that for all n,

$$h_{n+1}|h_n, \tag{10.7}$$

and since we assume $h_n < \infty$ for some n, (10.7) clearly implies (10.5) and (10.6).

We now demonstrate (10.7) in case (i) as follows. By (10.4) and (10.2), to prove (10.7) it suffices to prove that $h_{n+1}|h_{n,K}$ for all K. Choose τ such that (5.2) holds. There is a strictly positive probability that the first n points of $\mathcal{Z}_n$ lie in $B(K)$ while the last one lies outside $B(K+\tau)$. Hence by (5.2) and translation-invariance, the support of the distribution of $H(\mathcal{Z}^*_{n+1})$ contains the support of the distribution of $H(\mathcal{U}^*_{n,K}) + H(\{(0,T)\})$, where T is a $\mathbb{P}_{\mathcal{M}}$-distributed element of $\mathcal{M}$, independent of $U^*_{n,K}$. Hence by Lemma 10.1, $h_{n+1}|h_{n,K}$, so (10.7) holds as claimed in this case.

Now consider case (ii), where we assume $H = H^{(\xi)}$ with $\xi(\mathbf{x}; \mathcal{X})$ determined by the κ nearest neighbours. We claim that if $j \geq \kappa + 1$ and $\ell \geq \kappa + 1$ then

$$h_{j+\ell}|h_j \quad \text{and} \quad h_{j+\ell}|h_\ell. \tag{10.8}$$

By (10.2) and (10.4), to verify (10.8) it suffices to show that

$$h_{j+\ell}|h_{j,K} \quad \forall K > 0. \tag{10.9}$$

Given K, let B and B' be disjoint balls of radius K, distant more than $2K$ from each other. There is a positive probability that $\mathcal{Z}_{j+\ell}$ consists of j points in B and ℓ points in B', and if this happens then (since we assume $\min(j, \ell) > \kappa$)) the κ nearest neighbours of the points in B are also in B, while the κ nearest neighbours of the points in B' are also in B', so that $H(\mathcal{Z}^*_{j+\ell})$ is the sum of conditionally independent contributions from the points in B and those in B'. Hence the support of the distribution of $H(\mathcal{Z}^*_{j+\ell})$ contains the support of the distribution of $H(\mathcal{U}^*_{j,K}) + H(\tilde{\mathcal{U}}^*_{\ell,K})$, where $H(\tilde{\mathcal{U}}^*_{\ell,K})$ is defined to be a variable with the distribution of $H(\mathcal{U}^*_{\ell,K})$ independent of $H(\mathcal{U}^*_{j,K})$. Then (10.9) follows from Lemma 10.1.

Define

$$h' = \inf_{n \geq \kappa+1} h_n.$$

Then for all $\varepsilon > 0$ we can pick $j \geq \kappa + 1$ with $h_j \leq h' + \varepsilon$, and then by (10.8) we have $h_\ell \leq h' + \varepsilon$ for $\ell \geq j + \kappa + 1$. This demonstrates (10.5) for this case (with $h(H) = h'$), since we assume $h_n < \infty$ for some n. Moreover, if $h(H) > 0$, then in the argument just given we can take $\varepsilon < h(H)$ and then for $\ell \geq j + \kappa + 1$ we must

have $h_\ell | h_j$, which can happen only if $h_\ell = h_j$, so by (10.5), in fact $h_\ell = h_j = h(H)$. That is, we also have (10.6) for this case. □

Since we are in the setting of Section 5, we assume (as in Section 4) that f is an almost everywhere continuous probability density function on $\mathbb{R}^d$ with $f_{\max} < \infty$. The point process $\mathcal{X}_n \subset \mathbb{R}^d$ is a sample from this density, and the marked point process $\mathcal{X}_n^* \subset \mathbb{R}^d \times \mathcal{M}$ is obtained by giving each point of $\mathcal{X}_n$ a $\mathbb{P}_{\mathcal{M}}$-distributed mark. Recall also that we are given a sequence (r_n) with $\rho := \lim_{n\to\infty} nr_n^d \in (0, \infty)$. Recall from (5.1) that $H_n(\mathcal{X}^*) := H(r_n^{-1}\mathcal{X}^*)$ for a given translation-invariant H.

Our strategy for proving Theorem 5.1 goes as follows. First we choose μ, K as in Lemma 10.2. Then we choose constants $\beta \geq K$ and $m \geq \mu$ in a certain way (see below), and use the continuity of f to pick $\Theta(n)$ disjoint deterministic balls of radius βr_n such that f is positive and almost constant on each of these balls. We use a form of rejection sampling to make the density of points of $\mathcal{X}_n$ in each (unrejected) ball uniform. We also reject all balls which do not contain exactly m points of $\mathcal{X}_n$ in a certain 'good' configuration (of non-vanishing probability). The definition of 'good' is chosen in such a way that the contribution to H_n from inside an inner ball of radius Kr_n is shielded from everything outside the outer ball of radius βr_n. We end up with $\Theta(n)$ (in probability) unrejected balls, and the contributions to $H_n(\mathcal{X}_n^*)$ from the corresponding inner balls are independent (because of the shielding) and identically distributed (because of the uniformly distributed points) so the sum contribution of these inner balls can play the role of S_n in Theorem 2.1.

In the proof of Theorem 5.1, we need to consider certain functions, sets and sequences, defined for $\beta > 0$. For $x \in \mathbb{R}^d$ with $f(x) > 0$, define the function

$$g_{n,\beta}(x) := \frac{\inf\{f(y) : y \in B(x; \beta r_n)\}}{\sup\{f(y) : y \in B(x; \beta r_n)\}}, \tag{10.10}$$

and for $x \in \mathbb{R}^d$ with $f(x) > 0$ and $g_{n,\beta}(x) > 0$, and $z \in B(x; \beta r_n)$, define

$$p_{n,\beta}(x, z) := \frac{\inf\{f(y) : y \in B(x; \beta)\}}{f(z)}. \tag{10.11}$$

Since we assume f is almost everywhere continuous, the function $g_{n,\beta}$ converges almost everywhere on $\{x : f(x) > 0\}$ to 1. By Egorov's theorem (see e.g. [9]), given $\beta > 0$ there is a set A_β with $\int_{A_\beta} f(x)dx \geq 1/2$, such that $f(x)$ is bounded away from zero on A_β and $g_{n,\beta}(x) \to 1$ uniformly on A_β.

Since we assume (4.2) with $\rho > 0$ here, for n large enough $nr_n^d < 2\rho$. Set

$$\eta(\beta) := 2^{-(d+2)}\omega_d^{-1}\beta^{-d}f_{\max}^{-1}\rho^{-1}.$$

Given $\beta > 0$, we claim that for n large enough so that $nr_n^d < 2\rho$, we can (and do) choose points $x_{\beta,n,1}, \ldots, x_{\beta,n,\lfloor\eta(\beta)n\rfloor}$ in A_β with $|x_{\beta,n,j} - x_{\beta,n,k}| > 2\beta r_n$ for $1 \leq$

$j < k \leq \lfloor \eta(\beta) n \rfloor$. To see this we use a measure-theoretic version of the pigeonhole principle, as follows. Suppose inductively that we have chosen $x_{\beta,n,1}, \ldots, x_{\beta,n,k}$, with $k < \lfloor \eta(\beta) n \rfloor$. Then let $x_{\beta,n,k+1}$ be the first point, according to the lexicographic ordering, in the set $A_\beta \setminus \cup_{j=1}^{k} B(x_{\beta,n,j}; 2\beta r_n)$. This is possible, because this set is non-empty, because by subadditivity of measure,

$$\int_{\cup_{j=1}^{k} B(x_{\beta,n,j}; 2\beta r_n)} f(x)dx \leq k\omega_d (2\beta r_n)^d f_{\max} < \eta(\beta) n \omega_d (2\beta r_n)^d f_{\max}$$
$$= n r_n^d / (4\rho) < 1/2 \leq \int_{A_\beta} f(x)dx,$$

justifying the claim. Define the ball

$$B_{\beta,n,j} := B(x_{\beta,n,j}, \beta r_n); \qquad B^*_{\beta,n,j} := B(x_{\beta,n,j}, \beta r_n) \times \mathcal{M}.$$

The balls $B_{\beta,n,1}, \ldots, B_{\beta,n,\lfloor \eta(\beta) n \rfloor}$ are disjoint.

