UNIFORM MIXING TIME FOR RANDOM WALK ON LAMPLIGHTER GRAPHS

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Suppose that $\mathcal{G}$ is a finite, connected graph and X is a lazy random walk on $\mathcal{G}$. The lamplighter chain $X^\diamond$ associated with X is the random walk on the wreath product $\mathcal{G}^\diamond = \mathbf{Z}_2 \wr \mathcal{G}$, the graph whose vertices consist of pairs (f, x) where f is a labeling of the vertices of $\mathcal{G}$ by elements of $\mathbf{Z}_2$ and x is a vertex in $\mathcal{G}$. There is an edge between (f, x) and (g, y) in $\mathcal{G}^\diamond$ if and only if x is adjacent to y in $\mathcal{G}$ and $f(z) = g(z)$ for all $z \neq x, y$. In each step, $X^\diamond$ moves from a configuration (f, x) by updating x to y using the transition rule of X and then sampling both $f(x)$ and $f(y)$ according to the uniform distribution on $\mathbf{Z}_2$; $f(z)$ for $z \neq x, y$ remains unchanged. We give matching upper and lower bounds on the uniform mixing time of $X^\diamond$ provided $\mathcal{G}$ satisfies mild hypotheses. In particular, when $\mathcal{G}$ is the hypercube $\mathbf{Z}_2^d$, we show that the uniform mixing time of $X^\diamond$ is $\Theta(d2^d)$. More generally, we show that when $\mathcal{G}$ is a torus $\mathbf{Z}_n^d$ for $d \geq 3$, the uniform mixing time of $X^\diamond$ is $\Theta(dn^d)$ uniformly in n and d. A critical ingredient for our proof is a concentration estimate for the local time of random walk in a subset of vertices.

1. Introduction. Suppose that $\mathcal{G}$ is a finite graph with vertices $V(\mathcal{G})$ and edges $E(\mathcal{G})$, respectively. Let $\mathcal{X}(\mathcal{G}) = \{f \colon V(\mathcal{G}) \to \mathbf{Z}_2\}$ be the set of markings of $V(\mathcal{G})$ by elements of $\mathbf{Z}_2$. The wreath product $\mathcal{G}^\diamond = \mathbf{Z}_2 \wr \mathcal{G}$ is the graph whose vertices are pairs (f, x) where $f \in \mathcal{X}(\mathcal{G})$ and $x \in V(\mathcal{G})$. There is an edge between (f, x) and (g, y) if and only if $\{x, y\} \in E(\mathcal{G})$ and $f(z) = g(z)$ for all $z \notin \{x, y\}$. Suppose that P is a transition matrix for a Markov chain on $\mathcal{G}$. The lamplighter walk $X^\diamond$ (with respect to the transition matrix P) is the Markov chain on $\mathcal{G}^\diamond$ which moves from a configuration (f, x) by

1. picking y adjacent to x in $\mathcal{G}$ according to P, then

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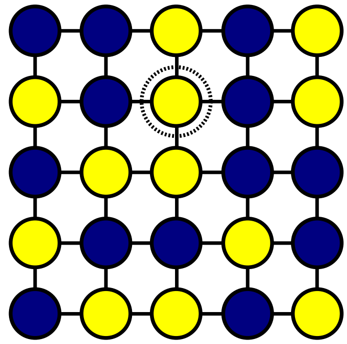

FIG 1. *A typical configuration of the lamplighter over a 5×5 planar grid. The colors indicate the state of the lamps and the dashed circle gives the position of the lamplighter.*

2. updating each of the values of $f(x)$ and $f(y)$ independently according to the uniform measure on $\mathbf{Z}_2$.

The lamp states at all other vertices in $\mathcal{G}$ remain fixed. It is easy to see that if P is ergodic and reversible with stationary distribution π_P then the unique stationary distribution of $X^\diamond$ is the product measure

$$\pi\big((f,x)\big) = \pi_P(x) 2^{-|\mathcal{G}|},$$

and $X^\diamond$ is itself reversible. In this article, we will be concerned with the special case that P is the transition matrix for the *lazy random walk* on $\mathcal{G}$ in order to avoid issues of periodicity. That is, P is given by

$$P(x,y) = \begin{cases} \frac{1}{2} \text{ if } x = y, \\ \frac{1}{2d(x)} \text{ if } \{x,y\} \in E(\mathcal{G}), \end{cases} \tag{1.1}$$

for $x, y \in V(\mathcal{G})$ and where $d(x)$ is the degree of x.

1.1. *Main Results.* Let P be the transition kernel for lazy random walk on a finite, connected graph $\mathcal{G}$ with stationary distribution π. The ϵ-*uniform mixing time* of $\mathcal{G}$ is given by

$$t_{\mathrm{u}}(\epsilon, \mathcal{G}) = \min\left\{ t \geq 0 : \max_{x,y \in V(\mathcal{G})} \left| \frac{P^t(x,y) - \pi(y)}{\pi(y)} \right| \leq \epsilon \right\}. \tag{1.2}$$

Throughout, we let $t_{\mathrm{u}}(\mathcal{G}) = t_{\mathrm{u}}((2e)^{-1}, \mathcal{G})$. The main result of this article is a general theorem which gives matching upper and lower bounds of $t_{\mathrm{u}}(\mathcal{G}^\diamond)$ provided $\mathcal{G}$ satisfies several mild hypotheses. One important special case of this result is the hypercube $\mathbf{Z}_2^d$ and, more generally, tori $\mathbf{Z}_n^d$ for $d \geq 3$. These examples are sufficiently important that we state them as our first theorem.

THEOREM 1.1. *There exists constants $C_1, C_2 > 0$ such that*

$$C_1 \le \frac{t_{\mathrm{u}}((\mathbf{Z}_2^d)^\diamond)}{d2^d} \le C_2 \text{ for all } d.$$

More generally,

$$C_1 \le \frac{t_{\mathrm{u}}((\mathbf{Z}_n^d)^\diamond)}{dn^{d+2}} \le C_2 \text{ for all } n \ge 2 \text{ and } d \ge 3.$$

Prior to this work, the best known bound [10] for $t_{\mathrm{u}}((\mathbf{Z}_2^d)^\diamond)$ was

$$C_1 d2^d \le t_{\mathrm{u}}((\mathbf{Z}_2^d)^\diamond) \le C_2 \log(d) d2^d$$

for $C_1, C_2 > 0$.

In order to state our general result, we first need to review some basic terminology from the theory of Markov chains. The *relaxation time* of P is

$$t_{\mathrm{rel}}(\mathcal{G}) = \frac{1}{1-\lambda_0} \tag{1.3}$$

where λ_0 is the second largest eigenvalue of P. The *maximal hitting time* of P is

$$t_{\mathrm{hit}}(\mathcal{G}) = \max_{x,y \in V(\mathcal{G})} \mathbf{E}_x[\tau_y], \tag{1.4}$$

where τ_y denotes the first time t that $X(t) = y$ and $\mathbf{E}_x$ stands for the expectation under the law in which $X(0) = x$. The Green's function $G(x, y)$ for P is

$$G(x,y) = \mathbf{E}_x\left[\sum_{t=0}^{t_{\mathrm{u}}(\mathcal{G})} \mathbf{1}_{\{X(t)=y\}}\right] = \sum_{t=0}^{t_{\mathrm{u}}(\mathcal{G})} P^t(x,y), \tag{1.5}$$

i.e. the expected amount of time X spends at y up to time t_{u} given $X(0) = x$. For each $1 \le n \le |\mathcal{G}|$, we let

$$G^*(n) = \max_{\substack{S \subseteq V(\mathcal{G}) \\ |S|=n}} \max_{z \in S} \sum_{y \in S} G(z,y). \tag{1.6}$$

This is the maximal expected time X spends in a set $S \subseteq V(\mathcal{G})$ of size n before the uniform mixing time. This quantity is related to the hitting time of subsets of $V(\mathcal{G})$. Finally, recall that $\mathcal{G}$ is said to be vertex transitive if for every $x, y \in V(\mathcal{G})$ there exists an automorphism φ of $\mathcal{G}$ with $\varphi(x) = y$. Our main result requires the following hypothesis.

ASSUMPTION 1.2. *$\mathcal{G}$ is a finite, connected, vertex transitive graph and X is a lazy random walk on $\mathcal{G}$. There exists constants $K_1, K_2, K_3 > 0$ such that*

(A) $t_{\mathrm{hit}}(\mathcal{G}) \leq K_1 |\mathcal{G}|$,
(B) $2K_2 (5/2)^{K_2} (G(x,y))^{K_2} \leq \exp\left(-\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right)$,
(C) $G^*(n^*) \leq K_3 (t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|)/(\log n^*)$

where $n^* = 4K_2 t_{\mathrm{u}}(\mathcal{G})/G(x,y)$ *for* $x, y \in V(\mathcal{G})$ *adjacent.*

The general theorem is:

THEOREM 1.3. *Let $\mathcal{G}$ be any graph satisfying Assumption 1.2. There exists constants C_1, C_2 depending only on K_1, K_2, K_3 such that*

$$C_1 \leq \frac{t_{\mathrm{u}}(\mathcal{G}^{\diamond})}{|\mathcal{G}|\big(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|\big)} \leq C_2 \tag{1.7}$$

The lower bound is proved in [10, Theorem 1.4]. The proof of the upper bound is based on the observation from [10] that the uniform distance to stationarity can be related to $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ where $\mathcal{U}(t)$ is the set of vertices in $\mathcal{G}$ which have not been visited by X by time t. Indeed, suppose that f is any initial configuration of lamps, let $f(t)$ be the state of the lamps at time t, and let g be an arbitrary lamp configuration. Let W be the set of vertices where $f \neq g$. Let $\mathcal{C}(t) = V(\mathcal{G}) \setminus \mathcal{U}(t)$ be the set of vertices which have been visited by X by time t. With $\mathbf{P}_{(f,x)}$ the probability under which $X^{\diamond}(0) = (f, x)$, we have that

$$\mathbf{P}_{(f,x)}[f(t) = g | \mathcal{C}(t)] = 2^{-|\mathcal{C}(t)|} \mathbf{1}_{\{W \subseteq \mathcal{C}(t)\}}.$$

Since the probability of the configuration g under the uniform measure is $2^{-|\mathcal{G}|}$, we therefore have

$$\frac{\mathbf{P}_{(f,x)}[f(t) = g]}{2^{-|\mathcal{G}|}} = \mathbf{E}_{(f,x)}\big[2^{|\mathcal{U}(t)|} \mathbf{1}_{\{W \subseteq \mathcal{C}(t)\}}\big]. \tag{1.8}$$

The right hand side is clearly bounded from above by $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ (the initial lamp configuration and position of the lamplighter no longer matters). On the other hand, we can bound (1.8) from below by

$$\mathbf{P}_{(f,x)}[W \subseteq \mathcal{C}(t)] \geq \mathbf{P}[|\mathcal{U}(t)| = 0] \geq 1 - (\mathbf{E}\big[2^{|\mathcal{U}(t)|}\big] - 1).$$

Consequently, to bound $t_{\mathrm{u}}(\epsilon, \mathcal{G}^{\diamond})$ it suffices to compute

$$\min\{t \geq 0 : \mathbf{E}[2^{|\mathcal{U}(t)|}] \leq 1 + \epsilon\} \tag{1.9}$$

since the amount of time it requires for X to subsequently uniformly mix after this time is negligible.

