THE MULTIPLICATIVE GOLDEN MEAN SHIFT HAS INFINITE HAUSDORFF MEASURE

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ABSTRACT. In an earlier work, joint with R. Kenyon, we computed the Hausdorff dimension of the "multiplicative golden mean shift" defined as the set of all reals in $[0,1]$ whose binary expansion (x_k) satisfies $x_k x_{2k} = 0$ for all $k \geq 1$. Here we show that this set has infinite Hausdorff measure in its dimension. A more precise result in terms of gauges in which the Hausdorff measure is infinite is also obtained.

1. INTRODUCTION

Consider the set

$$\Xi_G := \Big\{ x = \sum_{k=1}^{\infty} x_k 2^{-k} : \ x_k \in \{0,1\}, \ x_k x_{2k} = 0 \text{ for all } k \Big\}$$

which we call the "multiplicative golden mean shift". The reason for this term is that the set of binary sequences corresponding to the points of Ξ_G is invariant under the action of the semigroup of multiplicative positive integers $\mathbb{N}^*$: $M_r(x_k) = (x_{rk})$ for $r \in \mathbb{N}$. Fan, Liao, and Ma [3] showed that $\dim_M(\Xi_G) = \sum_{k=1}^{\infty} 2^{-k-1} \log_2 F_{k+1} = 0.82429\ldots$, where F_k is the k-th Fibonacci number: $F_1 = 1$, $F_2 = 2$, $F_{k+1} = F_{k-1} + F_k$, and raised the question of computing the Hausdorff dimension of Ξ_G.

Theorem 1.1 ([5, 6]). *We have* $\dim_H(\Xi_G) < \dim_M(\Xi_G)$. *In fact,*

$$\dim_H(\Xi_G) = -\log_2 p = 0.81137\ldots, \quad \text{where } p^3 = (1-p)^2, \quad 0 < p < 1. \tag{1.1}$$

Here we prove

Theorem 1.2. (i) *The set* Ξ_G *has infinite (not σ-finite) Hausdorff measure in its dimension. Moreover, let* $s = \dim_H(\Xi_G)$. *Then* $\mathcal{H}^{\phi}(\Xi_G) = \infty$ *for*

$$\phi(t) = t^s \exp\Big[-c \frac{|\log t|}{(\log|\log t|)^2} \Big] \tag{1.2}$$

provided that $c > 0$ is sufficiently small, and furthermore, Ξ_G is not σ-finite with respect to $\mathcal{H}^{\phi}$.
  (ii) *On the other hand, we have* $\mathcal{H}^{\psi_\theta}(\Xi_G) = 0$ *for*

$$\psi_\theta(t) = t^s \exp\Big[-\frac{|\log t|}{(\log|\log t|)^\theta} \Big], \tag{1.3}$$

provided that $\theta < 2$.

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Remarks. 1. In [6] we have pointed out a remarkable analogy between dimension properties of multiplicative shifts of finite type and self-affine carpets of Bedford and McMullen, see [1, 8], although we are not aware of any direct connection. The stated theorem provides further evidence of this: it exactly corresponds to Theorem 3 from the paper by the first-named author [9]. We should point out, however, that our proof requires many new elements; in particular, the recurrence relation from Lemma 3.2 below has no parallels in [9].

2. For self-affine carpets with non-uniform horizontal fibres, there is an elegant "soft" argument showing that the Hausdorff measure of the set in its dimension cannot be positive and finite [7], and more generally, this holds for any gauge [9]. It would be interesting to find a similar argument for the multiplicative golden mean shift as well.

3. We expect that similar results hold for other multiplicative shifts of finite type considered in [6]. Since the proofs are quite technical, we decided to focus on the most basic example of Ξ_G.

2. Preliminaries and the scheme of the proof

It is more convenient to work in the symbolic space $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$, with the metric

$$\varrho((x_k), (y_k)) = 2^{-\min\{n:\ x_n \neq y_n\}}.$$

It is well-known that the dimensions of a compact subset of $[0, 1]$ and the corresponding set of binary digit sequences in Σ_2 are equal (this is equivalent to replacing the covers by arbitrary interval with those by dyadic intervals), and the Hausdorff measures in the gauges that we are considering are comparable, up to a multiplicative constant. Thus, it suffices to work with the set X_G—the collection of all binary sequences (x_k) such that $x_k x_{2k} = 0$ for all k. Observe that

$$X_G = \left\{ \omega = (x_k)_{k=1}^{\infty} \in \Sigma_2 :\ (x_{i2^r})_{r=0}^{\infty} \in \Sigma_G \ \text{ for all } i \text{ odd} \right\} \tag{2.1}$$

where Σ_G is the usual (additive) golden mean shift:

$$\Sigma_G := \{(x_k)_{k=1}^{\infty} \in \Sigma_2,\ x_k x_{k+1} = 0, \ \text{ for all } k \geq 1\}.$$

We will use the Rogers-Taylor density theorem from [11]. We state it in the symbolic space Σ_2 where $[u]$ denotes the cylinder set of sequences starting with a finite "word" u and $x_1^n = x_1 \ldots x_n$. Given a continuous increasing function ϕ on $[0, \infty)$, with $\phi(0) = 0$, we consider the generalized Hausdorff measure with the gauge ϕ, denoted by $\mathcal{H}^{\phi}$, see e.g. [2, p.33] or [10, p.50] for the definition and basic properties.

Theorem 2.1 (Rogers and Taylor). *Let $\mathbb{P}$ be a finite Borel measure on Σ_2 and let Λ be a Borel set in Σ_2 such that $\mathbb{P}(\Lambda) > 0$. Let ϕ be any gauge function. If for all $x \in \Lambda$,*

$$\beta_1 \leq \liminf_{n \to \infty} \frac{\phi(2^{-n})}{\mathbb{P}[x_1^n]} \leq \beta_2 \tag{2.2}$$

(where β_1, β_2 may be zero or infinity), then

$$c_1 \beta_1 \mathbb{P}(\Lambda) \le \mathcal{H}^\phi(\Lambda) \le c_2 \beta_2 \mathbb{P}(\Lambda),$$

where c_1 and c_2 are positive and finite.

Corollary 2.2. *Let $\mathbb{P}$ be a finite Borel measure on Σ_2 and let Λ be a Borel set in Σ_2 such that $\mathbb{P}(\Lambda) > 0$. Let ϕ be any gauge function.*
   (i) *If for $\mathbb{P}$-a.e. $x \in \Lambda$*

$$\lim_{n \to \infty} (\log_2 \mathbb{P}[x_1^n] - \log_2 \phi(2^{-n})) = -\infty,$$

then $\mathcal{H}^\phi(\Lambda) = \infty$.
   (ii) *If for all $x \in \Lambda$*

$$\lim_{n \to \infty} (\log_2 \mathbb{P}[x_1^n] - \log_2 \phi(2^{-n})) = +\infty,$$

then $\mathcal{H}^\phi(\Lambda) = 0$.

For an odd i denote by $J(i) = \{2^r i\}_{r=0}^\infty$ the geometric progression with ratio 2 starting at i. Equation (2.1) says that $x \in X_G$ if and only if the "restriction" of x to every $J(i)$ belongs to Σ_G. We can define a measure on X_G by taking an infinite product of probability measures on each "copy" of Σ_G.