Let $W_1, W_2, W_3, \ldots$ be uniformly distributed random variables in $[0,1]$, independent of each other and of $(\mathbf{X}_j)_{j=1}^n$, where $\mathbf{X}_j = (X_j, T_j)$. For $k \in \mathbb{N}$, think of W_k as an extra mark attached to the point X_k. This is used in the rejection sampling procedure. Given β, if $X_k \in B_{\beta,n,j}$, let us say that the point X_k is *β-red* if the associated mark W_k is less than $p_{n,\beta}(x_{\beta,n,j}, X_k)$. Given that X_k lies in $B_{\beta,n,j}$ and is β-red, the conditional distribution of X_k is uniform over $B_{\beta,n,j}$.

Now let $m \in \mathbb{N}$, and suppose $\mathcal{S}$ is a measurable set of configurations of m points in $B(\beta)$ such that $P[\mathcal{U}_{m,\beta} \in \mathcal{S}] > 0$. The number m and the set $\mathcal{S}$ will be chosen so that given there are m points of $\mathcal{X}_n$ in ball $B_{\beta,n,j}$, and given their rescaled configuration of lies in the set $\mathcal{S}$, there is a subset of these m points which are 'shielded' from the rest of $\mathcal{X}_n$.

Given $\mathcal{S}$ (and by implication β and m), for $1 \leq j \leq \lfloor \eta(\beta) n \rfloor$, let $I_{\mathcal{S},n,j}$ be the indicator of the event that the following conditions hold:

- The point set $\mathcal{X}_n \cap B_{\beta,n,j}$ consists of m points, all of them β-red;
- The configuration $r_n^{-1}(-x_{\beta,n,j} + (\mathcal{X}_n \cap B_{\beta,n,j}))$ is in $\mathcal{S}$.

Let $N_{\mathcal{S},n} := \sum_{j=1}^{\lfloor \eta(\beta) n \rfloor} I_{\mathcal{S},n,j}$, and list the i for which $I_{\mathcal{S},n,j} = 1$ in increasing order as $J(\mathcal{S}, n, 1) \ldots, J(\mathcal{S}, n, N_{\mathcal{S},n})$.

Lemma 10.3. *Let $\beta > 0$, and $m \in \mathbb{N}$. Let $\mathcal{S}$ be a measurable set of configurations of m points in $B(\beta)$ such that $P[\mathcal{U}_{m,\beta} \in \mathcal{S}] > 0$. Then: (i) there exists $\delta > 0$ such that*

$$\limsup_{n \to \infty} \left(n^{-1} \log P[N_{\mathcal{S},n} < \delta n] \right) < 0, \tag{10.12}$$

and (ii) conditional on the values of $I_{\mathcal{S},n,i}$ for $1 \le i \le \lfloor \eta(\beta) n \rfloor$ and the configuration of $\mathcal{X}_n$ outside $B_{\beta,n,J(\mathcal{S},n,1)} \cup \cdots \cup B_{\beta,n,J(\mathcal{S},n,N_{\mathcal{S},n})}$, the joint distribution of the point sets

$$r_n^{-1}(-x_{\beta,n,J(\mathcal{S},n,1)} + (\mathcal{X}_n \cap B_{\beta,n,J(\mathcal{S},n,1)})), \ldots, r_n^{-1}(-x_{\beta,n,J(\mathcal{S},n,N_{\mathcal{S},n})} + (\mathcal{X}_n \cap B_{\beta,n,J(\mathcal{S},n,N_{\mathcal{S},n})}))$$

is that of $N_{\mathcal{S},n}$ independent copies of $\mathcal{U}_{m,\beta}$ each conditioned to be in $\mathcal{S}$.

Proof. Consider first the asymptotics for $\mathbb{E}[N_{\mathcal{S},n}]$. Given a finite point set $\mathcal{X} \subset \mathbb{R}^d$ and a set $B \subset \mathbb{R}^d$, let $\mathcal{X}(B)$ denote the number of points of $\mathcal{X}$ in B. Fix m. Since f is bounded away from zero and infinity on A_β and $g_{n,\beta} \to 1$ uniformly on A_β, we have uniformly over $x \in A_\beta$ that

$$n \int_{B(x;\beta r_n)} f(y) dy = n f(x) \int_{B(x;\beta r_n)} (f(y)/f(x)) dy \to \beta^d \omega_d \rho f(x)$$

Hence by binomial approximation to Poisson,

$$P[\mathcal{X}_n(B(x;\beta r_n)) = m] \to \frac{(\beta^d \omega_d \rho f(x))^m \exp(-\beta^d \omega_d \rho f(x))}{m!} \quad \text{as } n \to \infty,$$

and this convergence is also uniform over $x \in A_\beta$.

Given m points X_k in $B_{\beta,n,j}$, the probability that these are all β-red is at least $g_{n,\beta}(x)^m$ so exceeds $\frac{1}{2}$ if n is large enough, since $g_{n,\beta} \to 1$ uniformly on A_β.

Given that m of the points X_k lie in $B_{\beta,n,j}$, and given that they are all β-red, their spatial locations are independently uniformly distributed over $B_{\beta,n,j}$; hence the conditional probability that $r_n^{-1}(-x_{\beta,n,j} + (\mathcal{X}_n \cap B_{\beta,n,j}))$ lies in $\mathcal{S}$ is a strictly positive constant.

These arguments show that $\liminf_{n\to\infty} n^{-1}\mathbb{E}[N_{\mathcal{S},n}] > 0$. They also demonstrate part (ii) in the statement of the lemma.

Take $\delta > 0$ with $2\delta < \liminf_{n\to\infty} n^{-1}\mathbb{E}[N_{\mathcal{S},n}]$. We shall show that $P[N_{\mathcal{S},n} < \delta n]$ decays exponentially in n, using Lemma 8.2. The variable $N_{\mathcal{S},n}$ is a function of n independent identically distributed triples (marked points) (X_k, T_k, W_k).

Consider the effect of changing the value of one of the marked points ((X,T,W) to (X',T',W'), say). The change could affect the value of $I_{\mathcal{S},n,j}$ for at most two values of j, namely the j with $X \in B_{\beta,n,j}$ and the j' with $X' \in B_{\beta,n,j'}$. So by Lemma 8.2,

$$P[|N_{\mathcal{S},n} - \mathbb{E}\, N_{\mathcal{S},n}| > \delta n] \le 2\exp(-\delta^2 n/8),$$

and (10.12) follows. □

Proof of Theorem 5.1 under condition (i) (finite range interactions). Recall that $h(H)$ is given by (5.4). Since condition (i) includes the assumption that $h_{H(\mathcal{Z}_n^*)} < \infty$

for some n, by Lemma 10.2 we have $h(H) < \infty$. Let $b > 0$ with $h(H)|b$. Let $\varepsilon \in (0, b)$. Let $\mu \in \mathbb{N}$, and $K > 0$, be as given by Lemma 10.2. Then $h_{\mu,K} = h(H)$ if $h > 0$, or $h_{\mu,K} < \varepsilon$ if $h = 0$. Moreover $H(\mathcal{U}^*_{\mu,K})$ is integrable by assumption. Set

$$b_1 := \begin{cases} h_{\mu,K}\lfloor b/h_{\mu,K}\rfloor & \text{if } h_{\mu,K} > 0 \\ b & \text{if } h_{\mu,K} = 0. \end{cases} \tag{10.13}$$

Choose $\tau \in (0, \infty)$ such that (5.2) holds. We shall apply Lemma 10.3 with $\beta = K + \tau$. Let $\mathcal{S}$ be the set of configurations of μ points in $B(K + \tau)$ such that in fact all of the points are in $B(K)$. By Lemma 10.3, we can find $\delta > 0$ such that, writing N_n for $N_{\mathcal{S},n}$ we have exponential decay of $P[N_n < \delta n]$.