In order to establish (1.9), we will need to perform a rather careful analysis of the process by which $\mathcal{U}(t)$ is decimated by X. The key idea is to break the process of coverage into two different regimes, depending on the size of $\mathcal{U}(t)$. The main ingredient to handle the case when $\mathcal{U}(t)$ is large is the following concentration estimate of the local time

$$\mathcal{L}_S(t) = \sum_{s=0}^{t} \mathbf{1}_{\{X(s)\in S\}}$$

for X in $S \subseteq V(\mathcal{G})$.

PROPOSITION 1.4. *Let λ_0 be the second largest eigenvalue of P. Assume $\lambda_0 \geq \frac{1}{2}$ and fix $S \subseteq V(\mathcal{G})$. For $C_0 = 1/50$, we have that*

$$\mathbf{P}_\pi\left[\mathcal{L}_S(t) \leq t\frac{\pi(S)}{2}\right] \leq \exp\left(-C_0 t\frac{\pi(S)}{t_{\mathrm{rel}}(\mathcal{G})}\right). \tag{1.10}$$

Proposition 1.4 is a corollary of [7, Theorem 1]; we consider this sufficiently important that we state it here. By invoking Green's function estimates, we are then able to show that the local time is not concentrated on a small subset of S. The case when $\mathcal{U}(t)$ is small is handled via an estimate (Lemma 3.5) of the hitting time $\tau_S = \min\{t \geq 0 : X(t) \in S\}$ of S.

1.2. *Previous Work.* Suppose that μ, ν are probability measures on a finite measure space. Recall that the *total variation distance* between μ, ν is given by

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_A |\mu(A) - \nu(A)| = \frac{1}{2}\sum_x |\mu(x) - \nu(x)|. \tag{1.11}$$

The ϵ-*total variation mixing time* of P is

$$t_{\mathrm{mix}}(\epsilon, \mathcal{G}) = \min\left\{t \geq 0 : \max_{x\in V(\mathcal{G})} \|P^t(x,\cdot) - \pi\|_{\mathrm{TV}} \leq \epsilon\right\}. \tag{1.12}$$

Let $t_{\mathrm{mix}}(\mathcal{G}) = t_{\mathrm{mix}}((2e)^{-1}, \mathcal{G})$. It was proved [10, Theorem 1.4] by Peres and Revelle that if $\mathcal{G}$ is a regular graph such that $t_{\mathrm{hit}}(\mathcal{G}) \leq K|\mathcal{G}|$, there exists constants C_1, C_2 depending only on K such that

$$C_1|\mathcal{G}|(t_{\mathrm{rel}}(\mathcal{G}) + \log|\mathcal{G}|) \leq t_{\mathrm{u}}(\mathcal{G}^\diamond) \leq C_2|\mathcal{G}|(t_{\mathrm{mix}}(\mathcal{G}) + \log|\mathcal{G}|).$$

These bounds fail to match in general. For example, for the hypercube $\mathbf{Z}_2^d$, $t_{\rm rel}(\mathbf{Z}_2^d) = \Theta(d)$ [8, Example 12.15] while $t_{\rm mix}(\mathbf{Z}_2^d) = \Theta(d \log d)$ [8, Theorem 18.3]. Theorem 1.3 says that the lower from [10, Theorem 1.4] is sharp.

Before we proceed to the proof of Theorem 1.3, we will mention some other work on mixing times for lamplighter chains. The mixing time of $\mathcal{G}^\diamond$ was first studied by Häggström and Jonasson in [6] in the case of the complete graph K_n and the one-dimensional cycle $\mathbf{Z}_n$. Their work implies a total variation cutoff with threshold $\frac{1}{2}t_{\rm cov}(K_n)$ in the former case and that there is no cutoff in the latter. Here, $t_{\rm cov}(\mathcal{G})$ for a graph $\mathcal{G}$ denotes the expected number of steps required by lazy random walk to visit every site in $\mathcal{G}$. The connection between $t_{\rm mix}(\mathcal{G}^\diamond)$ and $t_{\rm cov}(\mathcal{G})$ is explored further in [10], in addition to developing the relationship between the relaxation time of $\mathcal{G}^\diamond$ and $t_{\rm hit}(\mathcal{G})$, and $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ and $t_{\rm u}(\mathcal{G}^\diamond)$. The results of [10] include a proof of total variation cutoff for $\mathbf{Z}_n^2$ with threshold $t_{\rm cov}(\mathbf{Z}_n^2)$. In [9], it is shown that $t_{\rm mix}((\mathbf{Z}_n^d)^\diamond) \sim \frac{1}{2}t_{\rm cov}(\mathbf{Z}_n^d)$ when $d \geq 3$ and more generally that $t_{\rm mix}(\mathcal{G}_n^\diamond) \sim \frac{1}{2}t_{\rm cov}(\mathcal{G}_n)$ whenever $(\mathcal{G}_n)$ is a sequence of graphs satisfying some uniform local transience assumptions.

The mixing time of $X^\diamond = (f, X)$ is typically dominated by the first coordinate f since the amount of time it takes for X to mix is negligible compared to that required by $X^\diamond$. We can sample from $f(t)$ by:

1. sampling the range $\mathcal{C}(t)$ of lazy random walk run for time t, then
2. marking the vertices of $\mathcal{C}(t)$ by iid fair coin flips.

Determining the mixing time of $X^\diamond$ is thus typically equivalent to computing the threshold t where the corresponding marking becomes indistinguishable from a uniform marking of $V(\mathcal{G})$ by iid fair coin flips. This in turn can be viewed as a statistical test for the uniformity of the uncovered set $\mathcal{U}(t)$ of X — if $\mathcal{U}(t)$ exhibits any sort of non-trivial systematic geometric structure then $X^\diamond(t)$ is not mixed. This connects this work to the literature on the geometric structure of the last visited points by random walk [1–3, 9].

1.3. *Outline.* The remainder of this article is structured as follows. In Section 2, we will give the proof of Theorem 1.1 by checking the hypotheses of Theorem 1.3. Next, in Section 3 we will collect a number of estimates regarding the amount of X spends in and requires to cover sets of vertices in $\mathcal{G}$ of various sizes. Finally, in Section 4, we will complete the proof of Theorem 1.3.

2. Proof of Theorem 1.1. We are going to prove Theorem 1.1 by checking the hypotheses of Theorem 1.3. We begin by noting that by [8, Corollary 12.12] and [8, Section 12.3.1], we have that

$$t_{\mathrm{rel}}(\mathbf{Z}_n^d) = \Theta(dn^2). \tag{2.1}$$

By [4, Example 2, Page 2155], we know that $t_{\mathrm{u}}(\mathbf{Z}_n) = O(n^2)$. Hence by [5, Theorem 2.10], we have that

$$t_{\mathrm{u}}(\mathbf{Z}_n^d) = O((d \log d) n^2). \tag{2.2}$$

The key to checking parts (A)–(C) of Assumption 1.2 are the Green's function estimates which are stated in Proposition 2.2 (low degree) and Proposition 2.6 (high degree). In order to establish these we will need to prove several intermediate technical estimates. We begin by recording the following facts about the transition kernel P for lazy random walk on a vertex transitive graph $\mathcal{G}$. First, we have that

$$P^t(x,y) \leq P^t(x,x) \text{ for all } x,y. \tag{2.3}$$

To see this, we note that for t even, the Cauchy-Schwarz inequality and the semigroup property imply

$$P^t(x,y) = \sum_z P^{t/2}(x,z)P^{t/2}(z,y) \leq \sqrt{P^t(x,x)P^t(y,y)} = P^t(x,x).$$

The inequality and final equality use the vertex transitivity of $\mathcal{G}$ so that $P(x,z) = P(z,x)$ and $P(x,x) = P(y,y)$. To get the same result for t odd, one just applies the same trick used in the proof of [8, Proposition 10.18(ii)]. Moreover, by [8, Proposition 10.18], we have that

$$P^t(x,x) \leq P^s(x,x) \text{ for all } s \leq t. \tag{2.4}$$

The main ingredient in the proof of Proposition 2.2, our low degree Green's function estimate, is the following bound for the return probability of a lazy random walk on $\mathbf{Z}^d$.