In order to compute $\dim_H(X_G)$, it was enough to take the same measure μ on each copy, see [5]. Given a probability measure μ on Σ_G, we define a probability measure on X_G by

$$\mathbb{P}_\mu[u] := \prod_{i \le n,\, i \text{ odd}} \mu[u|_{J(i)}], \tag{2.3}$$

where $u|_{J(i)}$ denotes the "restriction" of the word u to the subsequence $J(i)$. It was proved in [5, 6] that there is a unique probability measure μ on Σ_G such that $\dim_H(\mathbb{P}_\mu) = \dim_H(X_G)$. Denote by $\mu(r)$ the Markov (non-stationary) measure on Σ_G, with initial probabilities $(r, 1-r)$ and the matrix of transition probabilities $P = (P(i,j))_{i,j=0,1} = \begin{pmatrix} r & 1-r \\ 1 & 0 \end{pmatrix}$. Then $\mu = \mu(p)$, where $p^3 = (1-p)^2$. The measure $\mu(r)$ on cylinder sets can be explicitly written as follows:

$$\mu(r)([u_1 \ldots u_k]) = (1-r)^{N_1(u_1 \ldots u_k)} r^{N_0(u_1 \ldots u_k) - N_1(u_1 \ldots u_{k-1})}, \tag{2.4}$$

where $u \in \{0,1\}^k$ is a word admissible in Σ_G, i.e. if $u_j = 1$ then $u_{j+1} = 0$ for $j \le k-1$, and $N_i(u)$ denotes the number of symbols i in the word u. To verify (2.4), note that the probability of a 1 is always $1 - r$ (including the first position), and the probability of a 0 is r, except when it follows a 1, in which case its probability equals one.

For the lower bound, i.e. part (i) of Theorem 1.2, we have to "fine-tune" the measure $\mathbb{P}_\mu$ by taking a product of measures $\mu(p_k)$ on subsequences $J(i)$ with odd i such that $2^k \le i < 2^{k+1}$. It

is clear that we must have $\lim_{k\to\infty} p_k = p$; in fact, we will take $p_k = p + \frac{\delta}{k}$. More precisely, let

$$\mu_k = \mu(p_k), \quad \text{where} \ \ p_k = p + \frac{\delta}{k}, \ \ k \geq 1, \ \ p_0 = p, \tag{2.5}$$

and $\delta > 0$ is sufficiently small, so that $p_1 = p + \delta < 1$. Next, we define for $u \in \{0,1\}^n$, with $2^{\ell-1} < n \leq 2^\ell$,

$$\mathbb{P}_\delta[u] := \prod_{k=1}^{\ell} \ \prod_{\frac{n}{2^k} < i \leq \frac{n}{2^{k-1}}, \ i \ \text{odd}} \mu_{\ell-k}[u|_{J(i)}], \tag{2.6}$$

where $u|_{J(i)} = u_i \ldots u_{2^{k-1}i}$ is a word of length k. It is easy to see that $\mathbb{P}_\delta$ is a probability measure on X_G.

Without loss of generality we can (and will) use logarithms base 2 in (1.2) and (1.3). Theorem 1.2(i) immediately follows from Corollary 2.2(i) and the following proposition.

Proposition 2.3. *There exist constants $\delta > 0$ and $c > 0$ such that the measure $\mathbb{P}_\delta$ defined by (2.6) satisfies*

$$\lim_{n\to\infty} \left(\log_2 \mathbb{P}_\delta[x_1^n] - \log_2 \phi(2^{-n}) \right) = -\infty$$

for $\mathbb{P}_\delta$-a.e. $x \in X_G$, where ϕ is the gauge function from (1.2). Equivalently,

$$\lim_{n\to\infty} \left(\log_2 \mathbb{P}_\delta[x_1^n] + ns + \frac{cn}{(\log_2 n)^2} \right) = -\infty \tag{2.7}$$

for $\mathbb{P}_\delta$-a.e. $x \in X_G$, where $s = -\log_2 p = \dim_H(X_G)$.

For the upper bound of the Hausdorff measure, i.e. part (ii) of Theorem 1.2, it is enough to take the same measure $\mu = \mu(p)$ as in [5, 6], however, the proof is rather delicate; it follows the scheme of [9, Theorem 3(ii)], but with many modifications.

We will need a classical large deviation inequality, which we state in the generality needed for us.

Lemma 2.4 (Hoeffding's inequality [4]). *Let $\{X_i\}_{i\geq 1}$ be a sequence of independent random variables with expectation zero, such that $|X_i| \leq C$, and let $S_n = \sum_{i=1}^{n} X_i$. Then*

$$\mathbb{P}\left(S_n \geq tn \right) \leq \exp\left(-\frac{t^2 n}{2C^2} \right) \tag{2.8}$$

for all $t > 0$ and $n \geq 1$.

3. Lower estimates of Hausdorff measure

Here we prove Proposition 2.3. We start with a reduction.

Lemma 3.1. *If (2.7) holds for positive integers n satisfying*

$$n = 2^{\lfloor \ell/2 \rfloor} d, \quad \text{where} \ \ 2^{\ell-1} < n \leq 2^\ell, \ d \in \mathbb{N}, \tag{3.1}$$

with a constant $c > 0$, then (2.7) holds for all n with c replaced by $c/2$.

Proof. For a large integer $n \in (2^{\ell-1}, 2^{\ell}]$, let

$$d := \lfloor 2^{-\lfloor \ell/2 \rfloor} n \rfloor, \quad m := 2^{\lfloor \ell/2 \rfloor} d.$$

Then

$$n - \sqrt{2n} \le n - 2^{\lfloor \ell/2 \rfloor} < m \le n.$$

It is clear that m satisfies (3.1) (possibly with a different ℓ). Observe that

$$\log_2 \mathbb{P}_\delta[x_1^n] + ns + \frac{(c/2)n}{(\log_2 n)^2} \le \log_2 \mathbb{P}_\delta[x_1^m] + ms + \frac{cm}{(\log_2 m)^2} +$$
$$+ \quad s(n-m) + c\Big[\frac{n/2}{(\log_2 n)^2} - \frac{m}{(\log_2 m)^2}\Big].$$

Since

$$s(n-m) + c\Big[\frac{n/2}{(\log_2 n)^2} - \frac{m}{(\log_2 m)^2}\Big] \le s\sqrt{2n} + c\Big[\frac{n/2}{(\log_2 n)^2} - \frac{n - \sqrt{2n}}{(\log_2 n)^2}\Big] < 0$$

for large enough n, the claim follows. □   □

For $k \ge 1$ let α_k be the partition of Σ_G into cylinders of length k. For a measure μ on Σ_2 and a finite partition α, denote by $H^\mu(\alpha)$ the μ-entropy of the partition, with base 2 logarithms:

$$H^\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log_2 \mu(A).$$

Let n be such that (3.1) holds. In view of (2.6),

$$\log_2 \mathbb{P}_\delta[x_1^n] \le \sum_{k=1}^{\lfloor \ell/2 \rfloor} \sum_{\frac{n}{2^k} < i \le \frac{n}{2^{k-1}}, \ i \text{ odd}} \log_2 \mu_{\ell-k}[x_1^n|_{J(i)}]. \tag{3.2}$$