Let $V_1, V_2, \ldots,$ be random variables distributed as independent copies of $H(\mathcal{U}^*_{\mu,K})$, independently of $\mathcal{X}^*_n$. Set

$$S'_n := \sum_{\ell=1}^{\min(\lfloor \delta n\rfloor, N_n)} H_n(\mathcal{X}^*_n \cap B^*_{K+\tau,n,J(\mathcal{S},n,\ell)}); \qquad S_n = S'_n + \sum_{j=1}^{(\lfloor \delta n\rfloor - N_n)^+} V_j.$$

Thus, S'_n is the the total contribution to $H_n(\mathcal{X}^*_n)$ from points in $\cup_{\ell=1}^{\min(\lfloor \delta n\rfloor, N_n)} B^*_{K+\tau,n,J(\mathcal{S},n,\ell)}$.

By Part (ii) of Lemma 10.3, given that $N_n \geq \delta n$, for each ℓ we know that $r_n^{-1}(-x_{\beta,n,J(\mathcal{S},n,\ell)} + \mathcal{X}^*_n) \cap B^*(K+\tau)$ is conditionally distributed as $\mathcal{U}^*_{\mu,K+\tau}$ conditional on $\mathcal{U}^*_{\mu,K+\tau} \in \mathcal{S}$; in other words, distributed as $\mathcal{U}^*_{\mu,K}$. Therefore the distribution of S_n is that of the sum of $\lfloor \delta n\rfloor$ independent copies of $H(\mathcal{U}^*_{\mu,K})$, independent of the contribution of the other points. Let Y_n denote the contribution of the other points, i.e.

$$Y_n := H_n(\mathcal{X}^*_n) - S'_n.$$

Since the distribution of S_n, given the value of Y_n, does not depend on the value of Y_n, S_n is independent of Y_n.

By assumption $H_n(\mathcal{X}^*_n)$ and S_n are integrable. Clearly $n^{1/2}P[H_n(\mathcal{X}_n) \neq Y_n + S_n]$ is at most $n^{1/2}P[N_n < \delta n]$, which tends to zero by (10.12). Also by conditioning on N_n, we have that

$$\begin{aligned} n^{-1/2}\mathbb{E}\left[|H_n(\mathcal{X}^*_n) - (Y_n + S_n)|\right] &= n^{-1/2}\mathbb{E}\left[\left|\sum_{j=1}^{(\lfloor \delta n\rfloor - N_n)^+} V_j\right|\right] \\ &\leq n^{-1/2}\mathbb{E}\left[(\lfloor \delta n\rfloor - N_n)^+\right]\mathbb{E}\left[|V_1|\right] \\ &\leq n^{-1/2}\lfloor \delta n\rfloor P[N_n \leq \delta n]\mathbb{E}\left[|V_1|\right], \end{aligned} \tag{10.14}$$

which tends to zero by (10.12). This also shows that Y_n is integrable By the assumption (5.5),

$$\lfloor \delta n\rfloor^{-1/2}(H_n(\mathcal{X}^*_n) - \mathbb{E}\, H_n(\mathcal{X}^*_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \delta^{-1}\sigma^2), \tag{10.15}$$

and so, since $h_{\mu,K}|b_1$, Theorem 2.1 is applicable, and yields

$$\sup_{u\in\mathbb{R}}\left\{\left|(\delta n)^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b_1)]-\delta^{1/2}\sigma^{-1}b_1\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{(\delta n)^{1/2}(\delta^{-1}\sigma^2)^{1/2}}\right)\right|\right\}\to 0, \tag{10.16}$$

and dividing through by $\delta^{1/2}$ gives (5.6) in all cases where $b=b_1$. In general, suppose $b\neq b_1$. Then $h(H)=0$ (else $h_{\mu,K}=h(H)$ and $h(H)|b$ so $b=b_1$ by (10.13)), and hence $h_{\mu,K}<\varepsilon$. Since $b_1\le b$ by (10.13), we have that

$$\begin{aligned}&\inf_{u\in\mathbb{R}}\left\{n^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b)]-\sigma^{-1}b\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma}\right)\right\}\\ &\ge\inf_{u\in\mathbb{R}}\left\{n^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b_1)]-\sigma^{-1}b_1\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma}\right)\right\}\\ &\qquad+\sigma^{-1}(b_1-b)(2\pi)^{-1/2}\end{aligned}$$

so that by (10.16), since $b_1\ge b-\varepsilon$,

$$\liminf_{n\to\infty}\inf_{u\in\mathbb{R}}\left\{n^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b)]-\sigma^{-1}b\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma}\right)\right\}\ge-\frac{\varepsilon}{\sigma}(2\pi)^{-1/2}.$$

Similarly, setting $b_2:=h_{\mu,K}\lceil b/h_{\mu,K}\rceil$, we have that

$$\begin{aligned}&\sup_{u\in\mathbb{R}}\left\{n^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b)]-\sigma^{-1}b\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma}\right)\right\}\\ &\le\inf_{u\in\mathbb{R}}\left\{n^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b_2)]+\sigma^{-1}b_2\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma}\right)\right\}\\ &\qquad+\sigma^{-1}(b_2-b)(2\pi)^{-1/2}\end{aligned}$$

so that since $b_2-b\le\varepsilon$,

$$\limsup_{n\to\infty}\sup_{u\in\mathbb{R}}\left\{n^{1/2}P[H_n(\mathcal{X}_n^*)\in[u,u+b)]-\sigma^{-1}b\phi\left(\frac{u-\mathbb{E}\,H_n(\mathcal{X}_n^*)}{n^{1/2}\sigma}\right)\right\}\le\frac{\varepsilon}{\sigma}(2\pi)^{-1/2}.$$

Since $\varepsilon>0$ is arbitrarily small, this gives us (5.6). □

Proof of Theorem 5.1 under condition (ii). We now assume that H, instead of having finite range, is given by (5.3) with ξ depending only on the κ nearest neighbours. Again, by Lemma 10.2 we have that $h(H)$, given by (5.4), is finite.

Let $b>0$ with $h(H)|b$. Let $\varepsilon\in(0,b)$. Let $\mu\in\mathbb{N}$ and $K>0$, with $\mu\ge\kappa+1$, by as given by Lemma 10.2. Then $h_{\mu,K}=h(H)$ if $h(H)>0$, and $h_{\mu,K}<\varepsilon$ if $h(H)=0$.

Also, $H(\mathcal{U}^*_{\mu,K})$ integrable, by the integrability assumption in the statement of the result being proved.

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\nu$ be a minimal collection of open balls of radius K, each of them centred at a point on the boundary of $B(4K)$, such that their union contains the boundary of $B(4K)$. Let $\mathcal{B}_0$ be the ball $B(K)$.