Lemma 2.1. *Let $P(x,y;\mathbf{Z}^d)$ denote the transition kernel for lazy random walk on $\mathbf{Z}^d$. For all $t \geq 1$, we have that*

$$P^t(x,x;\mathbf{Z}^d) \leq \sqrt{2}\left(\frac{4d}{\pi}\right)^{d/2} \frac{1}{t^{d/2}} + e^{-t/8}. \tag{2.5}$$

Proof. To prove the lemma we first give an upper bound on the transition probabilities for a (non-lazy) simple random walk Y on $\mathbf{Z}^d$. One can easily give an exact formula for the return probability of Y to the origin of $\mathbf{Z}^d$ in $2t$ steps by counting all of the possible paths from 0 back to 0 of length $2t$ (here and hereafter, $P_{\rm NL}(x,y;\mathbf{Z}^d)$ denotes the transition kernel of Y):

$$P_{\rm NL}^{2t}\left(x,x;\mathbf{Z}^d\right) = \sum_{n_1+\cdots+n_d=t} \frac{(2t)!}{(n_1!)^2(n_2!)^2\cdots(n_d!)^2}\cdot\frac{1}{(2d)^{2t}}$$
$$= \frac{1}{(2d)^{2t}}\binom{2t}{t}\sum_{n_1+\cdots+n_d=t}\left(\frac{t!}{n_1!n_2!\cdots n_d!}\right)^2$$

We can bound the sum above as follows, using the multinomial theorem in the second step:

$$P_{\rm NL}^{2t}\left(x,x;\mathbf{Z}^d\right) \le \frac{1}{(2d)^{2t}}\binom{2t}{t}\left(\max_{n_1+\cdots+n_d=t}\frac{t!}{n_1!\cdots n_d!}\right)\sum_{n_1+\cdots+n_d=t}\frac{t!}{n_1!\cdots n_d!}$$
$$\le \frac{1}{(2d)^{2t}}\binom{2t}{t}\frac{t!}{[(\lfloor t/d\rfloor)!]^d}\cdot d^t.$$

Applying Stirlings formula to each term above, we consequently arrive at

$$P_{\rm NL}^{2t}\left(x,x;\mathbf{Z}^d\right) \le \frac{\sqrt{2}}{(2\pi)^{d/2}}\cdot\frac{d^{d/2}}{t^{d/2}} \tag{2.6}$$

We are now going to deduce from (2.6) a bound on the return probability for a lazy random walk X on $\mathbf{Z}^d$. We note that we can couple X and Y so that X is a random time change of Y: $X(t) = Y(N_t)$ where $N_t = \sum_{i=0}^t \xi_i$ and the (ξ_i) are iid with $\mathbf{P}[\xi_i=0] = \mathbf{P}[\xi_i=1] = \frac{1}{2}$ and are independent of Y. Note that N_t is distributed as a binomial random variable with parameters t and $1/2$. Thus,

$$P^t(x,x;\mathbf{Z}^d) = \sum_{i=0}^{t/2} P_{\rm NL}^{2i}(x,x;\mathbf{Z}^d)\mathbf{P}(N_t=2i)$$
$$\le \mathbf{P}(N_t < t/4) + \sqrt{2}\left(\frac{4d}{\pi}\right)^{d/2}\frac{1}{t^{d/2}},$$

where in the second term we used the monotonicity of the upper bound in (2.6) in t. The first term can be bounded from above by using the Hoeffding inequality. This yields the term $e^{-t/8}$ in (2.5). □

Throughout the rest of this section, we let $|x-y|$ denote the L^1 distance between $x,y\in\mathbf{Z}_n^d$.

PROPOSITION 2.2. *Let $G(x,y)$ denote the Green's function for lazy random walk on $\mathbf{Z}_n^d$. For each $\delta \in (0,1)$, there exists constants $C_1, C_2, C_3 > 0$ independent of n, d for $d \geq 3$ such that*

$$G(x,y) \leq \frac{C_1}{d}\left(\frac{4d}{\pi}\right)^{d/2}|x-y|^{1-d/2} + C_2(d\log d)\left(\frac{4d}{\pi}\right)^{d/2} n^{2-d(1-\delta/2)} + C_3\big(d^2\log d\big)n^2 e^{-n^\delta/2}$$

for all $x, y \in \mathbf{Z}_n^d$ distinct.

PROOF. Fix $\delta \in (0,1)$. We first observe that the probability that there is a coordinate in which the random walk wraps around the torus within $t < n^2$ steps can be estimated by using Hoeffding's inequality and a union bound by

$$d \cdot \mathbf{P}(Z(t) > n) = de^{-\frac{n^2}{2t}}$$

where $Z(t)$ is a one dimensional simple random walk on $\mathbf{Z}$. Let $k = |x-y|$. Applying (2.3) and (2.4) in the second step, and estimating the probability of wrapping around in time $n^{2-\delta}$ in the third term, we see that

$$G(x,y) = \sum_{t=k}^{t_{\mathrm{u}}} P^t(x,y) \leq \sum_{t=k}^{n^{2-\delta}} P^t(x,x;\mathbf{Z}^d) + t_{\mathrm{u}} P^{n^{2-\delta}}(x,x;\mathbf{Z}^d) + dt_{\mathrm{u}} e^{-\frac{n^\delta}{2}}. \tag{2.7}$$

We can estimate the sum on the right hand side above using Lemma 2.1, yielding the first term in the assertion of the lemma. Applying Lemma 2.1 again, we see that there exists a constant C_2 which does not depend on n, d such that the second term in the right side of (2.7) is bounded by

$$C_2(d\log d)\left(\frac{4d}{\pi}\right)^{d/2} n^{2-d(1-\delta/2)}. \tag{2.8}$$

Indeed, the factor $(d\log d)n^2$ comes from (2.2) and the other factor comes from Lemma 2.1. Combining proves the lemma. □

Proposition 2.2 is applicable when n is much larger than d. We now turn to prove Proposition 2.6, which gives us an estimate for the Green's function which we will use when d is large. Before we prove Proposition 2.6, we first need to collect the following estimates.

LEMMA 2.3. *Suppose that X is a lazy random walk on $\mathbf{Z}_n^d$ for $d \geq 8$ and that $|X(0)| = k \leq \frac{d}{8}$. For each $j \geq 0$, let τ_j be the first time t that $|X(t)| = j$. There exists a constant $C_k > 0$ depending only on k such that $\mathbf{P}[\tau_0 < \tau_{2k}] \leq C_k d^{-k}$. If, instead, $|X(0)| = 1$, then there exists a universal constant $p > 0$ such that $\mathbf{P}[\tau_0 < \tau_{2k}] \geq p$.*

PROOF. It clearly suffices to prove the result when X is non-lazy. Assume that $|X(t)| = j \in \{k, \ldots, 2k\}$. It is obvious that the probability that $|X|$ moves to $j+1$ in its next step is at least $1 - \frac{2k}{d}$. The reason is that the probability that the next coordinate to change is one of the coordinates of $X(t)$ whose value is 0 is at least $1 - \frac{2k}{d}$. Similarly, the probability that $|X|$ next moves to $j-1$ is at most $\frac{2k}{d}$. Consequently, the first result of the lemma follows from the Gambler's ruin problem (see, for example, [8, Section 17.3.1]). The second assertion of the lemma follows from the same argument. □

LEMMA 2.4. *Assume that $k \in \mathbf{N}$ and that $d = 2k \vee 3$. Suppose that X is a lazy random walk on $\mathbf{Z}^d$ and that $|X(0)| = 2k$. Let τ_k be the first time t that $|X(t)| = k$. There exists $p_k > 0$ depending only on k such that $\mathbf{P}[\tau_k = \infty] \geq p_k > 0$.*

PROOF. Let $\mathbf{P}_y$ denote the law under which X starts at y. Assume that $\mathbf{P}_y[\tau_k = \infty] = 0$ for some $y \in \mathbf{Z}^d$ with $|y| = 2k$. Suppose that $z \in \mathbf{Z}^d$ with $|z| = 2k$ and let τ_z be the first time that X hits z. Then since $\mathbf{P}_y[\tau_z < \tau_k] > 0$, it follows from the strong Markov property that $\mathbf{P}_z[\tau_k = \infty] = 0$. From this, it follows that the expected amount of time that X spends in $B(0,k)$ is infinite because it implies that on each successive hit to $\partial B(0,2k)$, X returns to $B(0,k)$ with probability 1. Since X is transient [**?** , Theorem 4.3.1], the expected amount of time that X spends in $B(0,k)$ is finite. This is a contradiction. □

LEMMA 2.5. *Assume that $k \in \mathbf{N}$ and $d \geq 2k \vee 3$. Suppose that X is a lazy random walk on $\mathbf{Z}_n^d$ and that $|X(0)| = 2k$. Let τ_k be the first time t that $|X(t)| = k$. There exists $p_k, c_k > 0$ depending only on k such that $\mathbf{P}[\tau_k > c_k d n^2] \geq p_k > 0$.*

PROOF. We first assume that $d = 2k \vee 3$. It follows from Lemma 2.4 that there exists a constant $p_{k,1} > 0$ depending only on k such that $\mathbf{P}[\tau_k > \tau_{n/4}] \geq p_{k,1}$. The local central limit theorem (see [**?** , Chapter 2]) implies that there exists constants $c_{k,1}, p_{k,2} > 0$ such that the probability that a random walk on $\mathbf{Z}_n^d$ moves more than distance $\frac{n}{4}$ in time $c_{k,1} n^2$ is at most $1 - p_{k,2}$. Combining implies the result for $d = 2k \vee 3$.

