DIMENSIONS OF SOME FRACTALS DEFINED VIA THE SEMIGROUP GENERATED BY 2 AND 3

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ABSTRACT. We compute the Hausdorff and Minkowski dimension of subsets of the symbolic space $\Sigma_m = \{0, ..., m-1\}^{\mathbb{N}}$ that are invariant under multiplication by integers. The results apply to the sets $\{x \in \Sigma_m : \forall k, \ x_k x_{2k} \cdots x_{nk} = 0\}$, where $n \geq 3$. We prove that for such sets, the Hausdorff and Minkowski dimensions typically differ.

1. INTRODUCTION

Let $m \geq 2$ be an integer. A widely studied issue in dynamics consists in computing the (Hausdorff, Minkowski,...) dimensions of subsets X of the symbolic space $\Sigma_m = \{0, \ldots, m-1\}^{\mathbb{N}}$. When X is a closed subset of Σ_m, invariant under the shift $x \mapsto mx$, by a well-known result of Furstenberg [5], both the Hausdorff and Minkowski dimensions of X coincide with the topological entropy of the shift on X divided by $\log m$. This theorem covers a lot of interesting examples. Unfortunately, as soon as the set is not invariant any more, many standard techniques fail, the most basic example of which is

$$X_2 = \{x = (x_k)_{k \geq 1} \in \Sigma_2 : \forall k \geq 1, \ x_k x_{2k} = 0\}.$$

In [8], the dimensions of X_2 were computed. In particular, it is shown that the Hausdorff dimension of X_2 is strictly smaller than its Minkowski dimension, this being a reflection of the "non-self-similarity" resulting from its definition.

In [8], the key property used to study X_2 is that this set, though not invariant under the shift, is nevertheless invariant under the action of multiplicative integers. More precisely,

$$x = (x_k)_{k \geq 1} \in X_2 \implies \forall i \in \mathbb{N}, \ (x_{in})_{n \geq 1} \in X_2.$$

In [8] the Hausdorff dimension of X_2 and of many more general sets invariant under the action of multiplicative integers were computed.

Date: June 22, 2012.

The techniques developed in [8], however, do not allow directly to determine the dimensions of sets such as

$$X_{2,3} = \{x = (x_n)_{n\geq 1} \in \Sigma_2 : \forall k \geq 1, \ x_k x_{2k} x_{3k} = 0\},$$

which is also invariant under the multiplication by integers. Roughly speaking, [8] relied on the fact that the condition $(x_j)_{j\geq 1} \in X_2$ "splits" into independent conditions along geometric progressions of ratio 2, namely, that the sequence $(x_{i2^k})_{k\geq 0}$ contains no two consecutive 1's for any odd i. In order to understand the structure of $X_{2,3}$, we will need to work with the semigroup generated by 2 and 3 instead of the cyclic semigroup $\{2^k\}_{k\geq 0}$.

Finding the dimensions of sets like $X_{2,3}$ is related to the general question of *multiple ergodic averages*: let $T : X \to X$ be a dynamical system, and $f : X^\ell \to \mathbb{R}$ a Hölder continuous potential ($\ell \geq 1$ being an integer). Classical questions concern the possible limits, for $x \in X$, of the multiple ergodic averages defined by

$$(1) \qquad\qquad S_n\, f(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x, T^{2k} x, ..., T^{\ell k} x),$$

when n goes to infinity. Furstenberg, see [6], introduced such non-conventional ergodic sums in his proof of the existence of arithmetic progressions of arbitrary length in sets of positive density (Szémeredi Theorem). A natural extension of classical multifractal analysis consists in investigating the (dimensions of the) sets

$$E_f(\alpha) := \{x : \lim_{n \to +\infty} S_n\, f(x) = \alpha\}.$$

These questions have been investigated by many authors, see e.g. [1, 7], and more recently in [10, 3, 8, 11, 4]. Our set $X_{2,3}$ is contained in, and can be shown to have the same dimension as, the set $E_f(0)$ in the simple case where $\ell = 3$ and $f(x, y, z) = x_1 y_1 z_1$. In fact, the paper [3], which raised the question of computing the Hausdorff dimension of X_2, was the motivation for [8], where $\dim_H(X_2)$ was determined. In turn, the authors of [4], building in part on [8], were able to compute the multifractal spectrum of certain "double" ergodic averages, that is, when $\ell = 2$ in (1). (Independently, some special cases were handled in [11].) We hope that the methods developed in the current paper will make it possible to perform a similar analysis for an arbitrary $\ell \in \mathbb{N}$.

The goal of this paper is to understand the structure of the sets, such as

$$(2) \qquad X^{(m)}_{n_1,\dots,n_r} := \{(x_k)_{k=1}^\infty \in \Sigma_m : \ x_{kn_1} x_{kn_2} \cdots x_{kn_r} = 0, \ \forall\, k \in \mathbb{N}\}\,,$$

where $n_1,\dots,n_r$ are arbitrary distinct positive integers, in particular,

$$(3) \qquad X_{2,3,\dots,n} = X^{(2)}_{2,3,\dots,n} = \{(x_k)_{k=1}^\infty \in \Sigma_2 : \ x_k x_{2k} \cdots x_{nk} = 0, \ \forall\, k \in \mathbb{N}\}\,.$$

First, we represent the Minkowski dimension as the sum of a series. For the Hausdorff dimension, we obtain, on the one hand, a variational formula; and on the other hand, a formula based on a system of nonlinear equations on an infinite tree. This tree has levels naturally indexed by a sub-semigroup of the multiplicative positive integers (e.g. the semigroup generated by 2 and 3). The formulas are complicated (more so than in [8]), but this seems unavoidable. In any case, they allow reasonably accurate numerical estimates. Perhaps more importantly, they yield a qualitative result: the Hausdorff dimension is strictly less than the Minkowski dimension for all sets of the form (2).