We shall apply Lemma 10.3 with $\beta = 5K$, with $m = (\nu+1)\mu$, and with $\mathcal{S}$ as follows. $\mathcal{S}$ is the set of configurations of $m = (\nu+1)\mu$ points in $B(\beta) = B(5K)$, such that each of $\mathcal{B}_1, \ldots, \mathcal{B}_\nu$ contains at least μ points, and $\cup_{i=1}^{\nu} \mathcal{B}_i$ contains exactly $\nu\mu$ points, and also the ball $\mathcal{B}_0$ contains exactly μ points (so that consequently there are no points in $B(5K) \setminus \cup_{i=0}^{\nu} \mathcal{B}_i$). A similar construction (using squares rather than balls, and with diagram) was given by Avram and Bertsimas [1] for a related problem.

With this choice of β and $\mathcal{S}$, let the locations $x_{\beta,n,j} = x_{5K,n,j}$, the balls $B_{\beta,n,j} = B_{5K,n,j}$, the indicators $I_{\mathcal{S},n,j}$, and the variables $N_{\mathcal{S},n}$ and $J(\mathcal{S}, n, \ell)$ be as described just before Lemma 10.3. By that result, we can (and do) choose $\delta > 0$ such that (10.12) holds.

For $1 \leq \ell \leq N_{\mathcal{S},n}$, the point process $r_n^{-1}(-x_{5K,n,J(\mathcal{S},n,\ell)} + (\mathcal{X}_n \cap B_{5K,n,J(\mathcal{S},n,\ell)}))$ has μ points within distance K of the origin, and also at least μ points in each of the balls $\mathcal{B}_1, \ldots, \mathcal{B}_\nu$.

Since $\mu \geq \kappa + 1$, for any point configuration in $\mathcal{S}$, each point inside $B(K)$ has its κ nearest neighbours also inside $B(K)$. Also none of the points in $B(5K) \setminus B(K)$ has any of its κ nearest neighbours in $B(K)$. Finally, any further added point outside $B(5K)$ cannot have any of its κ nearest neighbours inside $B(K)$, since the line segment from such a point to any point in $B(K)$ passes through the boundary of $B(4K)$ at a location inside some $\mathcal{B}_i$, and any of the μ or more points inside $\mathcal{B}_i$ are closer to the outside point than the point in $B(K)$ is. To summarise this discussion, the points in $B(K)$ are shielded from those outside $B(5K)$.

Given n, let $\mathcal{W}^{(1)}_{(\nu+1)\mu,5K}, \ldots, \mathcal{W}^{(\lfloor \delta n \rfloor)}_{(\nu+1)\mu,5K}$ be a collection of (marked) point processes which are each distributed as $\mathcal{U}^*_{(\nu+1)\mu,5K}$ conditioned on $\mathcal{U}^*_{(\nu+1)\mu,5K} \in \mathcal{S}$, independently of each other and of $\mathcal{X}^*_n$. For $1 \leq j \leq \lfloor \delta n \rfloor$ set $V_j := H(\mathcal{W}^{(j)}_{(\nu+1)\mu,5K} \cap B^*(K))$, so that $V_1, V_2, \ldots V_{\lfloor \delta n \rfloor}$ are random variables distributed as independent copies of $H(\mathcal{U}^*_{\mu,K})$, independent of $\mathcal{X}_n$. Define S'_n and S_n by

$$S'_n := \sum_{\ell=1}^{\min(\lfloor \delta n \rfloor, N_{\mathcal{S},n})} H_n(\mathcal{X}^*_n \cap B^*(x_{5K,n,J(\mathcal{S},n,\ell)}, Kr_n)); \quad S_n := S'_n + \sum_{j=1}^{(\lfloor \delta n \rfloor - N_{\mathcal{S},n})^+} V_j.$$

Also set $Y_n := H_n(\mathcal{X}^*_n) - S'_n$.

Thus S'_n is the total contribution to $H_n(\mathcal{X}^*_n)$ from points in $B^*(x_{5K,n,J(\mathcal{S},n,\ell)}; Kr_n)$, $1 \leq \ell \leq \min(\lfloor \delta n \rfloor, N_{\mathcal{S},n})$. On account of the shielding effect described above, S_n

is the sum of $\lfloor \delta n \rfloor$ independent copies of a random variable with the distribution of $H(\mathcal{U}^*_{\mu,K})$. Moreover, we assert that the distribution of S_n, given the value of Y_n, does not depend on the value of Y_n, and therefore S_n is independent of Y_n.

Essentially, this assertion holds because for any triple of sub-σ-algebras $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, if $\mathcal{F}_1 \vee \mathcal{F}_2$ is independent of $\mathcal{F}_3$ and $\mathcal{F}_1$ is independent of $\mathcal{F}_2$ then $\mathcal{F}_1$ is independent of $\mathcal{F}_2 \vee \mathcal{F}_3$ (here $\mathcal{F}_i \vee \mathcal{F}_j$ is the smallest σ-algebra containing both $\mathcal{F}_i$ and $\mathcal{F}_j$). In the present instance, to define these σ-algebras we first define the marked point processes $\mathcal{Y}_j$ for $1 \le j \le \lfloor \delta n \rfloor$ by

$$\mathcal{Y}_j := \begin{cases} r_n^{-1}(-x_{5K,n,J(\mathcal{S},n,j)} + (\mathcal{X}^*_n \cap B^*_{5K,n,J(\mathcal{S},n,j)})) & \text{if } 1 \le j \le \min(\lfloor \delta n \rfloor, N_{\mathcal{S},n}) \\ \mathcal{W}^{(j-N_{\mathcal{S},n})}_{(\nu+1)\mu,5K} & \text{if } N_{\mathcal{S},n} < j \le \lfloor \delta n \rfloor. \end{cases}$$

Take $\mathcal{F}_3$ to be the σ-algebra generated by the values of $J(\mathcal{S}, n, 1), \dots, J(\mathcal{S}, n, \min(\lfloor \delta n \rfloor, N_{\mathcal{S},n}))$ and the locations and marks of points of $\mathcal{X}_n$ outside the union of the balls $B_{5K,n,J(\mathcal{S},n,1)}, \dots, B_{5K,n,J(\mathcal{S},n,\min(\lfloor \delta n \rfloor, N_{\mathcal{S},n}))}$. Take $\mathcal{F}_2$ to be the σ-algebra generated by the point processes $\mathcal{Y}_j \cap B^*(5K) \setminus B^*(K), 1 \le j \le \lfloor \delta n \rfloor$. Take $\mathcal{F}_1$ to be the σ-algebra generated by the point processes $\mathcal{Y}_j \cap B^*(K), 1 \le j \le \lfloor \delta n \rfloor$. Then by Lemma 10.3 and the definition of $\mathcal{S}$, $\mathcal{F}_1 \vee \mathcal{F}_2$ is independent of $\mathcal{F}_3$ and $\mathcal{F}_1$ is independent of $\mathcal{F}_2$, so $\mathcal{F}_1$ is independent of $\mathcal{F}_2 \vee \mathcal{F}_3$. The variable S_n is measurable with respect to $\mathcal{F}_1$, and by shielding, the variable Y_n is measurable with respect to $\mathcal{F}_2 \vee \mathcal{F}_3$, justifying our assertion of independence.