Now we suppose that $d \geq 2k \vee 3$. Let $(X_1(t), \ldots, X_d(t))$ be the coordinates of $X(t)$. By re-ordering if necessary, we may assume without loss of generality that $X_{2k+1}(0), \ldots, X_d(0) = 0$. Let $Y(t) = (X_1(t), \ldots, X_{2k}(t))$. Then Y is a random walk on $\mathbf{Z}_n^{2k}$. Clearly, $|Y(0)| = 2k$ because $X(0)$ cannot have more than $2k$ non-zero coordinates. For each j, let τ_j^Y be the first time t that $|Y(t)| = j$. Then $\tau_k^Y \leq \tau_k$. For each t, let N_t denote the number of steps that X takes in the time interval $\{1, \ldots, t\}$ in which one of its first $2k$ coordinates is changed (in other words, N_t is the number of steps taken by Y). The previous paragraph implies that $\mathbf{P}[N_{\tau_k^Y} \geq c_{k,1} n^2] \geq p_{k,3} > 0$ for a constant $p_{k,3} > 0$ depending only on k. Since the probability that the first $2k$ coordinates are changed in any step is k/d (recall that X is lazy), the final result holds from a simple large deviations estimate. □

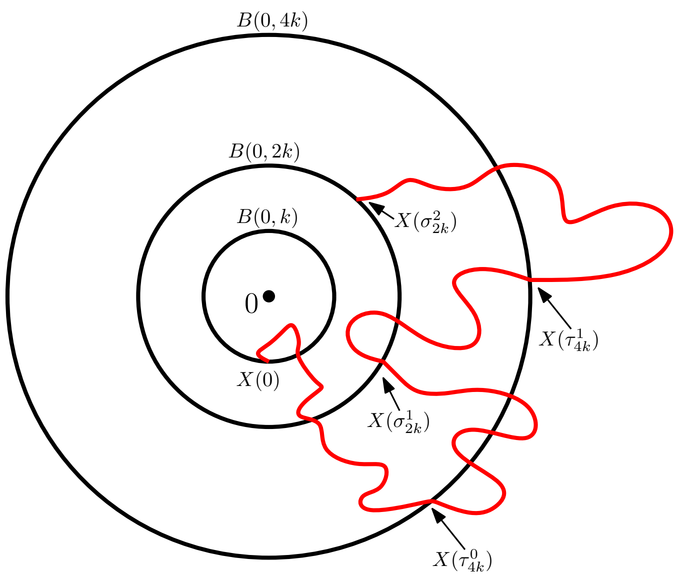

Fig 2. *Assume that $d \geq 8$ and that $k \in \mathbf{N}$ with $d \geq 8k$. Let X be a lazy random walk on $\mathbf{Z}_n^d$ and that $X(0) = x$ with $|x - y| = k$. In Proposition 2.6, we show that $G(x,y) \leq C_k d^{-k}$ where $C_k > 0$ is a constant depending only on k . By translation, we may assume without loss of generality that $|x| = k$ and $y = 0$. The idea of the proof is to first invoke Lemma 2.3 to show that X escapes to $\partial B(0,4k)$ with probability at least $1 - C_{k,1} d^{-k}$. We then decompose the path of X into successive excursions $\{X(\sigma_{2k}^j), \ldots, X(\tau_{4k}^j), \ldots, X(\sigma_{2k}^{j+1})\}$ between $\partial B(0,2k)$ back to itself through $\partial B(0,4k)$. By Lemma 2.3, we know that each excursion hits 0 with probability bounded by $C_{2k,1} d^{-2k}$ and Lemma 2.5 implies that each excursion takes length $c_k dn^2$ with probability at least $p_k > 0$. Consequently, the result follows from a simple stochastic domination argument.*

Now we are ready to prove our estimate of $G(x, y)$ when d is large.

PROPOSITION 2.6. *Suppose that $d \geq 8$. Let $G(x,y)$ denote the Green's function for lazy random walk on $\mathbf{Z}_n^d$. For each $k \in \mathbf{N}$ with $k \leq \frac{d}{8}$, there exists a constant $C_k > 0$ which does not depend on n, d such that*

$$G(x,y) \leq \frac{C_k}{d^k} \text{ for all } x, y \in \mathbf{Z}_n^d \text{ with } |x-y| \geq k.$$

PROOF. See Figure 2 for an illustration of the proof. By translation, we may assume without loss of generality that $y = 0$; let $k = |x|$. Let τ_0 be the first time t that $|X(t)| = 0$. The strong Markov property implies that

$$G(x,y) \leq \mathbf{P}[\tau_0 > t_{\mathrm{u}}] + (1 - \mathbf{P}[\tau_0 > t_{\mathrm{u}}])G(x,x).$$

Consequently, it suffices to show that for each $k \in \mathbf{N}$, there exists constants $C_k, C_0 > 0$ such that

$$\mathbf{P}[\tau_0 > t_{\mathrm{u}}] \leq \frac{C_k}{d^k} \text{ and} \tag{2.9}$$

$$G(x,x) \leq C_0. \tag{2.10}$$

We will first prove (2.9); the proof of (2.10) will be similar.

Let N be a geometric random variable with success probability $C_{2k}d^{-2k}$ where C_{2k} is the constant from Lemma 2.3. Let (ξ_j) be a sequence of independent random variables with $\mathbf{P}[\xi_j = c_{2k}dn^2] = p_{2k}$ and $\mathbf{P}[\xi_j = 0] = 1 - p_{2k}$ where c_{2k}, p_{2k} are the constants from Lemma 2.5 independent of N. We claim that τ_0 is stochastically dominated from below by $\sum_{j=1}^{N\zeta - 1} \xi_j$ where ζ is independent of N and (ξ_j) with $\mathbf{P}[\zeta = 0] = C_k d^{-k} = 1 - \mathbf{P}[\zeta = 1]$. Indeed, to see this we let $\sigma_k^0 = 0$ and let τ_{4k}^0 be the first time t that $|X(t)| = 4k$. For each $j \geq 1$, we inductively let σ_{2k}^j be the first time t after τ_{4k}^{j-1} that $|X(t)| = 2k$ and let τ_{4k}^j be the first time t after σ_{2k}^j that $|X(t)| = 4k$. Let $\mathcal{F}_t$ be the filtration generated by X. Lemma 2.3 implies that the probability that X hits 0 in $\{\sigma_{2k}^j, \ldots, \tau_{4k}^j\}$ given $\mathcal{F}_{\sigma_{2k}^j}$ is at most $C_{2k}d^{-2k}$ for each $j \geq 1$ where $C_{2k} > 0$ only depends on $2k$. This leads to the success probability in the definition of N above. The factor ζ is to take into account the probability that X reaches distance $2k$ before hitting 0. Moreover, Lemma 2.5 implies that $\mathbf{P}[\sigma_{2k}^j - \tau_{4k}^{j-1} \geq c_{2k}dn^2 | \mathcal{F}_{\tau_{4k}^j}] \geq p_{2k}$. This leads to the definition of the (ξ_j) above. This implies our claim.

To see (2.9) from our claim, an elementary calculation yields that

$$\mathbf{P}[N\zeta \leq C_{2k}^{-1}d^k] \leq \mathbf{P}[N \leq C_{2k}^{-1}d^k \text{ or } \zeta = 0] \leq 2d^{-k} + C_k d^{-k}.$$

We also note that

$$\mathbf{P}\left[\sum_{j=1}^{m} \xi_j \leq \frac{p_k c_k m n^2}{2}\right] \leq e^{-cm}$$

for some constant $c > 0$. Combining these two observations along with a union bound implies (2.9). To see (2.10), we apply a similar argument using the second assertion of Lemma 2.3. □

Now that we have proved Proposition 2.2 and Proposition 2.6, we are ready to check the criteria of Assumption 1.2.

2.1. *Part* (A). By [8, Proposition 1.14] with $\tau_x^+ = \min\{t \geq 1 : X(t) = x\}$, we have that $\mathbf{E}_x[\tau_x^+] = |\mathbf{Z}_n^d|$. Applying Proposition 2.6, we see that there exists constants $d_0, r > 0$ such that if $d \geq d_0$, then

$$G(x,y) \leq 1/2 \text{ for all } |x-y| \geq r. \tag{2.11}$$

Proposition 2.2 implies that there exists n_0 such that if $n \geq n_0$ and $3 \leq d < d_0$ then (2.11) likewise holds, possibly by increasing r (clearly, part (A) holds when $d \leq d_0$ and $n \leq n_0$; note also that we may assume without loss of generality that d_0, n_0 are large enough so that the diameter of the graph is at least $2r$). Let τ_r be the first time t that $|X(t) - X(0)| = r$. We observe that there exists $\rho_0 = \rho_0(r) > 0$ such that

$$\mathbf{P}_x[\tau_r < \tau_x^+] \geq \rho_0 \tag{2.12}$$

uniform in n, d since in each time step there are d directions in which $X(t)$ increases its distance from $X(0)$. By combining (2.11) with (2.12), we see that $\mathbf{P}_x[\tau_x^+ \geq t_{\mathrm{u}}(\mathcal{G})] \geq \rho_1 > 0$ uniform $d \geq d_0$. Let $\mathcal{F}_t$ be the filtration generated by X. We consequently have that

$$\begin{aligned}\mathbf{E}_x[\tau_x^+] \geq \mathbf{E}_x[\tau_x^+ \mathbf{1}_{\{\tau_x^+ \geq t_{\mathrm{u}}(\mathcal{G})\}}] &= \mathbf{E}_x\big[\mathbf{E}_x[\tau_x^+ | \mathcal{F}_{t_{\mathrm{u}}(\mathcal{G})}]\mathbf{1}_{\{\tau_x^+ \geq t_{\mathrm{u}}(\mathcal{G})\}}\big] \\ &\geq \mathbf{E}_x\big[\mathbf{E}_{X(t_{\mathrm{u}}(\mathcal{G}))}[\tau_x]\mathbf{1}_{\{\tau_x^+ \geq t_{\mathrm{u}}(G)\}}\big] \geq \rho_1\left(1 - \frac{1}{2e}\right)\mathbf{E}_\pi[\tau_x].\end{aligned}$$

That is, there exists $\rho_2 > 0$ uniform in $d \geq d_0$ such that $\mathbf{E}_x[\tau_x^+] \geq \rho_2 \mathbf{E}_\pi[\tau_x]$. Hence by [8, Lemma 10.2], we have that $t_{\mathrm{hit}}(\mathbf{Z}_n^d) \leq K_1 |\mathbf{Z}_n^d|$ where $K_1 = 2/\rho_2$ is a uniform constant.