The paper is organized as follows. In Section 2 we present precise statements of the results (Theorems 2.1 and 2.2). Section 3 contains some preliminary results, Sections 4, 5 and 6 give the proofs: the Minkowski dimension in Section 4, and the lower and upper bound for the Hausdorff dimension in Sections 5 and 6 respectively. Finally, Section 7 contains some numerical estimations and further examples.

2. STATEMENT OF RESULTS

Let $J \geq 2$ be an integer. Consider the semigroup $S = \langle p_1,\dots,p_J \rangle$ generated by distinct primes $p_1,\dots,p_J$. Denote by ℓ_k the k-th element of S, so that

$$S = \{\ell_k\}_{k=1}^\infty, \quad 1 = \ell_1 < \ell_2 < \dots$$

Notation. We write $(i, S) = 1$ if and only if $p_j \nmid i$ for all $j \leq J$ (in other words, i is mutually prime with all elements of S). Observe that

$$(4) \qquad \mathbb{N} = \bigsqcup \{iS : \ (i, S) = 1\}$$

is a disjoint union.

To each element $x = (x_k)_{k=1}^\infty \in \Sigma_m$, one can associate the subsequence $x|_{iS}$, viewed as an element of Σ_m, defined as

$$x|_{iS} := (x_{i\ell_k})_{k=1}^\infty.$$

Given a closed subset $\Omega \subset \Sigma_m$ let

$$(5) \qquad X_\Omega^{(S)} := \left\{ x = (x_k)_{k=1}^\infty \in \Sigma_m : \ x|_{iS} \in \Omega \ \text{ for all } i, \ (i, S) = 1 \right\}.$$

In this article, we obtain formulas for the Minkowski and Hausdorff dimensions of $X_\Omega^{(S)}$. Note that the case when $J = 1$ (the semigroup S is cyclic) was considered in [8]. Our main example is the set from the Introduction

$$X_{2,3} := \{(x_k)_{k=1}^\infty \in \Sigma_2 : \ x_k x_{2k} x_{3k} = 0, \ \forall\, k \in \mathbb{N}\} = X_\Omega^{(S)},$$

for which S is the semigroup generated by 2 and 3, and

$$\Omega = \{(\omega_k)_{k=1}^\infty \in \Sigma_2 : \ \forall\, i \geq 1, \ \omega_i \omega_j \omega_k = 0 \text{ if } 2\ell_i = \ell_j, \ 3\ell_i = \ell_k\}.$$

More generally, the sets $X_{2,3,\ldots,n}$ defined by (3), correspond to the case where S is the semigroup generated by all primes less than or equal to n and

$$\Omega = \left\{(\omega_k)_{k=1}^\infty \in \Sigma_2 : \ \omega_{i_1} \omega_{i_2} \cdots \omega_{i_n} = 0 \text{ if } j\ell_{i_1} = \ell_{i_j}, \ j = 1, \ldots, n\right\}.$$

Even more generally, our set-up includes the sets defined in (2):

$$X_{n_1,\ldots,n_r}^{(m)} = \{(x_k)_{k=1}^\infty \in \Sigma_m : \ x_{kn_1} x_{kn_2} \cdots x_{kn_r} = 0, \ \forall\, k \in \mathbb{N}\}$$

for arbitrary $n_1, \ldots, n_r \in \mathbb{N}$. In fact, $X_{n_1,\ldots,n_r}^{(m)} = X_\Omega^{(S)}$, where S is the semigroup generated by all prime factors of the numbers $n_1, \ldots, n_r$ and

$$\Omega = \left\{(\omega_k)_{k=1}^\infty \in \Sigma_m : \ \omega_{i_1} \omega_{i_2} \cdots \omega_{i_r} = 0 \text{ if } n_j \ell_{i_1} = \ell_{i_j}, \ j = 1, \ldots, r\right\}.$$

Throughout the paper, we fix the standard metric on Σ_m:

$$\varrho((x_k)_{k\geq 1}, (y_k)_{k\geq 1}) = m^{-\min\{n:\ x_n \neq y_n\}}.$$

All the dimensions are computed with respect to this metric. It is well-known that if we map Σ_m onto $[0, 1]$ via the base-m expansion, the dimensions of a subset of (Σ_m, ϱ) and its image on the real line (with respect to the Euclidean metric) are the same.

Next we continue with the general set-up and consider the tree of prefixes of the set Ω. It is a directed graph $\Gamma = \Gamma(\Omega)$ whose set of vertices is

$$V(\Gamma) = \mathrm{Pref}(\Omega) = \bigcup_{k=0}^\infty \mathrm{Pref}_k(\Omega),$$

where $\mathrm{Pref}_0(\Omega)$ has only one element, the empty word $\varnothing$, and

$$\mathrm{Pref}_k(\Omega) = \left\{ u \in \{0, \ldots, m-1\}^k, \ \Omega \cap [u] \neq \emptyset \right\}.$$

Here and below we denote by $[u]$ the cylinder set of all sequences starting with u. There is a directed edge from a prefix u to a prefix v if $v = ui$ for some $i \in \{0, \ldots, m-1\}$. In addition, there is an edge from $\varnothing$ to every $i \in \text{Pref}_1(\Omega)$. Clearly, $\Gamma(\Omega)$ is a tree, and there is at least one edge going out of every vertex. Denote

$$(6) \qquad\qquad A_k = |\text{Pref}_k(\Omega)|.$$

Let

$$(7) \qquad\qquad \gamma(S) := \sum_{k=1}^{\infty} \frac{1}{\ell_k}.$$

Observe that

$$(8) \qquad\qquad \gamma(S) = \prod_{j=1}^{J} \sum_{k=0}^{\infty} \frac{1}{p_j^k} = \prod_{j=1}^{J} \left(1 - \frac{1}{p_j}\right)^{-1}.$$