By the assumptions of the result being proved, $H_n(\mathcal{X}^*_n)$ and S_n are integrable. Clearly $n^{1/2} P[H_n(\mathcal{X}^*_n) \ne Y_n + S_n]$ is at most $n^{1/2} P[N_{\mathcal{S},n} < \delta n]$, which tends to zero. Also, as with (10.14) in Case (i), we have that $n^{-1/2} \mathbb{E}\left[|H_n(\mathcal{X}^*_n) - (Y_n + S_n)| \right]$ tends to zero by (10.12), and Y_n is integrable. By (5.5),

$$\lfloor \delta n \rfloor^{-1/2} (H_n(\mathcal{X}^*_n) - \mathbb{E}\, H_n(\mathcal{X}^*_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \delta^{-1}\sigma^2), \tag{10.17}$$

and so, since $h_{\mu,K} | b_1$, Theorem 2.1 is applicable with $Z_n = H_n(\mathcal{X}^*_n)$, yielding

$$\sup_{\{u \in \mathbb{R}\}} \left\{ \left| (\delta n)^{1/2} P[H_n(\mathcal{X}^*_n) \in [u, u + b_1)] - \delta^{1/2} \sigma^{-1} b_1 \phi\left(\frac{u - \mathbb{E}\, H_n(\mathcal{X}^*_n)}{(\delta n)^{1/2} \delta^{-1/2} \sigma} \right) \right| \right\} \to 0,$$

as $n \to \infty$. Multiplying through by $\delta^{-1/2}$ yields (5.6) for this case, when $b_1 = b$. If $b_1 \ne b$, we can complete the proof in the same manner as in the proof for Case (i). □

11 Proof of Theorems 5.2, 5.3 and 5.4

The proofs of Theorems 5.2, 5.3 and 5.4 all rely heavily on Theorem 2.3 of [20] so for convenience we state that result here in the form we shall use it. This requires some further notation, besides the notation we set up earlier in Section 5.

As before, we assume $\xi(\mathbf{x}, \mathcal{X}^*)$ is a translation invariant, measurable $\mathbb{R}$-valued function defined for all pairs $(\mathbf{x}, \mathcal{X}^*)$, where $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ is finite and $\mathbf{x}$ is an element of $\mathcal{X}^*$. We extend the definition of $\xi(\mathbf{x}, \mathcal{X}^*)$ to the case where $\mathcal{X}^* \subset \mathbb{R}^d \times \mathcal{M}$ and $\mathbf{x} \in (\mathbb{R}^d \times \mathcal{M}) \setminus \mathcal{X}^*$, by setting $\xi(\mathbf{x}, \mathcal{X}^*)$ to be $\xi(\mathbf{x}, \mathcal{X}^* \cup \{\mathbf{x}\})$ in this case. Recall that $H^{(\xi)}$ is defined by (5.3).

Let T be an $\mathcal{M}$-valued random variable with distribution $\mathbb{P}_{\mathcal{M}}$, independent of everything else. For $\lambda > 0$ let M_λ be a Poisson variable with parameter λ, independent of everything else, and let $\mathcal{P}_\lambda$ be the point process $\{X_1, \ldots, X_{M_\lambda}\}$, which is a Poisson point process with intensity $\lambda f(\cdot)$. Let $\mathcal{P}^*_\lambda := \{(X_1, T_1), \ldots, (X_{M_\lambda}, T_{M_\lambda})\}$ be the corresponding marked Poisson process.

Given $\lambda > 0$, we say ξ is λ-homoegeneously stabilizing if there is an almost surely finite positive random variable R such that with probability 1,

$$\xi((0,T); (\mathcal{H}^*_\lambda \cap B^*(0;R)) \cup \mathcal{Y}) = \xi((0,T); \mathcal{H}^*_\lambda \cap B^*(0;R))$$

for all finite $\mathcal{Y} \subset (\mathbb{R}^d \setminus B(0;R)) \times \mathcal{M}$. Recall that $\mathrm{supp}(f)$ denotes the support of f. We say that ξ is *exponentially stabilizing* if for $\lambda \geq 1$ and $x \in \mathrm{supp}(f)$ there exists a random variable $R_{x,\lambda}$ such that

$$\begin{aligned}&\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{P}^*_\lambda \cap B^*(x; \lambda^{-1/d}R_{x,\lambda})) \cup \mathcal{Y})\\ &\quad = \xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{P}^*_\lambda \cap B^*(x; \lambda^{-1/d}R_{x,\lambda})))\end{aligned}$$

for all finite $\mathcal{Y} \subset (\mathbb{R}^d \setminus B(x; \lambda^{-1/d}R_{x,\lambda}))) \times \mathcal{M}$, and there exists a finite positive constant C such that

$$P[R_{x,\lambda} > s] \leq C \exp(-C^{-1}s), \quad s \geq 1,\ \lambda \geq 1,\ f \in \mathrm{supp}(f).$$

For $k \in \mathbb{N} \cup \{0\}$, let $\mathcal{T}_k$ be the collection of all subsets of $\mathrm{supp}(f)$ with at most k elements. For $k \geq 1$ and $\mathcal{A} = \{x_1, \ldots, x_k\} \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$, let $\mathcal{A}^*$ be the corresponding marked point set $\{(x_1, T_1), \ldots, (x_k, T_k)\}$ where $T_1, \ldots, T_k$ are independent $\mathcal{M}$-valued variables with distribution $\mathbb{P}_{\mathcal{M}}$, independent of everything else. If $\mathcal{A} \in \mathcal{T}_0$ (so $\mathcal{A} = \emptyset$) let $\mathcal{A}^*$ also be the empty set.

We say that ξ is *binomially exponentially stabilizing* if there exist finite positive constants C, ε such that for all $x \in \mathrm{supp}(f)$ and all $\lambda \geq 1$ and $n \in \mathbb{N} \cap ((1-\varepsilon)\lambda, (1+\varepsilon)\lambda))$, and $\mathcal{A} \in \mathcal{T}_2$, there is a random variable $R_{x,\lambda,n,\mathcal{A}}$ such that

$$\begin{aligned}&\xi((\lambda^{1/d}x, T); \lambda^{1/d}((\mathcal{X}^*_n \cup \mathcal{A}^*) \cap B^*(x; \lambda^{-1/d}R_{x,\lambda,n,\mathcal{A}})) \cup \mathcal{Y})\\ &\quad = \xi((\lambda^{1/d}x, T); \lambda^{1/d}((\mathcal{X}^*_n \cup \mathcal{A}^*) \cap B^*(x; \lambda^{-1/d}R_{x,\lambda,n,\mathcal{A}})))\end{aligned} \tag{11.1}$$

for all finite $\mathcal{Y} \subset (\mathbb{R}^d \setminus B(x; \lambda^{-1/d}R_{x,\lambda,n,\mathcal{A}})) \times \mathcal{M}$, and such that all $\lambda \geq 1$ and all $n \in \mathbb{N} \cap ((1-\varepsilon)\lambda, (1+\varepsilon)\lambda))$, and all $x \in \mathrm{supp}(f)$ and all $\mathcal{A} \in \mathcal{T}_2$,

$$P[R_{x,\lambda,n,\mathcal{A}} > s] \leq C \exp(-C^{-1}s), \quad s \geq 1.$$

Given $p > 0$ and $\varepsilon > 0$, we consider the moments conditions

$$\sup_{\lambda \geq 1, x \in \mathrm{supp}(f), \mathcal{A} \in \mathcal{S}_1} \mathbb{E}\left[|\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{P}_\lambda^* \cup \mathcal{A}^*))|^p\right] < \infty \tag{11.2}$$

and

$$\sup_{\lambda \geq 1, n \in \mathbb{N} \cap ((1-\varepsilon)\lambda, (1+\varepsilon)\lambda), x \in \mathrm{supp}(f), \mathcal{A} \in \mathcal{S}_3} \mathbb{E}\left[|\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{X}_n^* \cup \mathcal{A}^*))|^p\right] < \infty. \tag{11.3}$$