Note that $x_1^n|_{J(i)}$ is a word of length k for $i \in (n/2^k, n/2^{k-1}]$, with i odd, which is a beginning of a sequence in Σ_G. Thus, $[x_1^n|_{J(i)}]$ is an element of the partition α_k. The random variables $x \mapsto \log_2 \mu_{\ell-k}[x_1^n|_{J(i)}]$ are i.i.d for $i \in (n/2^k, n/2^{k-1}]$, with i odd, and their expectation equals $-H^{\mu_{\ell-k}}(\alpha_k)$, by the definition of entropy. Note that there are $n/2^{k+1}$ odds in $(n/2^k, n/2^{k-1}]$. It is easy to see from (2.4) and (2.5) that

$$\big|\log_2 \mu_{\ell-k}[x_1^n|_{J(i)}]\big| \le Ck, \tag{3.3}$$

for $i \in (n/2^k, n/2^{k-1}]$, with some $C > 0$, independent of n and k. Let

$$S_{n/2^{k+1}} := \sum_{\frac{n}{2^k} < i \le \frac{n}{2^{k-1}}, \ i \text{ odd}} \log_2 \mu_{\ell-k}[x_1^n|_{J(i)}]$$

and $S^*_{n/2^{k+1}} := S_{n/2^{k+1}} + \frac{n}{2^{k+1}} H^{\mu_{\ell-k}}(\alpha_k)$, be the corresponding sum of centered (zero expectation) random variables. Then we have, for $k = 1, \dots \lfloor \ell/2 \rfloor$, and any $\varepsilon \in (0, \frac{1}{2})$, using (3.3) in Hoeffding's

inequality (2.8):

$$\mathbb{P}_\delta\left(x:\ S_{n/2^{k+1}} > \frac{-n}{2^{k+1}}H^{\mu_{\ell-k}}(\alpha_k) + \left(\frac{n}{2^{k+1}}\right)^{1-\varepsilon}\right)$$

$$=\ \mathbb{P}_\delta\left(x:\ S^*_{n/2^{k+1}} > \left(\frac{n}{2^{k+1}}\right)^{1-\varepsilon}\right)$$

$$\leq\ \exp\left[-\frac{(n/2^{k+1})^{1-2\varepsilon}}{2C^2k^2}\right].$$

Denote $b_\varepsilon = \sum_{k=1}^\infty 2^{-(k+1)\varepsilon}$. Now it follows from (3.2) that

$$\mathbb{P}_\delta\left(x:\ \log_2\mathbb{P}_\delta[x_1^n] > -n\sum_{k=1}^{\lfloor\ell/2\rfloor}\frac{H^{\mu_{\ell-k}}(\alpha_k)}{2^{k+1}} + b_\varepsilon n^{1-\varepsilon}\right) \tag{3.4}$$

$$\leq\ \mathbb{P}_\delta\left(x:\ \sum_{k=1}^{\lfloor\ell/2\rfloor}S^*_{n/2^{k+1}} > \sum_{k=1}^{\lfloor\ell/2\rfloor}\left(\frac{n}{2^{k+1}}\right)^{1-\varepsilon}\right)$$

$$\leq\ \sum_{k=1}^{\lfloor\ell/2\rfloor}\mathbb{P}_\delta\left(x:\ S^*_{n/2^{k+1}} > \left(\frac{n}{2^{k+1}}\right)^{1-\varepsilon}\right)$$

$$\leq\ \sum_{k=1}^{\lfloor\ell/2\rfloor}\exp\left[-\frac{n^{1-2\varepsilon}}{2C^22^{(k+1)(1-2\varepsilon)}k^2}\right]$$

$$\leq\ \ell\exp\left[-\frac{C'}{\ell^2}\left(\frac{n}{2^{\lfloor\ell/2\rfloor+1}}\right)^{1-2\varepsilon}\right]$$

$$\leq\ \log_2(2n)\exp\left[-\frac{C'}{\log_2^2(2n)}\left(\frac{n}{8}\right)^{\frac{1}{2}-\varepsilon}\right], \tag{3.5}$$

where we used that $\sqrt{8n} > 2^{1+\lfloor\ell/2\rfloor}$ and $\ell \leq \log_2(2n)$ by (3.1) in the last step. Since the last expression is summable in n, it follows from Borel-Cantelli that for $\mathbb{P}_\delta$-a.e. $x \in X_G$, the event in parentheses in Equation (3.4) holds only for finitely many n. This is the set of full $\mathbb{P}_\delta$ measure for which we will prove (2.7), for n satisfying (3.1).

Below we let $H(r) = -r\log_2 r - (1-r)\log_2(1-r)$.

Lemma 3.2. *We have, for any $r \in (0,1)$ and the measure $\mu(r)$ defined by (2.4),*

$$H^{\mu(r)}(\alpha_k) = H(r)F_{k-1}(r),\ k \geq 1, \tag{3.6}$$

where $F_0(x) = 1$, $F_1(x) = 1 + x$, and

$$F_k(x) = 1 + xF_{k-1}(x) + (1-x)F_{k-2}(x),\ k \geq 2. \tag{3.7}$$

Moreover, the polynomials $F_k(x)$ can be expressed as follows:

$$F_k(x) = \frac{(x-1)^{k+2} - (k+2)x + (2k+3)}{(x-2)^2}, \quad k \geq 0. \tag{3.8}$$

Proof. For $k = 1$ the formula (3.6) is trivially true. For $k \geq 2$ we have

$$H^{\mu(r)}(\alpha_k) = H^{\mu(r)}(\alpha_1) + H^{\mu(r)}(\alpha_k|\alpha_1) = H(r) + H^{\mu(r)}(\alpha_k|\alpha_1).$$

By the definition of conditional entropy and the properties of Σ_G, we have

$$H^{\mu(r)}(\alpha_k|\alpha_1) = rH^{\mu(r)}(\alpha_{k-1}) + (1-r)H^{\mu(r)}(\alpha_{k-2}).$$

(We set $H^{\mu(r)}(\alpha_0) = 0$ here.) Indeed, 0 in Σ_G can be followed by an arbitrary element of Σ_G, and 1 is followed by 0 and then by an arbitrary element of Σ_G. Now (3.6) and (3.7) are easily checked by induction. The explicit formula for $F_k(x)$ was found using that

$$F_k(x) - F_{k-1}(x) = 1 - (1-x)(F_{k-1}(x) - F_{k-2}(x)),$$

and can also be checked by induction. $\qquad\qquad\square\qquad\qquad\qquad\qquad\square$

Since $\mu_{\ell-k} = \mu(p_{\ell-k})$, we have by (3.6):

$$\sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H^{\mu_{\ell-k}}(\alpha_k)}{2^{k+1}} = \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H(p_{\ell-k})F_{k-1}(p_{\ell-k})}{2^{k+1}}. \tag{3.9}$$

Recall that $p_{\ell-k} = p + \frac{\delta}{\ell-k}$. Next we write the Taylor estimate at p, such that $p^3 = (1-p)^2$. We have $p \approx 0.56984 > \frac{1}{2}$, so it suffices to consider $x \in (\frac{1}{2}, 1)$. Below C_i denote positive absolute constants. It follows from (3.8) that

$$|F_k(x)| \leq C_1 k, \quad |F'(x)| \leq C_2 k, \quad |F''(x)| \leq C_3 k, \quad x \in (1/2, 1), \quad k \geq 1. \tag{3.10}$$