Theorem 2.1. *The Minkowski dimension of the set $X_\Omega^{(S)}$, defined by (5), equals*

$$
\begin{aligned}
\dim_M(X_\Omega^{(S)}) &= \gamma(S)^{-1} \sum_{k=1}^{\infty} \log_m(A_k) \left(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}}\right) \\
&= \gamma(S)^{-1} \left(1 + \sum_{k=1}^{\infty} \frac{\log_m(A_{k+1}/A_k)}{\ell_{k+1}}\right).
\end{aligned}
$$

The first formula for the Hausdorff dimension is obtained via a version of the variational principle. Let μ be a Borel probability measure on Ω. Denote by α_k the partition of Ω into cylinders of length k, so that $A_k = |\alpha_k|$. We consider the Shannon entropy of a finite partition, using logarithms based m:

$$H^\mu(\alpha) := -\sum_{B \in \alpha} \mu(B) \log_m \mu(B)$$

and the conditional entropy $H^\mu(\alpha|\beta)$ for two finite partitions. Define

$$
\begin{aligned}
s(\Omega, \mu) &:= \gamma(S)^{-1} \sum_{k=1}^{\infty} H^\mu(\alpha_k) \left(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}}\right) \\
(9) \qquad &= \gamma(S)^{-1} \left(H^\mu(\alpha_1) + \sum_{k=1}^{\infty} \frac{H^\mu(\alpha_{k+1}|\alpha_k)}{\ell_{k+1}}\right).
\end{aligned}
$$

Theorem 2.2. (i) *We have*

$$\dim_H(X_\Omega^{(S)}) = \sup_\mu s(\Omega, \mu),$$

where the supremum is over Borel probability measures on Ω.

(ii) *We have* $\dim_H(X_\Omega^{(S)}) = \dim_M(X_\Omega^{(S)})$ *if and only if the tree of prefixes of Ω is spherically symmetric, i.e. for every $k \in \mathbb{N}$, all prefixes of length k have the same (equal) number of continuations in* $\mathrm{Pref}_{k+1}(\Omega)$.

Corollary 2.3. *For every set of distinct $n_1, \dots, n_r \in \mathbb{N}$, with $r \geq 2$, we have $\dim_H(X_{n_1,\dots,n_r}^{(m)}) < \dim_M(X_{n_1,\dots,n_r}^{(m)})$ where the set $X_{n_1,\dots,n_r}^{(m)}$ is defined by (2).*

Proof (assuming Theorem 2.2). As explained above, $X_{n_1,\dots,n_r}^{(m)} = X_\Omega^{(S)} \subset \Sigma_m$, where S is the semigroup generated by all prime factors of $n_1, \dots, n_r$. Suppose that $n_1 < \dots < n_r$, and let $j \geq 2$ be such that $\ell_j = n_r$. Then every word with symbols $0, \dots, m-1$ of length $(j-1)$ is an allowed prefix of Ω. Observe that some words of length $(j-1)$ have only one continuation (by 0) to a word of length j, namely, those which have nonzeros in the positions corresponding to $n_i \in S$, for $i = 1, \dots, r-1$. However, all other words of length $(j-1)$ (and there will always be some) have m continuations to a word of length j. This shows that the tree $\mathrm{Pref}(\Omega)$ is not spherically symmetric and hence the Hausdorff dimension is strictly less than the Minkowski dimension by Theorem 2.2. $\qquad\square$

Another way to express the Hausdorff dimension of $X_\Omega^{(S)}$ is via a nonlinear system of equations.

Lemma 2.4. *Let Ω be a closed subset of Σ_m. Then there exists a vector*

$$\overline{t} = (t(u))_{u \in \mathrm{Pref}(\Omega)} \in [1, +\infty)^{\mathrm{Pref}(\Omega)},$$

such that

$$(10) \qquad t(\varnothing) \in [1, m], \quad t(u) \in [1, m^{\ell_k(\ell_{k+1}^{-1} + \ell_{k+2}^{-1} + \cdots)}], \ |u| = k, \ k \geq 1,$$

$$(11) \qquad\qquad t(\varnothing)^{\gamma(S)} = \sum_{j \in \mathrm{Pref}_1(\Omega)} t(j),$$

and

$$(12) \qquad t(u)^{\ell_{k+1}/\ell_k} = \sum_{j:\ uj \in \mathrm{Pref}_{k+1}(\Omega)} t(uj), \quad \forall\, u \in \mathrm{Pref}_k(\Omega), \ k \geq 1,$$

Using $\overline{t}$, it is easy to express the Hausdorff dimension.

Theorem 2.5. *We have*

$$\dim_H(X_\Omega^{(S)}) = \log_m t(\varnothing),$$

where $t(\varnothing)$ is from Lemma 2.4.

3. Preliminaries

Let us start with more notations. Denote

$$(13) \qquad \beta_n := |\{i \le n : \ (i, S) = 1\}|.$$

We need the following standard fact.