Theorem 11.1. *Suppose $H = H^{(\xi)}$ is induced by translation-invariant ξ. Suppose that ξ is $f(x)$-homogeneously stabilizing for Lebesgue-almost all $x \in \mathrm{supp}(f)$, and ξ is exponentially stabilizing, binomially exponentially stabilizing and for some $\varepsilon > 0$ and $p > 2$ satisfies* (11.2) *and* (11.3). *Suppose $f_{\max} < \infty$ and $\mathrm{supp}(f)$ is bounded. Suppose $(\lambda(n), n \geq 1)$ is a sequence taking values in $\mathbb{R}^+$ with $|\lambda(n) - n| = O(n^{1/2})$ as $n \to \infty$. Then there exists $\sigma \geq 0$ such that*

$$n^{-1/2}(H^{(\xi)}(\lambda(n)^{1/d}\mathcal{X}_n^*) - \mathbb{E}\, H^{(\xi)}(\lambda(n)^{1/d}\mathcal{X}_n^*)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2),$$

and $n^{-1}\mathrm{Var}(H^{(\xi)}(\lambda(n)^{1/d}\mathcal{X}_n^*) \to \sigma^2$ as $n \to \infty$.

Theorem 11.1 is a special case of Theorem 2.3 of [20], which also provides an expression for σ in terms of integrated two-point correlations; that paper considers random measures given by a sum of contributions from each point, whereas here we just consider the total measure. The sets Ω_∞ and (for all $\lambda \geq 1$) Ω_λ in [20] are taken to be $\mathrm{supp}(f)$. Our ξ is translation invariant, and these assumptions lead to some simplification of the notation in [20].

Proof of Theorem 5.2. The condition that $\xi(\mathbf{x}; \mathcal{X}^*)$ has finite range implies that $H = H^{(\xi)}$ has finite range interactions. Since ξ has finite range r, ξ is λ-homogeneously stabilizing for all $\lambda > 0$, exponentially stabilizing and binomially exponentially stabilizing (just take $R = r$, $R_{x,\lambda} = r$ and $R_{x,\lambda,n,\mathcal{A}} = r$).

We shall establish (5.5) by applying Theorem 11.1. We need to check the moments conditions (11.2) and (11.3) in the present setting. Since we assume that $f_{\max} < \infty$, for any $\lambda > 0$ and any $n \in \mathbb{N}$ with $n \leq 2\lambda$, and any $x \in \mathrm{supp}(f)$, the variable $\mathrm{card}(\mathcal{X}_n^* \cap B^*(x; r\lambda^{-1/d}))$ is binomially distributed with with mean at most $\omega_d f_{\max} 2r^d$. Hence by Lemma 8.1, there is a constant C, such that whenever $n \leq 2\lambda$ and $x \in \mathrm{supp}(f)$ we have

$$P[\mathrm{card}(\mathcal{X}_n^* \cap B^*(x; r\lambda^{-1/d})) > u] \leq C \exp(-u/C), \quad u \geq 1. \tag{11.4}$$

Moreover by (5.11) and the assumption that ξ has range r, for $\mathcal{A} \in \mathcal{T}_3$ we have

$$\mathbb{E}\left[\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{X}_n^* \cup \mathcal{A}^*))^4\right] \leq \gamma^4 \mathbb{E}\left[(4 + \mathrm{card}(\mathcal{X}_n^* \cap B^*(x; r\lambda^{-1/d})))^{4\gamma}\right]$$

so by (11.4) we can bound the fourth moments of $\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{X}_n^* \cup \mathcal{A}^*))$ uniformly over $(x, \lambda, n, \mathcal{A}) \in \mathrm{supp}(f) \times [1, \infty) \times \mathbb{N} \times \mathcal{T}_3$ with $n \leq 2\lambda$. This gives us (11.3) (for $p = 4$ and $\varepsilon = 1/2$) and (11.2) may be deduced similarly.

Hence, the assumptions of Theorem 11.1 are satisfied, with $\lambda(n)$ in that result given by $\lambda(n) = r_n^{-d}$. By Theorem 11.1, for some $\sigma \geq 0$ we have (5.5) and (5.9). Then by Theorem 5.1, we can deduce that $\sigma > 0$ and $h(H) < \infty$ and (5.6) holds whenever $h(H)|b$. □

Proof of Theorem 5.3. Under condition (5.2), the functional $H(\mathcal{X}^*)$ can be expressed as a sum of contributions from components of the geometric (Gilbert) graph $\mathcal{G}(\mathcal{X}, \tau)$, where $\mathcal{X} := \pi(\mathcal{X}^*)$ is the unmarked point set corresponding to $\mathcal{X}^*$ (recall that π denotes the canonical projection from $\mathbb{R}^d \times \mathcal{M}$ onto $\mathbb{R}^d$.) Hence, $H(\mathcal{X}^*)$ can be written as $H^{(\xi)}(\mathcal{X}^*)$ where $\xi(\mathbf{x}; \mathcal{X}^*)$ denotes the contribution to $H(\mathcal{X}^*)$ from the component containing $\pi(\mathbf{x})$, divided by the number of vertices in that component. Then $\xi(\mathbf{x}; \mathcal{X}^*)$ is unaffected by changes to $\mathcal{X}^*$ that do not affect the component of $\mathcal{G}(\mathcal{X}, \tau)$ containing $\pi(\mathbf{x})$, and we shall use this to demonstrate that the conditions of Theorem 11.1 hold, as follows (the argument is similar to that in Section 11.1 of [18]).

Consider first the homogeneous stabilization condition. For $\lambda > 0$, let $R(\lambda)$ be the maximum Euclidean distance from the origin of vertices in the graph $\mathcal{G}(\mathcal{H}_\lambda \cup \{0\}, \tau)$ that are pathwise connected to the origin, which by scaling (see the Mapping theorem in [15]) has the same distribution as τ times the maximum Euclidean distance from the origin of vertices in $\mathcal{G}(\mathcal{H}_{\tau^d\lambda} \cup \{0\}, 1)$, that are pathwise connected to the origin. Then $R(\lambda)$ is almost surely finite, for any $\lambda \in (0, \tau^{-d}\lambda_c)$.

Changes to $\mathcal{H}_\lambda$ at a distance more than $R(\lambda) + \tau$ from the origin do not affect the component of $\mathcal{G}(\mathcal{H}_\lambda \cup \{0\}, \tau)$ containing the origin and therefore do not affect $\xi((0, T); \mathcal{H}_\lambda^*)$. This shows that ξ is λ-homogeneously stabilizating for any $\lambda < \tau^{-d}\lambda_c$, and therefore by assumption (5.12) the homogeneous stabilization condition of Theorem 11.1 holds.

Next we consider the binomial stabilization condition. Let $x \in \mathrm{supp}(f)$. Let $R_{x,\lambda,n}$ be equal to τ plus the maximum Euclidean distance from $\lambda^{1/d}x$ of vertices in $\mathcal{G}(\lambda^{1/d}(\mathcal{X}_n \cup \{x\}), \tau)$ that are pathwise connected to $\lambda^{1/d}x$. Changes to $\mathcal{X}_n$ at a Euclidean distance greater than $\lambda^{-1/d}R_{x,\lambda,n}$ from x will have no effect on $\xi((\lambda^{1/d}x, T); \lambda^{1/d}\mathcal{X}_n^*)$.