Remark 2.7. *There is another proof of Part A which is based on eigenfunctions. In particular, we know that*

$$t_{\mathrm{hit}}(\mathbf{Z}_n^d) \leq 2\mathbf{E}_\pi[\tau_x] = 4\sum_i \frac{1}{1-\lambda_i}$$

where the λ_i are the eigenvalues of simple random walk on $\mathbf{Z}_n^d$ distinct from 1; the extra factor of 2 in the final equality accounts for the laziness of *the chain. The λ_i can be computed explicitly using [8, Lemma 12.11] and the form of the λ_i when $d = 1$ which are given in [8, Section 12.3]. The assertion follows by performing the summation which can be accomplished by approximating it by an appropriate integral.*

2.2. *Part* (B). It follows from Proposition 2.6 that there exist constants $C > 0$ and $d_0 \geq 3$ such that

$$G(x,y) \leq \frac{C}{d} \text{ for } x,y \in \mathbf{Z}_n^d \text{ with } |x-y| = 1 \tag{2.13}$$

provided $d \geq d_0$. Consequently, there exists $K \in \mathbf{N}$ which does not depend on $d \geq d_0$ such that

$$2K(5/2)^K G^K(x,y) = O\left(2K\left(\frac{5/2}{d}\right)^K\right) \tag{2.14}$$

It follows by combining (2.1) and (2.2) that we have that

$$\frac{t_{\mathrm{u}}(\mathbf{Z}_n^d)}{t_{\mathrm{rel}}(\mathbf{Z}_n^d)} = O(\log d). \tag{2.15}$$

Combining (2.14) with (2.15) shows that part (B) of Assumption 1.2 is satisfied provided we take $K_2 = K$ large enough. Moreover, (2.15) clearly holds if $3 \leq d < d_0$ by Proposition 2.2.

2.3. *Part* (C). We first note that it follows from (2.1), (2.2), Proposition 2.2, and Proposition 2.6 that there exists constants $C > 0$ such that n^* for $\mathbf{Z}_n^d$ is at most $Cd^2n^2\log d$ for all $d \geq 3$. To check this part, we need to show that there exists $K_3 > 0$ such that

$$G^*(n^*) \leq K_3\left(\frac{dn^2 + d\log n}{\log d + \log n}\right). \tag{2.16}$$

We are going to prove the result by considering the regimes of $d \leq \sqrt{\log n}$ and $d > \sqrt{\log n}$ separately.

Case 1: $d < \sqrt{\log n}$.

From (2.16) it is enough to show that $G^*(n^*) \leq Kdn^2/\log n$. We can bound $G^*(n^*)$ in this case as follows. Let $D = (d\log d\log n)^{1/(\frac{1}{2}d-1)}$. By Proposition 2.2, we can bound from above the expected amount of time that X starting at 0 in $\mathbf{Z}_n^d$ spends in the L^1 ball of radius D by summing radially:

$$\begin{aligned}&\sum_{k=1}^{D}\frac{C_1}{d}\left(\frac{4d}{\pi}\right)^{d/2}k^{1-d/2}\cdot 2d(2k)^{d-1}\\ \leq& C_1\left(\frac{16d}{\pi}\right)^{d/2}\sum_{k=1}^{D}k^{d/2}\leq\frac{C_2}{d}\left(\frac{16d}{\pi}\right)^{d/2}D^{1+d/2}\leq C_3n(d\log d\log n)^5\end{aligned}$$

for constants $C_1, C_2, C_3 > 0$, where we used that $d^{d/2} \leq n$. We also note that $2d(2k)^{d-1}$ is the size of the L^∞ ball of radius k. The exponent of 5 comes from the inequality

$$\frac{\frac{1}{2}d+1}{\frac{1}{2}d-1}\leq 5 \text{ for all } d\geq 3.$$

We can estimate $G^*(n)$ by dividing between the set of points which have distance at most D to 0 and those whose distance to 0 exceeds D by:

$$\begin{aligned}G^*(n^*) \leq& C_3n(d\log d\log n)^5 + C_4D^{1-\frac{1}{2}d}n^*\\ \leq& C_3n(d\log d\log n)^5 + \frac{C_4\cdot Cd^2n^2\log d}{d\log d\log n},\end{aligned}$$

where $C_4 > 0$ is a constant and we recall that $C > 0$ is the constant from the definition of n^*. This implies the desired result.

Case 2: $d \geq \sqrt{\log n}$.

In this case, we are going to employ Proposition 2.6 to bound $G^*(n^*)$. The number of points which have distance at most k to 0 is clearly $1 + (2d)^k$. Consequently, by Proposition 2.6, we have that

$$\begin{aligned}G^*(n^*) \leq& \left(C_0+\sum_{k=1}^{3}C_kd^{-k}(2d)^k\right)+C_4d^{-4}n^*\\ \leq& C_5+\frac{C_6(\log d)n^2}{d^2}\end{aligned}$$

for some constants $C_5, C_6 > 0$. Since $d^2 \geq \log n$, this is clearly dominated by the right hand side of (2.16) (with a large enough constant), which completes the proof in this case.

□

3. Coverage Estimates. Throughout, we assume that $\mathcal{G}$ is a finite, connected, vertex transitive graph and X is lazy random walk on $\mathcal{G}$ with transition matrix P and stationary measure π. For $S \subseteq V(\mathcal{G})$, we let $\mathcal{C}_S(t)$ be the set of vertices in S visited by X by time t and let $\mathcal{U}_S(t) = S \setminus \mathcal{C}_S(t)$ be the subset of S which X has not visited by time t. We let $\mathcal{C}(t) = \mathcal{C}_{V(\mathcal{G})}(t)$ and $\mathcal{U}(t) = \mathcal{U}_{V(\mathcal{G})}(t)$. We will use $\mathbf{P}_x, \mathbf{E}_x$ to denote the probability measure and expectation under which $X(0) = x$. Likewise, we let $\mathbf{P}_\pi, \mathbf{E}_\pi$ correspond to the case that X is initialized at stationarity. The purpose of this section is to develop a number of estimates which will be useful for determining the amount of time required by X in order to cover subsets S of $V(\mathcal{G})$. We consider two different regimes depending on the size of S. If S is large, we will estimate the amount of time it takes for X to visit $t_{\rm u}(\mathcal{G})$ distinct vertices in S. If S is small, we will estimate the amount of time it takes for X to visit $1/2$ of the vertices in S.

3.1. *Large Sets.* In this subsection, we will prove that the amount of time it takes for X to visit $t_{\rm u}(\mathcal{G})$ distinct elements of a large set of vertices $S \subseteq V(\mathcal{G})$ is stochastically dominated by a geometric random variable whose parameter depends on $t_{\rm u}(\mathcal{G})/t_{\rm rel}(\mathcal{G})$. The main result is:

Proposition 3.1. *Assume X satisfies part (B) of Assumption 1.2 with constants K_2, K_3. Let $S \subseteq V(\mathcal{G})$ consist of at least $2K_2 t_{\rm u}(\mathcal{G})/G(x,y)$ elements for $x, y \in V(\mathcal{G})$ adjacent and let*

$$t = \frac{2(K_2+2)t_{\rm u}(\mathcal{G})}{\pi(S)}.$$

There exists a universal constant $C > 0$ such that for every $x \in V(\mathcal{G})$, we have that

$$\mathbf{P}_x\left[\mathcal{C}_S(t) \le t_{\rm u}(\mathcal{G})\right] \le \exp\left(-C\frac{t_{\rm u}(\mathcal{G})}{t_{\rm rel}(\mathcal{G})}\right).$$

Recall that

$$\mathcal{L}_S(t) = \sum_{s=0}^{t} \mathbf{1}_{\{X(s)\in S\}}$$

is the amount of time that X spends in S up to time t. The proof consists of several steps. The first is Proposition 1.4, which we will deduce from [7, Theorem 1] shortly, which gives that the probability $\mathcal{L}_S(t)$ is less than $1/2$ its mean is exponentially small in t. Once we show that $\mathcal{L}_S(t)$ is large with high probability, in order to show that X visits many vertices in S, we need to rule out the possibility of X concentrating most of its local time in a small subset of S. This is accomplished in Lemma 3.2. We now proceed to the proof of Proposition 1.4.

Proof of Proposition 1.4. We rewrite the event

$$\left\{\mathcal{L}_S(t) \le t\frac{\pi(S)}{2}\right\} = \left\{\sum_{s=0}^{t} f(X_s) > t\left(1-\pi(S)+\frac{\pi(S)}{2}\right)\right\} \tag{3.1}$$

where $f(x) = \mathbf{1}_{S^c}(x)$. Let $\epsilon = \pi(S)/2$ and $\mu = \mathbf{E}_\pi[f(X(t))] = 1-2\epsilon$. The case $\epsilon \ge 1/4$ follows immediately from [7, Equation 3] in the statement of [7, Theorem 1], so we will only consider the case $\epsilon \in (0,1/4)$ here. Let $\overline{\mu} = 1-\mu = 2\epsilon$. For $x \in (0,1)$, let

$$I(x) = -x\log\left(\frac{\mu+\overline{\mu}\lambda_0}{1-2\overline{x}/(\sqrt{\Delta}+1)}\right) - \overline{x}\log\left(\frac{\overline{\mu}+\mu\lambda_0}{1-2x/(\sqrt{\Delta}+1)}\right)$$

where $\overline{x} = 1-x$ and

$$\Delta = 1 + \frac{4\lambda_0 x\overline{x}}{\mu\overline{\mu}(1-\lambda_0)^2}. \tag{3.2}$$

For $x \in [\mu, \mu+\epsilon] = [1-2\epsilon, 1-\epsilon]$, $\epsilon \in (0,1/4)$, and $\lambda_0 \ge 1/2$, we note that

$$\frac{1/2}{(1-\lambda_0)^2} \le \Delta \le \frac{20}{(1-\lambda_0)^2} \tag{3.3}$$

By [7, Theorem 1] and using the representation (3.1), we have that

$$\mathbf{P}_\pi\left[\mathcal{L}_S(t) \le t\epsilon\right] \le \exp(-I(\mu+\epsilon)t).$$

Since $I(\mu) = I'(\mu) = 0$ and $I''(x) = (\sqrt{\Delta}x\overline{x})^{-1}$ (see [7, Appendix B]), we can write

$$I(\mu+\epsilon) = \int_{\mu}^{\mu+\epsilon}\int_{\mu}^{x} \frac{1}{\sqrt{\Delta}y\overline{y}}\,dydx \tag{3.4}$$

where $\overline{y} = 1-y$. Inserting the bounds from (3.3), we thus see that the right side of (3.4) admits the lower bound

$$\frac{1-\lambda_0}{\sqrt{20}}\int_{1-2\epsilon}^{1-\epsilon}\int_{1-2\epsilon}^{x}\frac{1}{2\epsilon}\,dydx \ge \frac{(1-\lambda_0)\epsilon}{16\sqrt{5}}$$

for all $\epsilon \in (0,1/4)$ and $\lambda_0 \ge \frac{1}{2}$. □

As in the proof of Lemma 2.1, we couple X with a non-lazy random walk Y so that $X(t) = Y(N_t)$ where $N_t = \sum_{i=0}^{t} \xi_i$ and the (ξ_i) are iid with $\mathbf{P}[\xi_i = 0] = \mathbf{P}[\xi_i = 1] = \frac{1}{2}$ and are independent of Y. We let $\mathcal{L}_S^Y(t)$ denote the amount of time that $Y|_{[0,N_t]}$ spends in S (note that this differs slightly from the definition of $\mathcal{L}_x^Y(t)$ which appeared in Section 2). In other words, $\mathcal{L}_S^Y$ is the amount of that X spends in S by time t, not including those times where X does not move. The next lemma gives a lower bound on the probability that the number $\mathcal{C}_S(t)$ of distinct vertices X visits in a given set $S \subseteq V(\mathcal{G})$ by time t is proportional to $\mathcal{L}_S^Y(t)$. The lower bound for this probability will be given in terms of the Green's function $G(x,y)$ for X. Recall its definition from (1.5). Since X is a lazy random walk, we also have that

$$G(x,y) \leq G(x,x) \text{ for all } x,y \in V(\mathcal{G}). \tag{3.5}$$

This is a consequence of (2.3).