Lemma 3.1. *If $p_1, \ldots, p_J$ divide n, then*

$$\beta_n = \gamma(S)^{-1} n = n \prod_{j=1}^{J} \left(1 - \frac{1}{p_j}\right).$$

Proof. If $n = \prod_{j=1}^{J} p_j$, then $(i, S) = 1$ if and only if $(i, n) = 1$, hence $\beta_n = \phi(n)$, Euler's ϕ-function, for which the formula is well-known. In the general case, it remains to note that $(i, S) = 1$ if and only if $(i + \prod_{j=1}^{J} p_j, S) = 1$. $\qquad \square$

Recall that $S = \{\ell_k\}_{k=1}^{\infty}$, with $1 = \ell_1 < \ell_2 < \ldots$ We will denote

$$(14) \qquad B_k^{(n)} := \{i \in (n/\ell_{k+1}, n/\ell_k] \cap \mathbb{N} : \ (i, S) = 1\},$$

where n/ℓ_{k+1} and n/ℓ_k are not necessarily integers. For every n, let $K(n)$ be the unique integer such that

$$(15) \qquad \ell_{K(n)} \le n < \ell_{K(n)+1}.$$

Obviously, one has

$$(16) \qquad \beta_n = \sum_{k=1}^{K(n)} |B_k^{(n)}| \quad \text{and} \quad n = \sum_{k=1}^{K(n)} k|B_k^{(n)}|.$$

For a finite word u, we write

$$u|_{iS} := u_{i\ell_1} \ldots u_{i\ell_r}, \quad \text{where} \ \ i\ell_r \le |u| < i\ell_{r+1},$$

and for $x = (x_k)_{k \ge 1} \in \Sigma_m$ we denote

$$x_1^n := x_1 \ldots x_n.$$

We now prove Lemma 2.4.

Proof of Lemma 2.4. Consider the following compact set:

$$\Xi := [1, m] \times \prod_{k=1}^{\infty} [1, m^{\ell_k(\ell_{k+1}^{-1} + \ell_{k+2}^{-1} + \cdots)}]^{\mathrm{Pref}_k(\Omega)}.$$

Define a function $\Phi : \Xi \to [1, \infty)^{\mathrm{Pref}(\Omega)}$ by

$$\Phi(\overline{t})(u) = \Big(\sum_{uj \in \mathrm{Pref}_{k+1}(\Omega)} t(uj) \Big)^{\ell_k/\ell_{k+1}}, \quad u \in \mathrm{Pref}_k(\Omega), \ k \geq 1,$$

$$\Phi(\overline{t})(\varnothing) = \Big(\sum_{j=0}^{m-1} t(j) \Big)^{1/\gamma(S)}.$$

We claim that $\Phi(\Xi) \subset \Xi$. Indeed, if $t(u) \geq 1$ for all u, then clearly $\Phi(\overline{t})(u) \geq 1$. For the other inequality, we have, assuming that $\overline{t} \in \Xi$ and $u \in \mathrm{Pref}_k(\Omega)$, $k \geq 1$:

$$\Phi(\overline{t})(u) \leq m^{(1 + \ell_{k+1}(\ell_{k+2}^{-1} + \ell_{k+3}^{-1} + \cdots))\ell_k/\ell_{k+1}} = m^{\ell_k(\ell_{k+1}^{-1} + \ell_{k+2}^{-1} + \cdots)},$$

as desired. Finally,

$$\Phi(\overline{t})(\varnothing) \leq m^{(1 + \ell_1(\ell_2^{-1} + \ell_3^{-1} + \cdots))\gamma(S)^{-1}} = m^{\ell_1} = m,$$

by the definition of $\gamma(S)$ in (7), and the claim is verified.

Since Φ is continuous, it has a fixed point by the Tychonov fixed point theorem, which is the desired solution. (Alternatively, we can start with the vector of all 1's and iterate Φ. The operator Φ is monotone in each coordinate, hence there is a coordinate-wise limit, which will be a fixed point for Φ.) □

Hausdorff dimension will be computed with the help of the following lemma, essentially due to Billingsley, which we state in the symbolic space.

Lemma 3.2 (see Proposition 2.2 in [2])**.** *Let E be a Borel set in Σ_m and let ν be a finite Borel measure on Σ_m.*

(i) *If $\liminf_{n \to \infty}(-\frac{1}{n}) \log_2 \nu[x_1^n] \geq s$ for ν-a.e. $x \in E$, then $\dim_H(E) \geq s$.*

(ii) *If $\liminf_{n \to \infty}(-\frac{1}{n}) \log_2 \nu[x_1^n] \leq s$ for all $x \in E$, then $\dim_H(E) \leq s$.*

4. Minkowski dimension of $X_\Omega^{(S)}$

Proof of Theorem 2.1. We now compute the Minkowski dimension of $X_\Omega^{(S)}$. Recall that

$$\dim_M X_\Omega^{(S)} = \lim_{n \to +\infty} \frac{\log_m N_n(X_\Omega^{(S)})}{n},$$

where $N_n(X_\Omega^{(S)})$ is the number of words of length n that are prefixes of some $x \in X_\Omega^{(S)}$. This holds if the limit exists, one also defines the upper (resp. lower) dimension $\overline{\dim}_M$ and $\underline{\dim}_M$ by taking the liminf (resp. limsup) instead of the limit.

We need to estimate $N_n(X_\Omega^{(S)})$. Fix an integer $r \geq 1$. Considering integers n of the form $n = d(\prod_{j=1}^{r+1} \ell_j)(\prod_{i=1}^{J} p_i)$, for some $d \in \mathbb{N}$, is enough for the purpose of Minkowski dimension estimates.