Therefore,

$$\left| \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H(p_{\ell-k})F_{k-1}(p_{\ell-k})}{2^{k+1}} - \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H(p)F_{k-1}(p)}{2^{k+1}} - \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{(HF_{k-1})'(p)}{2^{k+1}} \cdot \frac{\delta}{\ell-k} \right|$$

$$\leq \quad C_4 \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{k}{2^{k+1}} \cdot \left(\frac{\delta}{\ell-k} \right)^2 \leq C_5 \frac{\delta^2}{\ell^2}. \tag{3.11}$$

Lemma 3.3. *We have*

$$\sum_{k=1}^{\infty} \frac{H(p)F_{k-1}(p)}{2^{k+1}} = s = -\log_2 p \tag{3.12}$$

and

$$\sum_{k=1}^{\infty} \frac{(HF_{k-1})'(p)}{2^{k+1}} = 0. \tag{3.13}$$

Proof. One can verify directly that $A(r) := H(r) \sum_{k=1}^{\infty} \frac{F_{k-1}(r)}{2^{k+1}} = \frac{2H(r)}{3-r}$, and this function achieves its maximum at p. Alternatively, this follows from [5], since $A(r)$ equals what was denoted $s(\mu)$ in [5], for $\mu = \mu(r)$. □   □

In view of (3.10), we have $|\sum_{k=\lfloor \ell/2 \rfloor+1}^{\infty} \frac{H(p)F_{k-1}(p)}{2^{k+1}}| \le C_6 \ell \cdot 2^{-\ell/2}$, hence (3.12) implies

$$\left| \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H(p)F_{k-1}(p)}{2^{k+1}} - s \right| \le C_6 \ell \cdot 2^{-\ell/2}. \tag{3.14}$$

Next, writing $\frac{1}{\ell-k} = \frac{1}{\ell} + \frac{k}{\ell^2} + \frac{k^2}{\ell^2(\ell-k)}$, we obtain

$$\sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{(HF_{k-1})'(p)}{2^{k+1}} \cdot \frac{\delta}{\ell-k}$$

$$= \frac{\delta}{\ell} \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{(HF_{k-1})'(p)}{2^{k+1}} + \frac{\delta}{\ell^2} \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{k(HF_{k-1})'(p)}{2^{k+1}} + \frac{\delta}{\ell^2} \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{k^2(HF_{k-1})'(p)}{2^{k+1}(\ell-k)}$$

$$=: \ S_1 + S_2 + S_3.$$

Using (3.10), by (3.13) we have

$$|S_1| \le \frac{\delta}{\ell} \left| \sum_{k=\lfloor \ell/2 \rfloor+1}^{\infty} \frac{(HF_{k-1})'(p)}{2^{k+1}} \right| \le C_7 \frac{\delta}{\ell} \cdot \frac{\ell}{2^{\ell/2}} = \frac{C_7 \delta}{2^{\ell/2}}, \tag{3.15}$$

and

$$|S_3| \le C_8 \frac{\delta}{\ell^3}. \tag{3.16}$$

Finally,

$$\left| S_2 - \frac{\delta}{\ell^2} \sum_{k=1}^{\infty} \frac{k(HF_{k-1})'(p)}{2^{k+1}} \right| \le C_9 \frac{\delta}{\ell^2} \cdot \frac{\ell^2}{2^{\ell/2}} = \frac{C_9 \delta}{2^{\ell/2}}. \tag{3.17}$$

Lemma 3.4. *We have*

$$\tau := \sum_{k=1}^{\infty} \frac{k(HF_{k-1})'(p)}{2^{k+1}} > 0.$$

The proof uses a (rigorous) numerical calculation, and we postpone it to the end of the section. Combining (3.9), (3.11), (3.14), (3.15), (3.16), and (3.17), we obtain

$$\left| \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H^{\mu_{\ell-k}}(\alpha_k)}{2^{k+1}} - s - \frac{\tau\delta}{\ell^2} \right| \le \frac{C_5 \delta^2}{\ell^2} + \frac{C_6 \ell}{2^{\ell/2}} + \frac{(C_7+C_9)\delta}{2^{\ell/2}} + \frac{C_8 \delta}{\ell^3}. \tag{3.18}$$

Now we can conclude the proof of the proposition. Let $x \in X_G$ be such that for all n sufficiently large, satisfying (3.1), we have

$$\log_2 \mathbb{P}_\delta[x_1^n] \leq -n \sum_{k=1}^{\lfloor \ell/2 \rfloor} \frac{H^{\mu_{\ell-k}}(\alpha_k)}{2^{k+1}} + b_\varepsilon n^{1-\varepsilon}.$$

Recall that this holds for $\mathbb{P}_\delta$-a.e. x by (3.5) and Borel-Cantelli Lemma. Then from (3.18) we obtain, keeping in mind that $n \in (2^{\ell-1}, 2^\ell]$:

$$
\begin{aligned}
\mathcal{S}_n(x) &:= \log_2 \mathbb{P}_\delta[x_1^n] + ns + \frac{cn}{(\log_2 n)^2} \\
&\leq \frac{cn}{(\log_2 n)^2} - \tau\delta \frac{n}{(\log_2 n)^2} + b_\varepsilon n^{1-\varepsilon} + \frac{C_5 \delta^2 n}{(\log_2 n)^2} \\
&+ C_6 \sqrt{n} \log_2 n + (C_7 + C_9)\delta\sqrt{n} + \frac{C_8 \delta n}{(\log_2 n)^3}.
\end{aligned}
$$

Now we choose a positive $\delta < \frac{\tau}{3C_5}$, which is possible by Lemma 3.4, so that $C_5 \frac{\delta^2 n}{(\log_2 n)^2} < \frac{1}{3} \frac{\tau \delta n}{(\log_2 n)^2}$, and then choose $c \in (0, \tau\delta/3)$, whence

$$\frac{cn}{(\log_2 n)^2} < \frac{1}{3} \cdot \tau\delta \frac{n}{(\log_2 n)^2}.$$

Then

$$\mathcal{S}_n(x) \leq -\frac{1}{3} \frac{\tau\delta n}{(\log_2 n)^2} + b_\varepsilon n^{1-\varepsilon} + C_6 \sqrt{n} \log_2 n + (C_7 + C_9)\delta\sqrt{n} + \frac{C_8 \delta n}{(\log_2 n)^3} \to -\infty,$$

as $n \to \infty$, and (2.7) follows. $\square$

Proof of Theorem 1.2(i). As already mentioned, $\mathcal{H}^\phi(\Xi_G) = \mathcal{H}^\phi(X_G) = \infty$ follows from the Rogers-Taylor density theorem (more precisely, from Corollary 2.2(i)). If $\mathcal{H}^\phi|_{\Xi_G}$ was σ-finite for some $c > 0$, we would have $\mathcal{H}^\phi(\Xi_G) = 0$ for all larger values of c, which is a contradiction. $\square$

Remark. It is clear, without any calculation, that there exists $\gamma > 0$, arbitrarily small, such that

$$\tau_\gamma := \sum_{k=1}^{\infty} \frac{k^{1+\gamma}(HF_{k-1})'(p)}{2^{k+1}} \neq 0.$$

This implies, by a minor modification of the argument, that $\mathcal{H}^{\phi_\gamma}(X_G) = \infty$ for the gauge function

$$\phi_\gamma(t) = t^s \exp\left[-c \frac{|\log t|}{(\log |\log t|)^{2+\gamma}}\right].$$

To this end, we need to take $p_k = p \pm \frac{\delta}{k^{1+\gamma}}$ in (2.5), where the sign is that of τ_γ. The details are left to the reader.