Using (5.12), let $\varepsilon \in (0, 1/2)$ with $(1 + \varepsilon)^2\tau^d f_{\max} < \lambda_c$. The Poisson point process $\mathcal{P}_{n(1+\varepsilon)} := \{X_1, \ldots, X_{M_{n(1+\varepsilon)}}\}$, is stochastically dominated by $\mathcal{H}_{nf_{\max}(1+\varepsilon)}$ (we say a point process $\mathcal{X}$ is stochastically dominated by a point process $\mathcal{Y}$ if there exist coupled point processes $\mathcal{X}', \mathcal{Y}'$ with $\mathcal{X}' \subset \mathcal{Y}'$ almost surely and $\mathcal{X}'$ having the distribution of $\mathcal{X}$ and $\mathcal{Y}'$ having the distribution of $\mathcal{Y}$). Hence by scaling, $\lambda^{1/d}\mathcal{P}_{n(1+\varepsilon)}$ is stochastically dominated by $\mathcal{H}_{nf_{\max}(1+\varepsilon)/\lambda}$, and hence we have for $n \leq \lambda(1 + \varepsilon)$

that $\lambda^{1/d}\mathcal{P}_{n(1+\varepsilon)}$ is stochastically dominated by $\mathcal{H}_{f_{\max}(1+\varepsilon)^2}$. Therefore for $u > 0$,

$$P[R_{x,\lambda,n} > u] \leq P[M_{n(1+\varepsilon)} < n] + P[R((1+\varepsilon)^2 f_{\max}) > u - \tau]. \tag{11.5}$$

By scaling, the second probability in (11.5) equals the probability that there is a path from the origin in $\mathcal{G}(\mathcal{H}_{\tau^d(1+\varepsilon)^2 f_{\max}} \cup \{0\}, 1)$ to a point at Euclidean distance greater than $\tau^{-1}u - 1$ from the origin. By the exponential decay for subcritical continuum percolation, (see e.g. Lemma 10.2 of [18]), this probability decays exponentially in u (and does not depend on n).

Let $\Delta := \mathrm{diam}(\mathrm{supp}(f))$ (here assumed finite). By Lemma 8.1, the first term in the right hand side of (11.5) decays exponentially in n. Hence, there is a finite positive constant C, independent of λ, such that provided we have $n > (1-\varepsilon)\lambda^{1/d}$ we have for all $u \leq \lambda^{1/d}(\Delta + \tau)$ that

$$P[M_{n(1+\varepsilon)} < n] \leq C \exp(-C^{-1}\lambda^{1/d}) \leq C \exp(-((\Delta+\tau)C)^{-1}u).$$

On the other hand $P[R_{x,\lambda,n} > u] = 0$ for $u > \lambda^{1/d}(\Delta+\tau)$. Combined with (11.5) this shows that there is a constant C such that for all $(x, n, \lambda, u) \in \mathrm{supp}(f) \times \mathbb{N} \times [1, , \infty)^2$ with $n \leq (1+\varepsilon)\lambda$, we have

$$P[R_{x,\lambda,n} > u] \leq C \exp(-u/C). \tag{11.6}$$

Now suppose $\mathcal{A} \in \mathcal{T}_3$, and $x \in \mathrm{supp}(f)$. Let $R_{x,\lambda,n,\mathcal{A}}$ be equal to τ plus the maximum Euclidean distance from $\lambda^{1/d}x$ of vertices in $\mathcal{G}(\lambda^{1/d}(\mathcal{X}_n \cup \mathcal{A} \cup \{x\}); \tau)$ that are pathwise connected to $\lambda^{1/d}x$. Changes to $\mathcal{X}_n \cup \mathcal{A}$ at a Euclidean distance greater than $\lambda^{-1/d}R_{x,\lambda,n,\mathcal{A}}$ from x will have no effect on $\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{X}_n^* \cup \mathcal{A}^*))$; that is, (11.1) holds. To check the tail behaviour of $R_{x,\lambda,n,\mathcal{A}}$, suppose for example that $\mathcal{A}$ has three elements, x_1, x_2 and x_3. Then it is not hard to see that

$$R_{x,\lambda,n,\mathcal{A}} \leq R_{x,\lambda,n} + R_{x_1,\lambda,n} + R_{x_2,\lambda,n} + R_{x_3,\lambda,n},$$

and likewise when $\mathcal{A}$ has fewer than three elements. Using this together with (11.6), it is easy to deduce that there is a constant C such that for all $(x, n, \mathcal{A}, \lambda, u) \in \mathrm{supp}(f) \times \mathbb{N} \times \mathcal{T}_3 \times [1, \infty)^2$ with $n \leq (1+\varepsilon)\lambda$, and we have

$$P[R_{x,\lambda,n,\mathcal{A}} > u] \leq C \exp(-u/C). \tag{11.7}$$

In other words, ξ is binomially exponentially stabilizing.

Next we check the moments condition (11.3), with $p = 4$ and using the same choice of ε as before. By our definition of ξ and the growth bound (5.13), we have for all $(x, n, \mathcal{A}, \lambda) \in \mathrm{supp}(f) \times \mathbb{N} \times \mathcal{T}_3 \times [1, \infty)^2$ with $n \leq \lambda(1+\varepsilon)$ that

$$\mathbb{E}\left[\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{X}_n^* \cup \mathcal{A}^*))^4\right] \leq \gamma^4 \mathbb{E}\left[(\mathrm{card}(\mathcal{C}) + \mathrm{diam}(\mathcal{C}))^{4\gamma}\right], \tag{11.8}$$

where $\mathcal{C}$ is the vertex set of the component of $\mathcal{G}(\lambda^{1/d}(\mathcal{X}_n \cup \mathcal{A} \cup \{x\}); \tau)$ containing $\lambda^{1/d}x$. By (11.7), there is a constant C such that for all $(x, n, \mathcal{A}, \lambda, u) \in (\mathrm{supp}(f) \times \mathbb{N} \times \mathcal{T}_3 \times [1, \infty)^2$ with $n \leq \lambda(1 + \varepsilon)$ we have

$$P[\mathrm{diam}(\mathcal{C}) > u] \leq C \exp(-u/C); \tag{11.9}$$

moreover,

$$P[\mathrm{card}(\mathcal{C}) > u] \leq P[\mathrm{diam}(\mathcal{C}) > u^{1/(2d)}] + P[\mathrm{card}(\mathcal{X}_n \cap B(x; \lambda^{-1/d}u^{1/(2d)})) > u - 4] \tag{11.10}$$

and the first term in the right hand side of (11.10) decays exponentially in $u^{1/(2d)}$ by (11.9). Since $\mathrm{card}(\mathcal{X}_n \cap B(x; \lambda^{-1/d}u^{1/(2d)}))$ is binomially distributed with

$$\mathbb{E}\left[\mathrm{card}(\mathcal{X}_n \cap B(x; \lambda^{-1/d}u^{1/(2d)}))\right] \leq u^{1/2}\omega_d f_{\max} n/\lambda,$$

by Lemma 8.1 there is a constant C such that for all (x, n, λ, u) with $n \leq \lambda(1 + \varepsilon)$ we have that

$$P[\mathrm{card}(\mathcal{X}_n \cap B(x; \lambda^{-1/d}u^{1/(2d)})) > u - 4] \leq C \exp(-C^{-1}u^{1/2}).$$

Thus by (11.10) there is a constant, also denoted C, such that for all $(x, n, \mathcal{A}, \lambda, u)$ with $n \leq \lambda(1 + \varepsilon)$ we have

$$P[\mathrm{card}(\mathcal{C}) > u] \leq C \exp(-C^{-1}u^{-1/(2d)}),$$

and combining this with (11.9) and using (11.8) gives us a uniform tail bound which is enough to ensure (11.3). The argument for (11.2) is similar.