Lemma 3.2. *Fix $S \subseteq V(\mathcal{G})$. For each positive integer k, we have that*

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) \geq \frac{\mathcal{L}_S^Y(t) - t_{\mathrm{u}}(\mathcal{G})}{k}\right] \geq 1 - \frac{t\pi(S)q^k(t)}{t_{\mathrm{u}}(\mathcal{G})}, \tag{3.6}$$

where

$$q(t) = (G(x,y) - 1) + (1 + (2e)^{-1})\frac{t - t_{\mathrm{u}}(\mathcal{G})}{|\mathcal{G}|}\mathbf{1}_{\{t > t_{\mathrm{u}}(\mathcal{G})\}}. \tag{3.7}$$

and x is adjacent to y.

Proof. For $t \geq t_{\mathrm{u}}(\mathcal{G})$, we have $P^t(x,y) \leq (1 + (2e)^{-1})\pi(y)$ by the definition of $t_{\mathrm{u}}(\mathcal{G})$. Thus by a union bound,

$$\mathbf{P}_x[\mathcal{L}_x^Y(t) > 1] \leq q(t).$$

Hence by the strong Markov property,

$$\mathbf{P}_x[\mathcal{L}_x^Y(t) > k] \leq q^k(t).$$

Observe

$$\mathbf{P}_\pi[\tau_x = s] \leq \mathbf{P}_\pi[X_s = x] \leq \pi(x). \tag{3.8}$$

Let

$$\mathcal{L}_{S,k}^Y(t) = \sum_{x \in S} \mathcal{L}_x^Y(t)\mathbf{1}_{\{\mathcal{L}_x^Y(t) > k\}}$$

be the total time that Y spends at points in S which it visits more than k times by time N_t. By (3.8), we have that

$$\mathbf{E}_\pi[\mathcal{L}^Y_{S,k}(t)] \leq \sum_{x\in S}\sum_{s=0}^{t} \mathbf{P}_\pi[\tau_x = s] q^k(t) \leq t\pi(S) q^k(t).$$

Applying Markov's inequality we have that

$$\mathbf{P}_\pi\left[\mathcal{L}^Y_{S,k}(t) \geq t_{\mathrm{u}}(\mathcal{G})\right] \leq \frac{\mathbf{E}_\pi[\mathcal{L}^Y_{S,k}(t)]}{t_{\mathrm{u}}(\mathcal{G})} \leq \frac{t\pi(S)q^k(t)}{t_{\mathrm{u}}(\mathcal{G})}$$

Observe

$$\mathcal{C}_S(t) = \sum_{x\in S} \mathbf{1}_{\{\mathcal{L}^Y_x(t)\geq 1\}} \geq \sum_{x\in S}\left(\mathbf{1}_{\{\mathcal{L}^Y_x(t)\geq 1\}} - \mathbf{1}_{\{\mathcal{L}^Y_x(t)>k\}}\right) \geq \frac{\mathcal{L}^Y_S(t) - \mathcal{L}^Y_{S,k}(t)}{k}.$$

Thus

$$\left\{\mathcal{L}^Y_{S,k}(t) < t_{\mathrm{u}}(\mathcal{G})\right\} \subseteq \left\{\mathcal{C}_S(t) \geq \frac{\mathcal{L}^Y_S(t) - t_{\mathrm{u}}(\mathcal{G})}{k}\right\}.$$

We arrive at

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) \geq \frac{\mathcal{L}^Y_S(t) - t_{\mathrm{u}}(\mathcal{G})}{k}\right] \geq 1 - \mathbf{P}_\pi\left[\mathcal{L}^Y_{S,k}(t) \geq t_{\mathrm{u}}(\mathcal{G})\right] \geq 1 - \frac{t\pi(S)q^k(t)}{t_{\mathrm{u}}(\mathcal{G})},$$

which completes the proof of the lemma. □

Proposition 1.4 gives a lower bound on the probability $\mathcal{L}_S(t)$ is proportionally lower than its expectation, Lemma 3.2 gives a lower bound on the probability X visits less than a positive fraction of $\mathcal{L}^Y_S(t) - t_{\mathrm{u}}(\mathcal{G})$ vertices in S by time t, and standard large deviations estimates bound the probability that $\mathcal{L}^Y_S(t)$ is proportionally smaller than $\mathcal{L}_S(t)$. By combining these two lemmas, we obtain the following result, which gives a lower bound on the rate at which X covers vertices in S.

LEMMA 3.3. *Fix $S \subseteq V(\mathcal{G})$. Then*

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) \leq \frac{t\pi(S) - 8t_{\mathrm{u}}(\mathcal{G})}{8k}\right] \leq \exp\left(-C_0 t\frac{\pi(S)}{t_{\mathrm{rel}}(\mathcal{G})}\right) + \exp\left(-\frac{1}{16}t\pi(S)\right) + \frac{t\pi(S)q^k(t)}{t_{\mathrm{u}}(\mathcal{G})} \tag{3.9}$$

where the constant C_0 is as in Proposition 1.4 and the function q is as in (3.7).

PROOF. We trivially have that

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) \geq \frac{t\pi(S) - 8t_{\mathrm{u}}(\mathcal{G})}{8k}\right] \geq \mathbf{P}_\pi\left[\mathcal{C}_S(t) \geq \frac{t\pi(S) - 8t_{\mathrm{u}}(\mathcal{G})}{8k},\ \mathcal{L}_S^Y(t) > \frac{t}{8}\pi(S)\right]$$
$$\geq \mathbf{P}_\pi\left[\mathcal{C}_S(t) > \frac{\mathcal{L}_S^Y(t) - t_{\mathrm{u}}(\mathcal{G})}{k},\ \mathcal{L}_S^Y(t) > \frac{t}{8}\pi(S)\right].$$

Therefore

$$\mathbf{P}_\pi\left[\mathcal{C}_S(t) \leq \frac{t\pi(S) - 8t_{\mathrm{u}}(\mathcal{G})}{8k}\right] \leq \mathbf{P}_\pi\left[\mathcal{L}_S^Y(t) \leq \frac{t}{8}\pi(S)\right] + \mathbf{P}_\pi\left[\mathcal{C}_S(t) \leq \frac{\mathcal{L}_S^Y(t) - t_{\mathrm{u}}(\mathcal{G})}{k}\right]$$

We can bound the second term from above by Lemma 3.2. The first term is bounded from above by

$$\mathbf{P}_\pi\left[\mathcal{L}_S^Y(t) \leq \frac{t}{8}\pi(S)\right] \leq \mathbf{P}_\pi\left[\mathcal{L}_S(t) \leq \frac{t}{2}\pi(S)\right] + \mathbf{P}_\pi\left[\mathcal{L}_S(t) > \frac{t}{2}\pi(S),\quad \mathcal{L}_S^Y(t) \leq \frac{t}{8}\pi(S)\right].$$

We can bound the first term using Proposition 1.4. Conditionally on $\{\mathcal{L}_S(t) > \frac{t}{2}\pi(S)\}$, we note that $\{\mathcal{L}_S^Y(t) \leq \frac{t}{8}\pi(S)\}$ occurs if X stays in place for at least $\frac{3t}{8}\pi(S)$ time steps. Consequently, standard large deviations estimates imply that the second term above is bounded by $\exp(-\frac{1}{16}t\pi(S))$. □

We can now easily complete the proof of Proposition 3.1 by ignoring the first $t_{\mathrm{u}}(\mathcal{G})$ units of time in order to reduce to the stationary case, then apply Assumption 1.2 in order to match the error terms in Lemma 3.3.

PROOF OF PROPOSITION 3.1. We first observe that

$$\mathbf{P}_x\left[\mathcal{C}_S(t) \leq t_{\mathrm{u}}(\mathcal{G})\right] \leq (1 + (2e)^{-1})\mathbf{P}_\pi[\mathcal{C}_S(t - t_{\mathrm{u}}(\mathcal{G})) \leq t_{\mathrm{u}}(\mathcal{G})].$$

With $\widetilde{t} = 2K_2 t_{\mathrm{u}}(\mathcal{G})/\pi(S)$ and using $|S| \geq 2K_2 t_{\mathrm{u}}(\mathcal{G})/(G(x,y) - 1)$ for $x, y \in V(\mathcal{G})$ adjacent, we see that

$$G(x,y) - 1 \leq q(\widetilde{t}) \leq \frac{5}{2}(G(x,y) - 1).$$

Combining this with part (B) of Assumption 1.2 implies

$$\frac{\widetilde{t}\pi(S)q^{K_2}(\widetilde{t})}{t_{\mathrm{u}}(\mathcal{G})} \leq 2K_2 q^{K_2}(\widetilde{t}) \leq \exp\left(-\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right). \tag{3.10}$$

Applying Lemma 3.3 gives the result. □

3.2. *Small Sets.* We will now give an upper bound on the rate at which X covers 1/2 the elements of a set of vertices $S \subseteq V(\mathcal{G})$, provided $|S|$ is sufficiently small.