Proof of Lemma 3.4. A numerical calculation (we used *Mathematica*) showed that

$$\sum_{k=1}^{12} \frac{k(HF_{k-1})'(p)}{2^{k+1}} \approx 0.187469.$$

Thus, we only need to estimate the remainder.

We have $(HF_{k-1})'(p) = H(p)F'_{k-1}(p) + H'(p)F_{k-1}(p)$. Recall that $p \approx 0.56984$, and a calculation gives

$$H(p) \approx 0.68336 < 0.7, \quad H'(p) \approx -0.281198, \quad \text{hence} \quad |H'(p)| < 0.3.$$

Recall (3.8) that $F_n(x) = (x-2)^{-2}[(x-1)^{n+2} - (n+2)x + (2n+3)]$, whence

$$0 < F_n(p) < 2^{-(n+2)} - (n+2)/2 + (2n+3) < 3 + 3n/2.$$

Further,

$$F'_n(p) = \frac{2((p-1)^{n+2} - (n+2)p + (2n+3))}{(p-2)^3} + \frac{(n+2)(p-1)^{n+1} - (n+2)}{(p-2)^2}.$$

Note that in the expression for $F'_n(p)$ the 1st term is positive and the 2nd term is negative. The first term, in absolute value, is less than $2(3 + 3n/2) = 3n + 6$, and the second term, in absolute value, is less than $n + 3$ for $n \geq 1$. Thus,

$$|F'_n(p)| < 3n + 6.$$

It follows (using a crude estimate) that

$$\left| \sum_{k=13}^{\infty} \frac{k(HF_{k-1})'(p)}{2^{k+1}} \right| < \sum_{k=13}^{\infty} \frac{(0.7(3k+3) + 0.3(3k+3)/2)k}{2^{k+1}} < \sum_{k=13}^{\infty} \frac{3k(k+1)}{2^{k+1}}.$$

Finally,

$$\begin{aligned}
\sum_{k=13}^{\infty} \frac{3k(k+1)}{2^{k+1}} &= (3/4)[(1-x)^{-1}x^{15}]''|_{x=1/2} \\
&= (3/4)[2^{-11} + 30 \cdot 2^{-12} + 15 \cdot 14 \cdot 2^{-12}] < 0.1 < 0.187469,
\end{aligned}$$

completing the proof of the claim that $\sum_{k=1}^{\infty} \frac{k(HF_{k-1})'(p)}{2^{k+1}} > 0$. □   □

4. Upper bound for Hausdorff measure

First we give a short proof of a weaker result: $\mathcal{H}^{\psi}(X_G) = 0$ where

$$\psi(t) = t^s \exp\left[-\frac{|\log_2 t|}{g(\log_2 |\log_2 t|)}\right] \tag{4.1}$$

where g is increasing and $\int^{\infty} \frac{dt}{g(t)} = \infty$; in particular, this includes $\psi = \psi_\theta$ from (1.3) with $\theta = 1$.

Proof. We use the measure $\mathbb{P}_\mu$ from (2.3), where $\mu = \mu(p)$, $p^3 = (1-p)^2$, as in [5]. Consider **any** point $x \in X_G$. Then we obtain from (2.6), as in [5], for n even:

$$
\begin{aligned}
\mathbb{P}_\mu[x_1^n] &= (1-p)^{N_1(x_1^n)} p^{N_0(x_1^n)-N_1(x_1^{n/2})} \\
&= p^n p^{N_0(x_1^{n/2})-N_0(x_1^n)/2},
\end{aligned}
\tag{4.2}
$$

in view of $1 - p = p^{3/2}$, $N_1(x_1^n) = n - N_0(x_1^n)$. Note that $\log_2 \psi(2^{-n}) = -ns - \frac{n}{(\ln 2)g(\log_2 n)}$. In view of $s = -\log_2 p$, we have

$$
\frac{\log_2 \mathbb{P}_\mu[x_1^n] - \log_2 \psi(2^{-n})}{n} = \frac{s}{2}\left(\frac{N_0(x_1^{n/2})}{n/2} - \frac{N_0(x_1^n)}{n}\right) + \frac{1}{(\ln 2)g(\log_2 n)}.
\tag{4.3}
$$

Denote

$$
b_j := \frac{\log_2 \mathbb{P}_\mu[x_1^{2^j}] - \log_2 \psi(2^{-2^j})}{2^j} = \frac{s}{2}\left(\frac{N_0(x_1^{2^{j-1}})}{2^{j-1}} - \frac{N_0(x_1^{2^j})}{2^j}\right) + \frac{1}{(\ln 2)g(j)}.
$$

Then

$$
b_1 + \cdots + b_\ell = \frac{s}{2}\left(N_0(x_1^1) - \frac{N_0(x_1^{2^\ell})}{2^\ell}\right) + \sum_{j=1}^{\ell} \frac{1}{(\ln 2)g(j)} \to +\infty, \; \ell \to \infty,
$$

by the assumption on the function g. It follows that $\limsup 2^j b_j = +\infty$, hence

$$
\limsup_{n\to\infty}(\log_2 \mathbb{P}_\mu[x_1^n] - \log_2 \psi(2^{-n})) = +\infty,
$$

and we obtain $\mathcal{H}^\psi(X_G) = 0$ by Corollary 2.2(ii). □ □

Obtaining the same result for ψ_θ from (1.3) with $1 < \theta < 2$ is more delicate. Our proof follows the scheme of the proof of [9, Theorem 3(ii)], but we have to make a number of modifications. The following lemma is a version of [9, Lemma 5] in the form convenient for us.

Lemma 4.1. (i) *Let* $1 < \eta < 2$. *Suppose that* $\{\gamma(n)\}_{n=1}^\infty$ *is a real sequence such that*

$$
C_1 := \sup_n |\gamma(n) - \gamma(n-1)| < \infty
\tag{4.4}
$$

and for all $n \geq n_0$,

$$
\gamma(n) \geq \frac{\gamma(2n)}{2} + \frac{n}{(\log_2(2n))^\eta}.
\tag{4.5}
$$

Then either there exists $c > 0$ *such that for all* $n \geq n_0$

$$
\gamma(2n) \geq c\frac{2n}{(\log_2(2n))^{\eta-1}},
\tag{4.6}
$$

or there exists $\varepsilon > 0$ *such that for infinitely many* n,

$$
\gamma(2n) \leq -\varepsilon n \quad and \quad \gamma(n) - \frac{\gamma(2n)}{2} \leq \frac{n}{\log_2(2n)}.
\tag{4.7}
$$

(ii) *For any real sequence* $\{\gamma(n)\}_{n=1}^{\infty}$ *satisfying (4.4),*

$$\gamma(n) - \frac{\gamma(2n)}{2} < \frac{n}{\log_2(2n)} \tag{4.8}$$

for infinitely many n.