Thus our ξ satisfies all the assumptions of Theorem 11.1, and we can deduce (5.5) and (5.9) for some $\sigma \geq 0$ by applying that result with $\lambda(n) = r_n^{-d}$. Then by applying Theorem 5.1, we can deduce that $\sigma > 0$ and $h(H) < \infty$ and (5.6) holds whenever $h(H)|b$. □

Proof of Theorem 5.4. Suppose the hypotheses of Theorem 5.4 hold, and assume without loss of generality that $\xi(\mathbf{x}, \mathcal{X}^*) = 0$ whenever $\mathcal{X}^* \setminus \{\mathbf{x}\}$ has fewer than κ elements. We assert that under these hypotheses, there exists a constant C such that for all $(x, n, \lambda, u) \in \mathrm{supp}(f) \times \mathbb{N} \times [1, \infty)^2$ with $n \in [\lambda/2, 3\lambda/2]$ and $n \geq \kappa$, we have

$$P[\lambda^{1/d}R_\kappa((x, T); \mathcal{X}_n^*) > u] \leq C \exp(-C^{-1}u). \tag{11.11}$$

Indeed, if $\mathrm{supp}(f)$ is a compact convex region in $\mathbb{R}^d$ and f is bounded away from zero on $\mathrm{supp}(f)$, then (11.11) is demonstrated in Section 6.3 of [20], while if $\mathrm{supp}(f)$

is a compact d-dimensional submanifold-with-boundary of $\mathbb{R}^d$, and f is bounded away from zero on $\mathrm{supp}(f)$, then (11.11) comes from the proof of Lemma 6.1 of [24].

It is easy to see that ξ is λ-homogeneously stabilizing for all $\lambda > 0$. Also, for any $(x, \mathcal{A}) \in (\mathrm{supp}(f) \times \mathcal{T}_3)$ we obviously have $R_\kappa((x,T); \mathcal{X}^* \cup \mathcal{A}^*) \leq R_\kappa((x,T); \mathcal{X}^*)$ and hence by (11.11), ξ is binomially exponentially stabilizing, and exponential stabilization comes from a similar estimate with a Poisson sample.

We need to check the moments conditions to be able to deduce (5.5) via Theorem 11.1. With γ as in the growth bound (5.14), we claim that there is a constant C such that for any $\mathcal{A} \in \mathcal{T}_3$, any $x \in \mathrm{supp}(f)$, and any $u > 0$, and for all $(x, n, \mathcal{A}, \lambda, u) \in \mathrm{supp}(f) \times \mathbb{N} \times \mathcal{T}_3 \times [1, \infty)^2$ with $\lambda/2 \leq n \leq 3\lambda/2$, and $n \geq \kappa$, we have

$$P[|\xi((\lambda^{1/d}x, T); \lambda^{1/d}(\mathcal{X}_n^* \cup \mathcal{A}^*))| > u] \leq P[\gamma(1 + \lambda^{1/d} R_\kappa((x,T), \mathcal{X}_n^*))^\gamma > u] \leq C \exp(-C^{-1} u^{1/\gamma}). \quad (11.12)$$

Indeed, the first bound comes from the (5.14), and the second bound comes from (11.11). Using (11.12), we can deduce the moments bound (11.3) for $p = 4$ and $\varepsilon = 1/2$. We can derive (11.2) similarly. Thus Theorem 11.1 is applicable, and enables us to deduce (5.5) and (5.9) for some $\sigma \geq 0$, in the present setting. Then by using Theorem 5.1, we can deduce that $\sigma > 0$ and $h(H) > 0$ and (5.6) holds whenever $h(H)|b$. □

Acknowledgments. We thank the Oberwolfach Mathematical Research Instiute for hosting the 2008 workshop 'New Perspectives in Stochastic Geometry', at which this work was started. We also thank Antal Járai for helpful discussions.

References

[1] Avram, F. and Bertsimas, D. (1993). On central limit theorems in geometrical probability. *Ann. Appl. Probab.* **3**, 1033-1046.

[2] Baryshnikov, Yu., Penrose, M. D. and Yukich, J. E. (2009). Gaussian limits for generalized spacings. *Ann. Appl. Probab.* **19**, 158–185.

[3] Baryshnikov, Yu. and Yukich, J. E. (2005). Gaussian limits for random measures in geometric probability. *Ann. Appl. Probab.* **15**, 213–253.

[4] Bender, E. A. (1973). Central and local limit theorems applied to asymptotic enumeration. *J. Combinatorial Theory* **A 15**, 91–111.

[5] Bickel, P. J. and Breiman, L. (1983). Sums of functions of nearest neighbor distances, moment bounds, limit theorems and a goodness of fit test. *Ann. Probab.* **11**, 185–214.

[6] Breiman, L. (1992). *Probability.* SIAM, Philadelphia.

[7] Chatterjee, S. (2008). A new method of normal approximation, *Ann. Probab.* **36**, 1584-1610.

[8] Davis, B. and McDonald, D. (1995). An elementary proof of the local central limit theorem. *J. Theoret. Probab.* **8**, 693–701.

[9] Durrett, R. (1996). *Probability: Theory and Examples.* 2nd Edition, Wadsworth, Belmont, CA.

[10] Evans, D., and Jones, A. J. (2002), A proof of the gamma test. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **458**, 2759–2799.

[11] Feller, W. (1966). *An Introduction to Probability Theory and its Applications. Vol. II.* John Wiley & Sons, New York.

[12] Grimmett, G. and Marstrand, J.M. (1990) The supercritical phase of percolation is well behaved. *Proc. Royal Soc. London A* **430**, 439-457.

[13] Heinrich, L. and Molchanov, I.S. (1999). Central limit theorem for a class of random measures associated with germ-grain models. *Adv. Appl. Probab.* **31**, 283–314.

[14] Henze, N. (1988). A multivariate two-sample test based on the number of nearest neighbor type coincidences. *Ann. Statist.* **16**, 772–783.

[15] Kingman, J. F. C. (1993). *Poisson Processes.* Oxford University Press, Oxford.

[16] Levina, E. and Bickel, P. J. (2005), Maximum likelihood estimation of intrinsic dimension, in *Advances in NIPS*, **17**, Eds. L. K. Saul, Y. Weiss, L. Bottou.

[17] Leonenko, N., Pronzato, L. and Savani, V. (2008). A class of Rényi information estimators for multidimensional densities, *Ann. Statist.* **36**, 2153–2182.

[18] Penrose, M. (2003). *Random Geometric Graphs.* Oxford University Press, Oxford.

[19] Penrose, M. D. (2001). A central limit theorem with applications to percolation, epidemics and Boolean models. *Ann. Probab.* **29**, 1515–1546.

[20] Penrose, M. D. (2007). Gaussian limits for random geometric measures. *Electron. J. Probab.* **12**, 989–1035.

[21] Penrose, M.D. and Yukich, J.E. (2001). Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11**, 1005-1041.

[22] Penrose, M. D. and Yukich, J. E. (2002). Limit theory for random sequential packing and deposition. *Ann. Appl. Probab.* **12**, 272–301.

[23] Penrose, M. D. and Yukich, J. E. (2003). Weak laws of large numbers in geometric probability. *Ann. Appl. Probab.* **13**, 277–303.

[24] Penrose, M. D. and Yukich, J. E. (2011). Limit theory for point processes in manifolds. Preprint, ArXiv:1104.0914

[25] Schreiber, T., Penrose, M. D., and Yukich, J. E. (2007) Gaussian limits for multidimensional random sequential packing at saturation. *Comm. Math. Phys.* **272**, 167–183.

[26] Vakhania, N. N. (1993). Elementary Proof of Polya's Characterization Theorem and of the Necessity of Second Moment in the CLT. *Theory Probab. Appl.* **38**, 166–168.