By the definition (5), we have $x \in X_\Omega^{(S)}$ if and only if $x|_{iS} \in \Omega$ for all i, $(i, S) = 1$. It follows that x_1^n is a beginning (prefix) of some $x \in X_\Omega^{(S)}$ if and only if

$$x_1^n|_{iS} \in \mathrm{Pref}_k(\Omega), \quad \forall\, i \in B_k^{(n)}, \ k = 1, \ldots, K(n),$$

where $B_k^{(n)}$ is defined in (14). Thus, using the definition (6) of A_k, we obtain

$$N_n(X_\Omega^{(S)}) = \prod_{k=1}^{K(n)} A_k^{|B_k^{(n)}|}.$$

By the choice of n, n/ℓ_k and n/ℓ_{k+1} are integers for every $k \leq r$. By Lemma 3.1, one sees that

$$(17) \qquad |B_k^{(n)}| = \beta_{\frac{n}{l\ell_k}} - \beta_{\frac{n}{\ell_{k+1}}} = \gamma(S)^{-1} n \Big(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \Big) \quad \text{for } k \leq r,$$

hence,

$$\frac{1}{n} \log_m N_n(X_\Omega^{(S)}) \geq \sum_{k=1}^{r} \frac{|B_k^{(n)}|}{n} \log_m A_k = \gamma(S)^{-1} \sum_{k=1}^{r} \log_m(A_k) \Big(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \Big).$$

We obtain that

$$(18) \qquad \underline{\dim}_M(X_\Omega^{(S)}) \geq \gamma(S)^{-1} \sum_{k=1}^{r} \log_m(A_k) \Big(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \Big).$$

On the other hand, for $r+1 \leq k \leq K(n)$, A_k is bounded from above by m^k. This yields

$$(19) \qquad N_n(X_\Omega^{(S)}) \leq \prod_{k=1}^{r} A_k^{|B_k^{(n)}|} \cdot \prod_{k=r+1}^{K(n)} m^{k|B_k^{(n)}|} = \prod_{k=1}^{r} A_k^{|B_k^{(n)}|} \cdot m^{n - \sum_{k=1}^{r} k|B_k^{(n)}|},$$

where (16) has been used. We have

$$\sum_{k=1}^{r} k|B_k^{(n)}| = \gamma(S)^{-1} n \sum_{k=1}^{r} \Big(\frac{k}{\ell_k} - \frac{k}{\ell_{k+1}} \Big) = \gamma(S)^{-1} n \Big(-\frac{r}{\ell_{r+1}} + \sum_{k=1}^{r} \frac{1}{\ell_k} \Big).$$

Thus,

$$n - \sum_{k=1}^{r} k|B_k^{(n)}| = \gamma(S)^{-1} n\Big(\frac{r}{\ell_{r+1}} + \sum_{i=r+1}^{\infty} \frac{1}{\ell_i}\Big).$$

It follows from (19), again using (17), that

$$\overline{\dim}_M(X_\Omega^{(S)}) \le \gamma(S)^{-1} \left(\sum_{k=1}^{r} \log_m(A_k)\Big(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}}\Big) + \Big(\frac{r}{\ell_{r+1}} + \sum_{i=r+1}^{\infty} \frac{1}{\ell_i}\Big) \right),$$

and letting $r \to \infty$ here and in (18) yields the desired formula.   □

5. Lower bound for $\dim_H X_\Omega^{(S)}$ in Theorem 2.2(i)

Given a probability measure μ on Ω, we set

$$(20) \qquad \mathbb{P}_\mu[u] := \prod_{i \le |u|,\, (i,S)=1} \mu[u|_{iS}].$$

It follows from (5) that $\mathbb{P}_\mu$ extends to a Borel probability measure supported on $X_\Omega^{(S)}$.

Fix a probability measure μ on Ω. Recall that α_k is the partition of Ω into cylinders of length k. We are going to demonstrate that for every $r \in \mathbb{N}$,

$$(21) \qquad \liminf_{n \to \infty} \frac{-\log_m \mathbb{P}_\mu[x_1^n]}{n} \ge \gamma(S)^{-1} \sum_{k=1}^{r} H^\mu(\alpha_k)\Big(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}}\Big) \quad \text{for } \mathbb{P}_\mu\text{-a.e. } x.$$

Then, letting $r \to \infty$ will yield $\dim_H(X_\Omega^{(S)}) \ge s(\Omega, \mu)$ by Billingsley's Lemma 3.2(ii).

Let us fix an integer $r \ge 1$. Again, to verify (21), we can restrict ourselves to the integers of the form

$$(22) \qquad n = d \cdot \Big(\prod_{j=1}^{r+1} \ell_j \Big) \cdot \Big(\prod_{i=1}^{J} p_i \Big), \quad d \in \mathbb{N}.$$

In view of (20), we have

$$(23) \qquad \mathbb{P}_\mu[x_1^n] = \prod_{k=1}^{K(n)} \prod_{i \in B_n^{(k)}} \mu[x_1^n|_{iS}],$$

hence, when n is large enough,

$$(24) \qquad \mathbb{P}_\mu[x_1^n] \le \prod_{k=1}^{r} \prod_{i \in B_k^{(n)}} \mu[x_1^n|_{iS}].$$

Note that $x_1^n|_{iS}$ is a word of length k for $i \in B_k^{(n)}$, which is a beginning of a sequence in Ω. Thus, $[x_1^n|_{iS}]$ is an element of the partition α_k.

The random variables $x \mapsto -\log_m \mu[x_1^n|_{iS}]$ are i.i.d. for $i \in B_k^{(n)}$ (hence, there are $|B_k^{(n)}|$ of them), and their expectation equals $H^\mu(\alpha_k)$, by the definition of entropy. Fixing k with $k \leq r$ and taking n of the form (22), since $|B_k^{(n)}|$ tends to infinity as d tends to infinity, we get an infinite sequence of i.i.d. random variables. Therefore, by a version of the Law of Large Numbers,

$$(25) \qquad \text{for } \mathbb{P}_\mu\text{-a.e. } x, \ \forall \, k \leq r, \quad \frac{1}{|B_k^{(n)}|} \sum_{i \in B_k^{(n)}} -\log_m \mu[x_1^n|_{iS}] \ \longrightarrow \ H^\mu(\alpha_k)$$

as n (and thus d) tends to infinity. Using (24), we deduce that

$$\frac{-\log_m \mathbb{P}_\mu[x_1^n]}{n} \geq \frac{1}{n} \sum_{k=1}^{r} \sum_{i \in B_k^{(n)}} -\log_m \mu[x_1^n|_{iS}].$$

In view of (17), we obtain

$$\frac{-\log_m \mathbb{P}_\mu[x_1^n]}{n} \geq \sum_{k=1}^{r} \gamma(S)^{-1} \Big(\frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \Big) \sum_{i \in B_k^{(n)}} \frac{-\log_m \mu[x_1^n|_{iS}]}{|B_k^{(n)}|}.$$

Taking the liminf as n tends to infinity and using (25), we confirm (21) for $\mathbb{P}_\mu$-a.e. x, completing the proof.