Proof. (i) Iterating (4.5) we obtain for $n \geq n_0$ and $m \geq 1$:

$$\gamma(n) \geq \frac{\gamma(2^m n)}{2^m} + n \cdot \sum_{j=1}^{m} \frac{1}{(j + \log_2 n)^\eta} . \tag{4.9}$$

Case 1: $\gamma(n) \geq 0$ for all $n \geq n_0$. Then (4.9) implies for $n \geq n_0$:

$$\gamma(n) \geq n \sum_{j=1}^{\infty} \frac{1}{(j + \log_2 n)^\eta} \geq c \frac{n}{(\log_2 n)^{\eta-1}} ,$$

whence (4.6) holds.

Case 2: there exists $n_1 \geq n_0$ such that $\gamma(n_1) < -\varepsilon < 0$. Then (4.5) implies $\gamma(2n_1) \leq 2\gamma(n_1) < -2\varepsilon$, and inductively, $\gamma(2^m n_1) \leq -2^m \varepsilon$ for all $m \geq 1$. Moreover, for infinitely many m we have

$$\gamma(2^{m-1} n_1) - \frac{\gamma(2^m n_1)}{2} \leq \frac{2^{m-1} n_1}{\log_2(2^m n_1)} ,$$

since otherwise,

$$\frac{\gamma(2^{m-1} n_1)}{2^{m-1}} - \frac{\gamma(2^m n_1)}{2^m} > \frac{n_1}{m + \log n_1}, \quad m \geq m_0 + 1,$$

and then taking the sum over m from $m_0 + 1$ to ℓ yields

$$\frac{\gamma(2^{m_0} n_1)}{2^{m_0}} - \frac{\gamma(2^{m_0+\ell} n_1)}{2^{m_0+\ell}} \to \infty, \quad \ell \to \infty,$$

which is a contradiction, since $|\gamma(i)| \leq C_1 i$ by (4.4). Thus, (4.7) holds for infinitely many $n = 2^m n_1$, as desired.

(ii) If the claim is not true, then (4.5) holds for $n \geq n_0$ with $\eta = 1$, for some $n_0 \in \mathbb{N}$. Then we obtain (4.9) with $\eta = 1$. But $\gamma(2^m n) \geq -C_1 2^m n$ by (4.4), and we get a contradiction letting $m \to \infty$. $\qquad\qquad\qquad\qquad\square\qquad\qquad\qquad\qquad\qquad\qquad\square$

We still use the measure $\mathbb{P}_\mu$ from (2.3), as in [5], so by (4.2), keeping in mind that $s = -\log_2 p$, we have

$$\log_2 \mathbb{P}_\mu[x_1^{2n}] + s(2n) = [N_0(x_1^{2n})/2 - N_0(x_1^n)]s. \tag{4.10}$$

Observe that

$$N_0(x_1^n) = \sum_{k=1}^{\ell+1} \sum_{\frac{n}{2^k} < i \leq \frac{n}{2^{k-1}},\ i \text{ odd}} N_0(x_1^n|_{J_i}), \quad 2^{\ell-1} < n \leq 2^\ell. \tag{4.11}$$

By the definition of the measure $\mathbb{P}_\mu$, the random variables $N_0(x_1^n|_{J_i})$ are i.i.d. for odd $i \in (\frac{n}{2^k}, \frac{n}{2^{k-1}}]$. Note that $|x_1^n|_{J_i}| = k$ for such i, and the distribution of these random variables is the distribution

of $N_0(u)$, $|u| = k$, where $\{u_i\}$ is the Markov chain corresponding to μ. By the definition of $\mu = \mu(p)$,

$$\mathbb{E}\big[N_0[u]\big] = \sum_{j=0}^{k-1} (\mathbf{p}P^j)_0, \quad |u| = k,$$

where $\mathbf{p} = (p, 1-p)$ and $P = \begin{pmatrix} p & 1-p \\ 1 & 0 \end{pmatrix}$. Since P has left eigenvectors $\boldsymbol{\pi} = (\frac{1}{2-p}, \frac{1-p}{2-p})$ and $\boldsymbol{\tau} = (1, -1)$ corresponding to the eigenvalues 1 and $p-1$, respectively, we have

$$(\mathbf{p}P^j)_0 = \frac{1}{2-p}\big[1 - (p-1)^{j+2}\big], \ j \geq 0,$$

hence

$$\mathbb{E}\big[N_0[u]\big] = \frac{k}{2-p} - \frac{1}{2-p} \sum_{j=0}^{k-1} (p-1)^{j+2} = \frac{k}{2-p} - \frac{(1-(p-1)^k)(p-1)^2}{(2-p)^2} =: L_k. \qquad (4.12)$$

Lemma 4.2. *We have*

$$\left| \frac{\mathbb{E}\big[N_0(x_1^{2n})\big]}{2} - \mathbb{E}\big[N_0(x_1^n)\big] \right| \leq C(\log_2 n)^2, \quad n \in \mathbb{N},$$

for some $C > 0$, where x has the law of $\mathbb{P}_\mu$.

Proof. Denote by $\mathbb{Z}_{\mathrm{odd}}(a, b]$ the set of odd integers in the interval $(a, b]$, where $a < b$ are reals. We have from (4.11) and (4.12):

$$\mathbb{E}\big[N_0(x_1^n)\big] = \sum_{k=1}^{\ell+1} \#\mathbb{Z}_{\mathrm{odd}}(\tfrac{n}{2^k}, \tfrac{n}{2^{k-1}}] \cdot L_k.$$

Note that $\mathbb{Z}_{\mathrm{odd}}(\frac{n}{2^{\ell+1}}, \frac{n}{2^\ell}] = \{1\}$ if $n = 2^\ell$, and it is empty otherwise. It follows that

$$\frac{\mathbb{E}\big[N_0(x_1^{2n})\big]}{2} - \mathbb{E}\big[N_0(x_1^n)\big] = \sum_{k=1}^{\ell+1} \Big(\frac{\#\mathbb{Z}_{\mathrm{odd}}(\frac{n}{2^{k-1}}, \frac{n}{2^{k-2}}]}{2} - \#\mathbb{Z}_{\mathrm{odd}}(\tfrac{n}{2^k}, \tfrac{n}{2^{k-1}}] \Big) \cdot L_k + d \cdot L_{\ell+2}, \quad (4.13)$$

where $d \in \{0, 1/2\}$. It is easy to see that

$$\left| \#\mathbb{Z}_{\mathrm{odd}}(a, b] - \Big(\frac{b-a}{2}\Big) \right| \leq 1, \quad \text{for } \ 0 < a < b, \qquad (4.14)$$

hence, taking (4.12) into account,

$$\left| \frac{\mathbb{E}\big[N_0(x_1^{2n})\big]}{2} - \mathbb{E}\big[N_0(x_1^n)\big] \right| \leq 2 \sum_{k=1}^{\ell+2} L_k \leq C' \sum_{k=1}^{\ell+2} k \leq C'' \ell^2 \leq C(\log_2 n)^2.$$

$$\square$$

$\square$

Proof of Theorem 1.2(ii). In order to show that $\mathcal{H}^{\psi_\theta}(X_G) = 0$, we cover X_G by three subsets: B, L, and Λ, defined as follows. Let

$$B := \left\{ x \in X_G : \exists\, \eta > \theta, \ \frac{N_0(x_1^{2n})}{2} - N_0(x_1^n) > \frac{-2n}{(\log_2(2n))^\eta} \text{ for infinitely many } n \right\}. \quad (4.15)$$