PROPOSITION 3.4. *Fix $S \subseteq V(\mathcal{G})$, let $s = |S|$, and assume that*

$$t_{\mathrm{u}}(\mathcal{G}) \leq \frac{|\mathcal{G}|}{4s}.$$

There exists constants $C_2, C_3 > 0$ such that

$$\mathbf{P}_x\left[\mathcal{C}_S(C_2|\mathcal{G}|G^*(s)) \leq \frac{s}{2}\right] \leq \exp(-C_3 s)$$

for all $x \in V(\mathcal{G})$.

The main step in the proof of Proposition 3.4 is the next lemma, which gives an upper bound on the hitting time for S. Its proof is based on the following observation. Suppose that $S \subseteq V(\mathcal{G})$ and $\tau_S = \min\{t \geq 0 : X(t) \in S\}$. Let Z be a non-negative random variable with $Z\mathbf{1}_{\{\tau_S > t\}} = 0$ and $\mathbf{E}_x[Z\mathbf{1}_{\{\tau_S \leq t\}}] > 0$. Then we have that

$$\mathbf{P}_x[\tau_S \leq t] = \frac{\mathbf{E}_x[Z]}{\mathbf{E}_x[Z|\tau_S \leq t]}. \tag{3.11}$$

We will take Z to be the amount of time X spends in S.

LEMMA 3.5. *Fix $S \subseteq V(\mathcal{G})$ and let $s = |S|$. Assume that*

$$t_{\mathrm{u}}(\mathcal{G}) \leq \frac{|\mathcal{G}|}{2s}.$$

There exists a universal constant $\rho_0 > 0$ such that $x \in V(\mathcal{G})$ we have

$$\mathbf{P}_x\left[\tau_S \leq \frac{|\mathcal{G}|}{s}\right] \geq \frac{\rho_0}{G^*(s)}.$$

PROOF. Let us introduce $E = \left\{\tau_S \leq \frac{|\mathcal{G}|}{s}\right\}$. Observe that

$$\mathbf{P}_x[E] \geq \frac{\mathbf{E}_x\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)\right]}{\mathbf{E}_x\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)|E\right]}$$

We can bound the numerator from below as follows:

$$\begin{aligned}\mathbf{E}_x\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)\right] &\geq (1-(2e)^{-1})\mathbf{E}_\pi\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}-t_{\mathrm{u}}(\mathcal{G})\right)\right]\\ &\geq (1-(2e)^{-1})\pi(S)\left(\frac{|\mathcal{G}|}{s}-t_{\mathrm{u}}(\mathcal{G})\right)\geq \frac{1}{4}.\end{aligned} \tag{3.12}$$

Let $\mathcal{L}_S(u,t)=\mathcal{L}_S(t)-\mathcal{L}_S(u-1)$ be the number of times in the set $\{u,\ldots,t\}$ that X spends in S. Then we can express the denominator as the sum

$$\begin{aligned}&\mathbf{E}_x\left[\mathcal{L}_S\left(\tau_S,\tau_S+t_{\mathrm{u}}(\mathcal{G})\right)|E\right]+\mathbf{E}_x\left[\mathcal{L}_S\left(\tau_S+t_{\mathrm{u}}(\mathcal{G})+1,\frac{|\mathcal{G}|}{s}\right)|E\right]\\ =:&D_1+D_2.\end{aligned}$$

We have

$$D_2\leq (1+(2e)^{-1})\mathbf{E}_\pi\left[\mathcal{L}_S\left(\frac{|\mathcal{G}|}{s}\right)\right]\leq 2.$$

We will now bound D_1. By the strong Markov property, we have that

$$\begin{aligned}D_1\leq \max_{z\in S}\mathbf{E}_z[\mathcal{L}_S(t_{\mathrm{u}}(\mathcal{G}))] &= \max_{z\in S}\mathbf{E}_z\sum_{t=0}^{t_{\mathrm{u}}(\mathcal{G})}\mathbf{1}_{\{X(t)\in S\}}\\ &= \max_{z\in S}\sum_{y\in S}G(z,y)\leq G^*(s).\end{aligned}$$

Putting everything together completes the proof. □

The remainder of the proof of Proposition 3.4 is based on a simple stochastic domination argument.

Proof of Proposition 3.4. Let $C_2>0$; we will fix its precise value at the end of the proof. That X visits at least $s/2$ points in S by the time $C_2|\mathcal{G}|G^*(s)$ with probability exponentially close to 1 in s follows from a simple large deviation estimate of a binomial random variable. Namely, we run the chain for $C_2G^*(s)s$ rounds, each of length $|\mathcal{G}|/s$. We let $S_0=S$ and inductively let $S_i=S_{i-1}\setminus\{x\}$ if X hits x in the ith round for $i\geq 1$. If $|S_i|\geq s/2$, the hypotheses of Lemma 3.5 hold. In this case, the probability that X hits a point in S_i in the ith round is at least $\rho_0/G^*(s)>0$. Thus by stochastic domination, we have that

$$\mathbf{P}[\mathcal{C}_S(C_2|\mathcal{G}|G^*(s))<s/2]\leq \mathbf{P}\left[Z<s/2\right]$$

where $Z \sim \mathrm{BIN}(C_2 G^*(s)s, \rho_0/G^*(s))$. By picking C_2 large enough ($C_2 > 1/\rho_0$ will do, say) and applying the Chernoff bound, we see that

$$\mathbf{P}\left[\mathcal{C}_S(C_2|\mathcal{G}|G^*(s)) < s/2\right] \le \exp(-C_3 s) \tag{3.13}$$

for some constant C_3 (one can check that $C_3 = \frac{1}{8}$ suffices). This estimate also holds if $s = 1$. In this case we cover the point with constant probability in $C_2|\mathcal{G}|$ steps. □

4. Proof of Theorem 1.3. Throughout this section, we shall assume that X is a lazy random walk on a graph $\mathcal{G}$ which satisfies Assumption 1.2. Recall that $\mathcal{U}(t)$ is the set of vertices of $\mathcal{G}$ which X has not visited by time t. We will use the notation $\mathbf{P}_x, \mathbf{E}_x$ for the probability measure and expectation under which $X(0) = x$. Likewise, we let $\mathbf{P}_\pi, \mathbf{E}_\pi$ correspond to the case that X is initialized at stationarity. We will now work towards completing the proof of Theorem 1.3 by applying the results of the previous section to describe the process by which X covers $V(\mathcal{G})$. We will study the process of coverage in two different regimes: before and after $\mathcal{U}(t)$ contains at least n^* vertices (recall the definition of n^* from part (C) of Assumption 1.2). To this end, we let

$$\begin{aligned} r &= \max\{i : |\mathcal{G}| - i t_{\mathrm{u}}(\mathcal{G}) \ge n^*\},\\ \widetilde{r} &= \lfloor \log_2(|\mathcal{G}| - r t_{\mathrm{u}}(\mathcal{G}))\rfloor \end{aligned}$$

and

$$\begin{aligned} s_i &= |\mathcal{G}| - i t_{\mathrm{u}}(\mathcal{G}), \quad i = 0, \ldots, r,\\ s_{r+i} &= \left\lfloor \frac{s_r}{2^i} \right\rfloor \quad i = 1, \ldots, \widetilde{r} - 1,\\ s_{r+\widetilde{r}} &= 0. \end{aligned}$$

We also define the stopping times

$$T_i = \min\{t \ge 1 : |\mathcal{U}(t)| \le s_i\}, \quad i = 1, \ldots, r + \widetilde{r}.$$

Lemma 4.1. *There exists constants C_4, C_5 such that for each $1 \le i \le r$ and all $x \in V(\mathcal{G})$, we have that*

$$\mathbf{P}_x\left[|\mathcal{U}(t)| > s_i\right] \le \exp\left(\frac{s_i}{t_{\mathrm{rel}}(\mathcal{G})}\left(C_4 \log|\mathcal{G}| - \frac{C_5}{|\mathcal{G}|} t\right)\right). \tag{4.1}$$

PROOF. For each $i \in \{1, \ldots, r\}$, we let

$$t_i = \frac{2(K_2+2)t_{\mathrm{u}}(\mathcal{G})|\mathcal{G}|}{s_i}$$

Proposition 3.1 implies that

$$\mathbf{P}_x[|\mathcal{U}(t+t_i)| \le s_{i+1} \mid |\mathcal{U}(t)| \in (s_{i+1}, s_i]] \ge 1 - \exp\left(-C\frac{t_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right).$$

Consequently, it follows that there exists independent variables $Z_j \sim \mathrm{GEO}(1-\exp(-Ct_{\mathrm{u}}(\mathcal{G})/t_{\mathrm{rel}}(\mathcal{G})))$ such that $T_j - T_{j-1}$ is stochastically dominated by $t_j Z_j$ for all $j \in \{1, \ldots, r\}$. Thus for $\theta_i > 0$, we have that

$$\begin{aligned} \mathbf{P}_x[|\mathcal{U}(t)| > s_i] = \mathbf{P}_x[T_i > t] = \mathbf{P}_x\left[\sum_{j=1}^{i} T_j - T_{j-1} > t\right] \\ \le e^{-\theta_i t}\prod_{j=1}^{i}\mathbf{E}_x[e^{\theta_i t_j Z_j}]. \end{aligned} \tag{4.2}$$

Note that for every $\beta \in (0,1)$ there exists $\alpha = \alpha(\beta) > 0$ such the moment generating function of a $\mathrm{GEO}(p)$ random variable satisfies

$$\frac{pe^x}{1-(1-p)e^x} \le e^{\alpha x} \text{ provided } (1-p)e^x \le \beta. \tag{4.3}$$

Choosing

$$\theta_i = \frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_i t_{\mathrm{rel}}(\mathcal{G})}$$

we have that

$$\theta_i t_j = \frac{Ct_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})} \cdot \frac{t_j}{t_i} = \frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_{\mathrm{rel}}(\mathcal{G})} \cdot \frac{s_i}{s_j}.$$