6. Upper bound for $\dim_H X_\Omega^{(S)}$, and Theorem 2.5

To find the upper bound, we will construct an explicit measure on $X_\Omega^{(S)}$ which has the right dimension. Since we will be able to compute the Hausdorff dimension of this measure μ (it will be $\log_m t_\varnothing$), this will allow us to conclude.

In view of Lemma 2.4, we can define the probability measure μ on Ω such that

$$(26) \qquad \mu[j] := \frac{t(j)}{t(\varnothing)^{\gamma(S)}} \quad \text{for all } j \in \mathrm{Pref}_1(\Omega),$$

$$(27) \quad \forall \, k \geq 1, \qquad \mu[uj|u] := \frac{t(uj)}{t(u)^{\ell_{k+1}/\ell_k}} \quad \text{for all } u \in \mathrm{Pref}_k(\Omega), \ uj \in \mathrm{Pref}_{k+1}(\Omega).$$

Thus, for every $u = u_1 \ldots u_k \in \mathrm{Pref}_k(\Omega)$,

$$(28) \qquad \mu[u] = t(\varnothing)^{-\gamma(S)} t(u_1)^{1 - \frac{\ell_2}{\ell_1}} \cdots t(u_1 \ldots u_{k-1})^{1 - \frac{\ell_k}{\ell_{k-1}}} t(u).$$

We are going to use Billingsley's Lemma 3.2(i), for which we need to estimate the $\liminf_{n \to \infty} \frac{-\log_m \mathbb{P}_\mu[x_1^n]}{n}$ from above, **for every** $x \in X_\Omega^{(S)}$.

We will assume throughout the proof that $p_1, \ldots, p_J$ divide n.

Recalling (23), we need to estimate $\prod_{i \in B_k^{(n)}} \mu[x_1^n|_{iS}]$. By (28), we have

$$\prod_{i \in B_k^{(n)}} \mu[x_1^n|_{iS}] = \prod_{i \in B_k^{(n)}} \mu[x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k}]$$

$$= \prod_{i \in B_k^{(n)}} t(\varnothing)^{-\gamma(S)} t(x_{i\ell_1})^{1-\frac{\ell_2}{\ell_1}} t(x_{i\ell_1} x_{i\ell_2})^{1-\frac{\ell_3}{\ell_2}} \cdots t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_{k-1}})^{1-\frac{\ell_k}{\ell_{k-1}}}$$

$$\times t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})$$

$$= t(\varnothing)^{-|B_k^{(n)}|\gamma(S)} \prod_{i \in B_k^{(n)}} t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k}) \prod_{k'=1}^{k-1} t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_{k'}})^{1-\frac{\ell_{k'+1}}{\ell_{k'}}}.$$

Using (16), the product can be rewritten as

$$\mathbb{P}_\mu[x_1^n] = t(\varnothing)^{-\beta_n \gamma(S)} \times \left(\prod_{k=1}^{K(n)} \prod_{i \in B_k^{(n)}} \prod_{k'=1}^{k} t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_{k'}}) \right)$$

$$\times \left(\prod_{k=1}^{K(n)} \prod_{i \in B_k^{(n)}} \prod_{k'=1}^{k-1} t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_{k'}})^{1-\frac{\ell_{k'+1}}{\ell_{k'}}} \right).$$

Observe that if k is given in $\{1, \ldots, K(n)\}$ and $(i, S) = 1$ with $i \leq n/\ell_k$, then the term $t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})$ appears exactly once in the first product above. Similarly, if k is given in $\{1, \ldots, K(n) - 1\}$ and $(i, S) = 1$ with $i \leq n/\ell_{k+1}$, then the term $t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})^{1-\frac{\ell_{k+1}}{\ell_k}}$ appears once in the second product above. We deduce that

(29)
$$\mathbb{P}_\mu[x_1^n] = t(\varnothing)^{-\beta_n \gamma(S)} \prod_{k=1}^{K(n)} \prod_{\substack{i \leq n/\ell_k \\ (i,S)=1}} t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})$$

$$\times \prod_{k=1}^{K(n)-1} \prod_{\substack{i \leq n/\ell_{k+1} \\ (i,S)=1}} t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})^{-\ell_{k+1}/\ell_k}.$$

Therefore, taking Lemma 3.1 into account, we have

$$\frac{-\log_m \mathbb{P}_\mu[x_1^n]}{n} = \log_m t(\varnothing) + \sum_{k=1}^{K(n)-1} \sum_{\substack{i \le n/\ell_{k+1} \\ (i,S)=1}} \frac{\ell_k^{-1} \log_m t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})}{n/\ell_{k+1}}$$

$$(30) \qquad - \sum_{k=1}^{K(n)} \sum_{\substack{i \le n/\ell_k \\ (i,S)=1}} \frac{\ell_k^{-1} \log_m t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})}{n/\ell_k} \,.$$

Denote

$$(31) \qquad \mathrm{Av}_k(x,s) := \sum_{\substack{i \le s \\ (i,S)=1}} \frac{\ell_k^{-1} \log_m t(x_{i\ell_1} x_{i\ell_2} \ldots x_{i\ell_k})}{s},$$

where $s > 0$ is not necessarily in $\mathbb{N}$. Then (30) becomes

$$(32) \qquad \frac{-\log_m \mathbb{P}_\mu[x_1^n]}{n} = \log_m t(\varnothing) + \sum_{k=1}^{K(n)-1} \mathrm{Av}_k\left(x, \frac{n}{\ell_{k+1}}\right) - \sum_{k=1}^{K(n)} \mathrm{Av}_k\left(x, \frac{n}{\ell_k}\right).$$