Denote

$$N_0^*(x_1^n) := N_0(x_1^n) - \mathbb{E}\big[N_0(x_1^n)\big]$$

and let

$$L := \Big\{ x \in X_G : \ \exists\, \varepsilon > 0, \quad \frac{N_0(x_1^{2n})}{2} - N_0(x_1^n) \geq \frac{-2n}{\log_2(2n)} \quad \text{and}$$
$$N_0^*(x_1^{2n}) \leq -\varepsilon n \quad \text{for infinitely many } n \Big\}. \quad (4.16)$$

Finally, let $\Lambda = X_G \setminus (L \cup B)$. It suffices to verify that each of the three sets B, L, Λ has zero $\mathcal{H}^{\psi_\theta}$-measure (indeed, L and Λ even have zero $\mathcal{H}^s$-measure).

Step 1: $\mathcal{H}^{\psi_\theta}(B) = 0$. Let B_η be the set of $x \in X_G$ such that the condition in (4.15) holds for a fixed η. Thus, $B = \bigcup_{\eta>\theta} B_\eta = \bigcup_{\eta\in\mathbb{Q},\ \eta>\theta} B_\eta$, and it is enough to show that $\mathcal{H}^{\psi_\theta}(B_\eta) = 0$. We have from (4.10) and the definition of ψ_θ for all $x \in B_\eta$:

$$\begin{aligned} \log_2 \mathbb{P}_\mu[x_1^{2n}] - \log_2 \psi_\theta(2^{-2n}) &= s\Big(\frac{N_0(x_1^{2n})}{2} - N_0(x_1^n) \Big) + \frac{2n}{(\ln 2)(\log_2(2n))^\theta} \\ &> \frac{-2ns}{(\log_2(2n))^\eta} + \frac{2n}{(\ln 2)(\log_2(2n))^\theta} \end{aligned}$$

for infinitely many n. Since $\eta > \theta$, it follows that

$$\limsup_{n\to\infty} \big(\log_2 \mathbb{P}_\mu[x_1^{2n}] - \log_2 \psi_\theta(2^{-2n})\big) = +\infty,$$

hence $\mathcal{H}^{\psi_\theta}(B_\eta) = 0$ by Theorem 2.1.

Step 2: $\mathcal{H}^{\psi_\theta}(L) = 0$. Denote by $L(\varepsilon)$ the set of points $x \in X_G$ which satisfy the condition in (4.16) for a given $\varepsilon > 0$. For $\varepsilon > 0$ and $n \in \mathbb{N}$ let $\mathcal{L}_n(\varepsilon)$ be the set of words u of length $2n$ for which the condition in (4.16) holds. (Note that this condition depends only on the first $2n$ symbols of x; thus, (4.16) holds for all $x \in [u]$.) If $u \in \mathcal{L}_n(\varepsilon)$ then by (4.10) and (4.16),

$$\log_2(\mathbb{P}_\mu[u]\, 2^{2ns}) \geq \frac{-2sn}{\log_2(2n)},$$

hence

$$2^{-2ns} \leq \exp\left(\frac{2sn(\ln 2)}{\log_2(2n)} \right) \mathbb{P}_\mu[u]. \quad (4.17)$$

By the definition of $\mathcal{L}_n(\varepsilon)$ we have

$$\sum_{u\in\mathcal{L}_n(\varepsilon)} \mathbb{P}_\mu[u] \leq \mathbb{P}_\mu\big(x : \ N_0^*(x_1^{2n}) \leq -\varepsilon n\big). \quad (4.18)$$

The following lemma is a consequence of large deviation estimates; it will be used in the last step of the proof as well.

Lemma 4.3. *There exist $c_2, c_3 > 0$ such that for all $t > 0$ and $n \in \mathbb{N}$,*

$$\mathbb{P}_\mu\big(x : \ |N_0^*(x_1^{2n})| \geq tn\big) \leq c_2 \exp(-c_3 t^2 n).$$

Proof. We have by (4.11),

$$N_0^*(x_1^{2n}) = \sum_{k=1}^{\ell+1} S_{A_k}^*, \quad \text{where} \quad A_k = \#\mathbb{Z}_{\text{odd}}(\tfrac{n}{2^k}, \tfrac{n}{2^{k-1}}],$$

$$S_{A_k}^* := \sum_{\frac{n}{2^k} < i \leq \frac{n}{2^{k-1}}, \ i \text{ odd}} N_0^*(x_1^n|_{J_i}),$$

and

$$N_0^*(x_1^n|_{J_i}) = N_0(x_1^n|_{J_i}) - \mathbb{E}[N_0(u)] \quad \text{for} \quad |u| = k \quad \text{and} \quad i \in \mathbb{Z}_{\text{odd}}(\tfrac{n}{2^k}, \tfrac{n}{2^{k-1}}].$$

Now,

$$\mathbb{P}_\mu\Big(\Big|\sum_{k=1}^{\ell+1} S_{A_k}^*\Big| \geq tn\Big) \leq \sum_{k=1}^{\ell+1} \mathbb{P}_\mu\Big(\Big|S_{A_k}^*\Big| \geq \frac{tn}{k(k+1)}\Big),$$

since $\sum_{k=1}^\infty \frac{1}{k(k+1)} = 1$. Note that $S_{A_k}^*$ is a sum of A_k independent random variables, which are bounded by k in modulus, hence by Hoeffding's inequality (2.8),

$$\mathbb{P}_\mu\Big(\Big|S_{A_k}^*\Big| \geq \frac{tn}{k(k+1)}\Big) = \mathbb{P}_\mu\Big(|S_{A_k}^*| \geq A_k \cdot \frac{tn}{k(k+1)A_k}\Big)$$

$$\leq 2\exp\Big[-\frac{t^2 n^2}{2k^4(k+1)^2 A_k}\Big].$$

Observe that $A_k \leq \frac{n}{2^{k+1}} + 1 \leq \frac{n}{2^k}$ by (4.14), hence

$$\mathbb{P}_\mu\Big(\Big|S_{A_k}^*\Big| \geq \frac{tn}{k(k+1)}\Big) \leq 2\exp\Big[-\frac{t^2 n \cdot 2^k}{2k^4(k+1)^2}\Big],$$

and, therefore,

$$\mathbb{P}_\mu\big(x : \ |N_0^*(x_1^{2n})| \geq tn\big) \leq 2\sum_{k=1}^\infty \exp\Big[-\frac{t^2 n \cdot 2^k}{2k^4(k+1)^2}\Big] \leq c_2 \exp(-c_3 t^2 n),$$

for some positive c_2, c_3, as desired. $\square$ $\square$

Combining (4.17), (4.18) and Lemma 4.3, with $t = \varepsilon$, yields

$$\sum_{u \in \mathcal{L}_n(\varepsilon)} (2^{-2n})^s \leq c_2 \exp\Big[\frac{2sn(\ln 2)}{\log_2(2n)} - c_3\varepsilon^2 n\Big].$$

The right-hand side of this inequality is summable in n, so by choosing large n_0 we can make the sum

$$\sum_{n \geq n_0} \sum_{u \in \mathcal{L}_n(\varepsilon)} (2^{-2n})^s = \sum_{n \geq n_0} \sum_{u \in \mathcal{L}_n(\varepsilon)} (\mathrm{diam}[u])^s$$

arbitrarily small. But for any n_0, the union $\bigcup_{n=n_0}^\infty \bigcup_{u \in \mathcal{L}_n(\varepsilon)} [u]$ forms a cover of $L(\varepsilon)$, proving that $\mathcal{H}^s(L(\varepsilon)) = 0$. Finally, $L = \bigcup_{\varepsilon \in \mathbb{Q}} L(\varepsilon)$, so we obtain that $\mathcal{H}^s(L) = 0$ and certainly $\mathcal{H}^{\psi_\theta}(L) = 0$.