Hence as $s_i \le s_j$ for all $i, j \in \{1, \ldots, r\}$ with $j \le i$, we have

$$\exp\left(\frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_{\mathrm{rel}}(\mathcal{G})} \cdot \frac{s_i}{s_j} - \frac{Ct_{\mathrm{u}}(\mathcal{G})}{t_{\mathrm{rel}}(\mathcal{G})}\right) \le \exp\left(-\frac{Ct_{\mathrm{u}}(\mathcal{G})}{2t_{\mathrm{rel}}(\mathcal{G})}\right) \le \exp(-C/2).$$

Let $\alpha = \alpha(e^{-C/2})$ as in (4.3). Consequently, we can bound the product of exponential moments in (4.2) by

$$\begin{aligned} \log \prod_{j=1}^{i}\mathbf{E}_x[e^{\theta_i t_j Z_j}] \le \alpha\sum_{j=1}^{i}\theta_i t_j = \frac{\alpha C t_{\mathrm{u}}(\mathcal{G}) s_i}{2t_{\mathrm{rel}}(\mathcal{G})}\sum_{j=1}^{i}\frac{1}{s_j} \\ = \frac{\alpha C s_i}{2t_{\mathrm{rel}}(\mathcal{G})}\sum_{j=1}^{i}\frac{1}{|\mathcal{G}|/t_{\mathrm{u}}(\mathcal{G}) - j} \le \frac{\alpha C s_i}{2t_{\mathrm{rel}}(\mathcal{G})}\log|\mathcal{G}|. \end{aligned}$$

Inserting this expression into (4.2) gives (4.1). □

LEMMA 4.2. *There exists constants C_6, C_7 such that for all $1 \leq i \leq \widetilde{r}$ and $x \in V(\mathcal{G})$, we have that*

$$\begin{aligned}\mathbf{P}_x\left[|\mathcal{U}(t)| > s_{r+i}\right] \\ \leq \mathbf{P}_x[|\mathcal{U}(t/2)| > s_r] + \exp\left(s_{r+i-1}\left(C_6 i - \frac{C_7}{|\mathcal{G}|G^*(n^*)}t\right)\right).\end{aligned} \tag{4.4}$$

PROOF. Let

$$q_{r+j} = C_2|\mathcal{G}|G^*(s_{r+j})$$

where C_2 is as in Proposition 3.4. Proposition 3.4 implies that

$$\mathbf{P}_x[|\mathcal{U}(t+q_{r+j})| \leq s_{r+j+1} \mid |\mathcal{U}(t)| \in (s_{r+j+1}, s_{r+j}]] \geq 1 - \exp(-C_3 s_{r+j})$$

for $j \in \{1, \cdots, \widetilde{r}\}$. Consequently, there exists independent random variables $Z_{r+j} \sim \text{GEO}(1 - \exp(-C_3 s_{r+j}))$ such that $T_{r+j} - T_{r+j-1}$ is stochastically dominated by $q_{r+j} Z_{r+j}$. We have that

$$\begin{aligned}\mathbf{P}_x[|\mathcal{U}(t)| > s_{r+i}] = \mathbf{P}_x[T_{r+i} > t] \\ \leq \mathbf{P}_x\left[T_r > \frac{t}{2}\right] + \mathbf{P}_x\left[\sum_{j=1}^{i} T_{r+j} - T_{r+j-1} > \frac{t}{2}\right] =: I_1 + I_2\end{aligned} \tag{4.5}$$

Using that $I_1 = \mathbf{P}[|\mathcal{U}(t/2)| > s_r]$ gives the first term in (4.4). We now turn to bound I_2. Fixing $\theta_{r+i} > 0$, we have

$$I_2 \leq e^{-\theta_{r+i}t/2} \prod_{j=1}^{i} \mathbf{E}_x\left[e^{\theta_{r+i} q_{r+j} Z_{r+j}}\right]. \tag{4.6}$$

With the particular choice

$$\theta_{r+i} = \frac{C_3}{2C_2} \frac{s_{r+i}}{|\mathcal{G}|G^*(n^*)}$$

we have that

$$\exp(\theta_{r+i} q_{r+j} - C_3 s_{r+j}) \leq \exp(-C_3/2) =: \beta < 1.$$

Here, we used that if $n \leq m$ then $G^*(n) \leq G^*(m)$. Thus by (4.3) there exists $\alpha = \alpha(\beta) > 0$ such that we can bound the exponential moments in (4.6) by

$$\log \prod_{j=1}^{i} \mathbf{E}_x\left[e^{\theta_{r+i} q_{r+j} Z_{r+j}}\right] \leq \alpha \theta_{r+i} \sum_{j=1}^{i} q_{r+j} = \frac{\alpha C_3}{2} i s_{r+i}$$

Inserting this bound into (4.6) gives the second term in (4.4). □

LEMMA 4.3. *There are constants $C_8, C_9, C_{10} > 0$ such that for*

$$t = (1+a)C_8|\mathcal{G}|(t_{\rm rel}(\mathcal{G}) + \log|\mathcal{G}|)$$

and every $x \in V(\mathcal{G})$ we have

$$\mathbf{E}_x\left[2^{|\mathcal{U}(t)|}\right] \le 1 + C_9 \exp\left(-aC_{10}\log(n^*)\right). \tag{4.7}$$

PROOF. We can write

$$\mathbf{E}_x\left[2^{|\mathcal{U}(t)|}\right] \le 1 + \sum_{i=1}^{r+\widetilde{r}} 2^{s_{i-1}} \mathbf{P}\left[|\mathcal{U}(t)| > s_i\right].$$

For $i \le r$, we have that $s_{i-1} = s_i + t_{\rm u}(\mathcal{G})$. By Lemma 4.1, we have that

$$2^{s_i + t_{\rm u}(\mathcal{G})}\mathbf{P}[|\mathcal{U}(t)| > t] \le \exp\left((s_i + t_{\rm u}(\mathcal{G}))\log 2 + \frac{s_i}{t_{\rm rel}(\mathcal{G})}\left(C_4\log|\mathcal{G}| - \frac{C_5}{|\mathcal{G}|}t\right)\right).$$

By taking C_8 (in the statement) large enough, this is in turn bounded from above by

$$\exp\left(-as_i\left(1 + \frac{\log|\mathcal{G}|}{t_{\rm rel}(\mathcal{G})}\right)\right). \tag{4.8}$$

For $r + i \in \{r+1, \ldots, r+\widetilde{r}\}$ we have from (4.4) that

$$\begin{aligned}&2^{s_{r+i-1}}\mathbf{P}_x[|\mathcal{U}(t)| > s_{r+i}]\\ \le& 2^{s_{r+i-1}}\mathbf{P}_x[|\mathcal{U}(t)| > \tfrac{t}{2}] + \exp\left(s_{r+i-1}\left((C_6 + \log 2)i - \frac{C_7}{|\mathcal{G}|G^*(n^*)}t\right)\right).\end{aligned}$$

The first term admits the same bound as (4.8) with $i = r$, possibly by increasing C_8 if necessary. Using that $i \le \log_2|n^*|$, by increasing C_8 if necessary, from condition (C) it is easy to see that the second term admits the bound

$$\exp\left(-as_{r+i}\frac{\log|\mathcal{G}| + t_{\rm rel}(\mathcal{G})}{G^*(n^*)}\right). \tag{4.9}$$

Applying condition (C) again, we see that (4.9) is bounded from above by

$$\exp(-as_{r+i}\log(n^*)).$$

Putting together the estimates we get that for $i \in \{1 \ldots \widetilde{r}\}$

$$\begin{aligned}&2^{s_{r+i-1}}\mathbf{P}_x[|\mathcal{U}(t)| > s_{r+i}]\\ &\le \exp\left(-as_r\left(1 + \frac{\log|\mathcal{G}|}{t_{\rm rel}(\mathcal{G})}\right)\right) + \exp\left(-as_{r+i}\log(n^*)\right)\end{aligned} \tag{4.10}$$

Summing (4.8) and (4.10) gives (4.7) (the dominant term in the summation comes from when $s_{r+i} = 1$) which proves the lemma. □

PROOF OF THEOREM 1.3. This is a consequence of Lemma 4.3 and the relationship between $t_{\mathrm{u}}(\mathcal{G}^{\diamond})$ and $\mathbf{E}[2^{|\mathcal{U}(t)|}]$ given in (1.9). □

References.

[1] BRUMMELHUIS, M. AND HILHORST, H. (1991). Covering of a finite lattice by a random walk. *Physica A. 176*, 387–408.

[2] DEMBO, A., PERES, Y., ROSEN, J., AND ZEITOUNI, O. (2004). Cover times for Brownian motion and random walks in two dimensions. *Ann. Math.* **160**, 2, 433–464.

[3] DEMBO, A., PERES, Y., ROSEN, J., AND ZEITOUNI, O. (2006). Late points for random walk in two dimensions. *Ann. Probab. 34*, 219–263.

[4] DIACONIS, P. AND SALOFF-COSTE, L. (1993). Comparison techniques for random walk on finite groups. *Ann. Probab.* **21**, 4, 2131–2156. MR1245303 (95a:60009)

[5] DIACONIS, P. AND SALOFF-COSTE, L. (1996). Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.* **6**, 3, 695–750. MR1410112 (97k:60176)

[6] HÄGGSTRÖM, O. AND JONASSON, J. (1997). Rates of convergence of lamplighter processes. *Stochastic Processes and their Applications 67*, 227–249.

[7] LEON, C. A. AND PERRON, F. (2004). Optimal hoeffding bounds for discrete reversible Markov chains. *Annals of Applied Probability* **14**, 2, 958–970.

[8] LEVIN, D., PERES, Y., AND WILMER, E. (2008). *Markov Chains and Mixing Times.* American Mathematical Society.

[9] MILLER, J. AND PERES, Y. (2011). Uniformity of the uncovered set of random walk and cutoff for lamplighter chains. *Annals of Probability*.

[10] PERES, Y. AND REVELLE, D. (2004). Mixing times for random walks on finite lamplighter groups. *Electronic Journal of Probability 9*, 825–845.

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