Next, observe that by Lemma 2.4 and the fact that $t(u) \in [1, m^{\ell_k(\ell_{k+1}^{-1}+\ell_{k+2}^{-1}+\cdots)}]$ if $|u| = k$, we have

$$(33) \qquad \mathrm{Av}_k(x,s) \le \sum_{\substack{i \le s \\ (i,S)=1}} \frac{(\ell_{k+1}^{-1} + \ell_{k+2}^{-1} + \cdots)}{s} \le \sum_{i=k+1}^{\infty} \ell_i^{-1} \quad \text{for all } s > 0,$$

hence for $r < K(n)$ we have

$$\sum_{k=r+1}^{K(n)} \mathrm{Av}_k\left(x, \frac{n}{\ell_k}\right) < \sum_{k=r+1}^{\infty} \sum_{i=k+1}^{\infty} \ell_i^{-1}$$

$$(34) \qquad \qquad\qquad < \sum_{i=r+2}^{\infty} i\ell_i^{-1} =: \mathcal{E}_{r+2} \to 0, \quad \text{as } r \to \infty.$$

The convergence $\sum_{i=1}^{\infty} i\ell_i^{-1} < \infty$ is clear, since ℓ_i grows faster than any polynomial. The same holds for $\sum_{k=r+1}^{K(n)} \mathrm{Av}_k\left(x, \frac{n}{\ell_{k+1}}\right)$.

We will estimate from above the averages

$$\mathcal{A}(x,n,M) := \frac{1}{M} \sum_{j=1}^{M} \frac{-\log_m \mathbb{P}_\mu[x_1^{n\ell_j}]}{n\ell_j}.$$

Lemma 6.1. *For all $x \in X_\Omega^{(S)}$, $\liminf_{M\to\infty} \mathcal{A}(x,n,M) \le \log_m t(\varnothing)$.*

Proof. Fix $\varepsilon > 0$ and choose $r \in \mathbb{N}$ such that $\mathcal{E}_{r+2} < \varepsilon$. Choose $n \in \mathbb{N}$ of the form (22). By (32) and (34),

$$(35) \quad \mathcal{A}(x, n, M) \leq \log_m t(\varnothing) + \frac{1}{M} \sum_{k=1}^{r} \sum_{j=1}^{M} \left(\operatorname{Av}_k\left(x, \frac{n\ell_j}{\ell_{k+1}}\right) - \operatorname{Av}_k\left(x, \frac{n\ell_j}{\ell_k}\right) \right) + 2\varepsilon.$$

We are going to argue that the majority of the terms in each of the interior sums above cancel out. In fact, when ℓ_{k+1} divides ℓ_j, there exists a unique $j' < j$ such that $\frac{\ell_{j'}}{\ell_k} = \frac{\ell_j}{\ell_{k+1}}$, hence the term $\operatorname{Av}_k\left(x, \frac{n\ell_j}{\ell_{k+1}}\right)$ cancels out with the term $\operatorname{Av}_k\left(x, \frac{n\ell_{j'}}{\ell_k}\right)$. For this to happen, all we need is that ℓ_{k+1} divide ℓ_j.

We will show that this occurs for most of the terms in the sum above.

Lemma 6.2. *For any $r \in \mathbb{N}$, let $F_M = \{j \leq M : \ell_{k+1}|\ell_j, \ \forall k = 1, \ldots, r\}$. Then*

$$(36) \quad \lim_{M \to \infty} M^{-1}|F_M| = 1.$$

Proof. We have

$$S = \left\{ p_1^{m_1} \cdot \ldots \cdot p_J^{m_J} : m_1, \ldots, m_J \in \mathbb{N} \cup \{0\} \right\}.$$

Let $C = C(r) \in \mathbb{N}$ be such that $\ell_2, \ldots, \ell_{r+1}$ all divide $\prod_{j=1}^{J} p_j^C$. Let us define

$$
E_s = \left\{ (m_1, \ldots, m_J) \in \mathbb{Z}_+^J : \sum_{j=1}^{J} m_j \log p_j \leq \log(s) \right\}
$$

$$
\widetilde{E}_s = \left\{ (m_1, \ldots, m_J) \in \mathbb{Z}_+^J : \min_j m_j \geq C, \ \sum_{m=1}^{J} m_j \log p_j \leq \log(s) \right\}.
$$

For $s = \ell_M$, we have $|E_{\ell_M}| = M$ and

$$
\widetilde{E}_{\ell_M} \subset \left\{ (m_1, \ldots, m_J) \in \mathbb{Z}_+^J : \ell := p_1^{m_1} \cdots p_J^{m_J} \leq \ell_M \text{ and } \ell_2, \ldots, \ell_{r+1} \mid \ell \right\} \subset E_{\ell_M},
$$

from which we deduce that $|\widetilde{E}_{\ell_M}| \leq |F_M| \leq M = |E_{\ell_M}|$. Then (36) follows from

$$
\lim_{s \to +\infty} \frac{|E_s|}{|\widetilde{E}_s|} = 1,
$$

which is clear. □

From Lemma 6.2, we choose $M > 0$ so large that $M(1 - \varepsilon) \leq |F_M| \leq M$. As said above, when $j \in F_M$, the term $\operatorname{Av}_k\left(x, \frac{n\ell_j}{\ell_{k+1}}\right)$ cancels out with some term $\operatorname{Av}_k\left(x, \frac{n\ell_{j'}}{\ell_k}\right)$ in (35). The cardinality of the remaining terms is thus less than