Step 3. For $\eta \in (1,2)$, $\varepsilon \in (0, \eta)$, and $c > 0$, let $\Lambda(\eta, \varepsilon, c)$ be the set of $x \in X_G$ such that for n sufficiently large we have

$$\frac{N_0(x_1^{2n})}{2} - N_0(x_1^n) \leq \frac{-2n}{(\log_2(2n))^\eta} \tag{4.19}$$

and

$$N_0^*(x_1^{2n}) \geq c \frac{2n}{(\log_2(2n))^{\eta-1}}, \tag{4.20}$$

but for infinitely many n,

$$\frac{N_0(x_1^{2n})}{2} - N_0(x_1^n) > \frac{-2n}{(\log_2(2n))^{\eta-\varepsilon}}. \tag{4.21}$$

By Lemma 4.1(ii), applied to $\{N_0(x_1^n)\}_{n \geq 1}$, (4.21) certainly holds for $\varepsilon = \eta - 1$.

We claim that

$$X_G \setminus (B \cup L) \subset \bigcup_{\eta \in (1,2)} \bigcup_{c > 0} \bigcup_{\varepsilon \in (0, 2-\eta)} \Lambda(\eta, \varepsilon, c). \tag{4.22}$$

Indeed, for $x \in X_G \setminus B$ let η^* be the infimum of η for which (4.19) holds for n sufficiently large (note that $x \notin B$ means such η exists). Then $\eta^* \in [1, 2)$ by Lemma 4.1(ii), and (4.19) holds with $\eta = \eta^* + \frac{2-\eta^*}{3}$ for n sufficiently large, whereas (4.21) holds for $\varepsilon \in (\eta - \eta^*, 2 - \eta) = (\frac{2-\eta^*}{3}, \frac{2(2-\eta^*)}{3})$. Let

$$\gamma(n) := N_0^*(x_1^n), \quad n \geq 1.$$

It is clear tthat

$$|\gamma(n+1) - \gamma(n)| \leq 2, \quad n \geq 1.$$

It follows from (4.19) and Lemma 4.2 that

$$\gamma(n) - \frac{\gamma(2n)}{2} \geq \frac{2n}{(\log_2(2n))^\eta} - C(\log_2 n)^2 \geq \frac{n}{(\log_2(2n))^\eta}$$

for n sufficiently large. Thus, the sequence $\{\gamma(n)\}_{n \geq 1}$ satisfies the assumptions of Lemma 4.1(i). By Lemma 4.1(i), either there exists $c > 0$ such that for all n sufficiently large

$$N_0^*(x_1^{2n}) \geq c \frac{2n}{(\log_2(2n))^{\eta-1}},$$

which together with the above yields that $x \in \Lambda(\eta, \varepsilon, c)$, or else there exists $\varepsilon > 0$ such that for infinitely many n,

$$N_0^*(x_1^{2n}) \leq -\varepsilon n \quad \text{and} \quad N_0^*(x_1^n) - \frac{N_0^*(x_1^{2n})}{2} \leq \frac{n}{\log_2(2n)},$$

hence by Lemma 4.2,

$$N_0(x_1^n) - \frac{N_0(x_1^{2n})}{2} \leq \frac{n}{\log_2(2n)} + c(\log_2 n)^2 \leq \frac{2n}{\log_2(2n)}$$

for infinitely many n, so that $x \in L$, proving the claim. Since the union in (4.22) can be taken over rational η, c, ε, it suffices to show that $\mathcal{H}^s(\Lambda(\eta, \varepsilon, c)) = 0$ for $\varepsilon \in (0, 2 - \eta)$.

Let $\Gamma_n(\eta, \varepsilon, c)$ be the collection of words u of length $2n$ for which (4.19), (4.20), and (4.21) hold (as before, this is well defined). If $u \in \Gamma_n(\eta, \varepsilon, c)$, then by (4.10) and (4.21)

$$\log_2(\mathbb{P}_\mu[u] \, 2^{2ns}) \geq \frac{-2ns}{(\log_2(2n))^{\eta-\varepsilon}},$$

hence

$$2^{-2ns} \leq \exp\Big(\frac{2ns(\ln 2)}{\log_2(2n)^{\eta-\varepsilon}}\Big) \mathbb{P}_\mu[u]. \tag{4.23}$$

By the definition of $\Gamma_n(\eta, \varepsilon, c)$ and Lemma 4.3, with $t = \frac{2c}{(\log_2(2n))^{\eta-1}}$,

$$\sum_{u \in \Gamma_n(\eta, \varepsilon, c)} \mathbb{P}_\mu[u] \;\leq\; \mathbb{P}_\mu\Big(x: \; N_1^*(x_1^{2n}) \geq c\frac{2n}{(\log_2(2n))^{\eta-1}}\Big)$$

$$\leq\; c_2 \exp\Big(-\frac{\widetilde{c}n}{(\log_2(2n))^{2\eta-2}}\Big),$$

with $\widetilde{c} = 4c_3 c^2$. Combining this with (4.23) yields

$$\sum_{u \in \Gamma_n(\eta, \varepsilon, c)} 2^{-2ns} \leq \exp c_2\Big[\frac{2ns(\ln 2)}{\log_2(2n)^{\eta-\varepsilon}} - \frac{\widetilde{c}n}{(\log_2(2n))^{2\eta-2}}\Big].$$

Recall that $\varepsilon < 2 - \eta$, and therefore $\eta - \varepsilon > 2\eta - 2$ and the right-hand side of the last inequality is summable in n. It follows that, by taking n_1 sufficiently large, the sum

$$\sum_{n \geq n_1} \sum_{u \in \Gamma_n(\eta, \varepsilon, c)} 2^{-2ns}$$

can be made arbitrarily small. Since for any n_1, the union

$$\bigcup_{n \geq n_1} \bigcup_{u \in \Gamma_n(\eta, \varepsilon, c)} [u]$$

covers $\Lambda(\eta, \varepsilon, c)$, this implies $\mathcal{H}^s(\Lambda(\eta, \varepsilon, c)) = 0$ (and hence $\mathcal{H}^{\psi_\theta}(\Lambda(\eta, \varepsilon, c)) = 0$), completing the proof. $\qquad \square \qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

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