$2M\varepsilon$, and each remaining term is upper bounded above by $\gamma(S)$, see (33), hence (35) yields

$$\liminf_{M\to\infty} \mathcal{A}(x, n, M) \le \log_m t(\varnothing) + 2\varepsilon\gamma(S) + 2\varepsilon.$$

for all $x \in X_\Omega^{(S)}$. Letting ε go to zero yields the result. $\qquad\square$

From Lemma 6.1, we conclude that $\liminf_{j\to\infty} \frac{-\log_m \mathbb{P}_\mu[x_1^j]}{j} \le \log_m t(\varnothing)$ for all $x \in X_\Omega^{(S)}$. It is key that this inequality holds true for all $x \in X_\Omega^{(S)}$, not only for almost all $x \in X_\Omega^{(S)}$. Now we conclude by Billingsley's Lemma 3.2(i) that $\dim_H X_\Omega^{(S)} \le \log_m t(\varnothing)$, and the upper bound in Theorem 2.5 is proved.

Proof of the lower bound in Theorem 2.5. We deduce the lower bound $\dim_H(X_\Omega^{(S)}) \ge \log_m t(\varnothing)$ from the lower bound in Theorem 2.2(i) and the following lemma, which asserts that the measure constructed in (26) and (27) is "optimal".

Lemma 6.3. *The measure μ on Ω defined by (26) and (27) satisfies*

$$(37) \qquad\qquad s(\Omega, \mu) = \log_m t(\varnothing).$$

Proof. We have by (26),

$$
\begin{aligned}
H^\mu(\alpha_1) &= -\sum_{j=0}^{m-1} \frac{t(j)}{t(\varnothing)^{\gamma(S)}} \log_m\Big(\frac{t(j)}{t(\varnothing)^{\gamma(S)}}\Big) \\
&= \gamma(S)\log_m t(\varnothing) - \sum_{j=0}^{m-1} \frac{t(j)}{t(\varnothing)^{\gamma(S)}} \log_m t(j) \\
&= \gamma(S)\log_m t(\varnothing) - \sum_{j=0}^{m-1} \mu[j] \log_m t(j).
\end{aligned}
$$

Further,

$$
\begin{aligned}
H^\mu(\alpha_{k+1}|\alpha_k) &= \sum_{[u]\in\alpha_k} \mu[u]\Big(-\sum_{j:\,[uj]\in\alpha_{k+1}} \frac{t(uj)}{t(u)^{\ell_{k+1}/\ell_k}} \log_m \frac{t(uj)}{t(u)^{\ell_{k+1}/\ell_k}}\Big) \\
&= \sum_{[u]\in\alpha_k} \mu[u]\Big(\frac{\ell_{k+1}}{\ell_k}\log_m t(u) - \sum_{j:\,[uj]\in\alpha_{k+1}} \mu[uj|u]\log_m t(uj)\Big) \\
&= \frac{\ell_{k+1}}{\ell_k}\sum_{[u]\in\alpha_k} \mu[u]\log_m t(u) - \sum_{[v]\in\alpha_{k+1}} \mu[v]\log_m t(v).
\end{aligned}
$$

Thus,

$$\frac{H^{\mu}(\alpha_{k+1}|\alpha_k)}{\ell_{k+1}} = \frac{1}{\ell_k} \sum_{[u]\in\alpha_k} \mu[u] \log_m t(u) - \frac{1}{\ell_{k+1}} \sum_{[v]\in\alpha_{k+1}} \mu[v] \log_m t(v).$$

Now it is clear that the sum in (9) "telescopes". Note also that

$$\frac{1}{\ell_k} \sum_{[u]\in\alpha_k} \mu[u] \log_m t(u) \leq \sum_{i=k+1}^{\infty} \ell_i^{-1} \to 0, \quad \text{as } k \to \infty$$

by Lemma 2.4. It follows that $s(\Omega, \mu) = \log_m t_\varnothing$, as desired. This completes the proof of the lemma, and of Theorem 2.5.   □

All that remains to prove is the part (ii) of Theorem 2.2.

Proof of Theorem 2.2.(ii) Every term in the expression for $\dim_M(X_\Omega^{(S)})$ in Theorem 2.1 dominates the corresponding term in (9), with equality if and only if μ assigns the same measure to each cylinder of length k, for every k. This is true for the "natural" uniform measure, when $\text{Pref}(\Omega)$ is spherically symmetric, and cannot hold otherwise.   □

7. Numerics and further examples

In this section we introduce a "geometric" argument used to determine the Minkowski dimension of several examples, and which allowed us to write an algorithm to produce the values of A_k for large values of k. The main idea is that we use a triangular arrangement of the sets iS, for $(i, S) = 1$.

Let $p < q \in \mathbb{N}$; $(p, q) = 1$ and $S = \langle p, q \rangle$ be the semigroup generated by p, q. Let $F : \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \mathbb{R}$ be a function depending only on the first coordinates, i.e. $F(x, y, z) = F(x_1, y_1, z_1)$. We are interested in the sets

$$(38) \qquad X_F := \{x \in \Sigma_2 : F(x_\ell, x_{p\ell}, x_{q\ell}) = 0 \ \forall \ell \in \mathbb{N}\}.$$

In $\mathbb{N}^2$ we consider the infinite triangular matrix

$$\Delta := \{p^{n-m}q^m : n \geq m \geq 0\}.$$

For instance, when $p = 2$ and $q = 3$, the matrix Δ (and S itself) can be represented as

$$
\begin{array}{ccccccccccc}
 & & & & & & 2187 & & \ldots & & \\
 & & & & & 729 & 1458 & 3916 & \ldots & & \\
 & & & & 243 & 686 & 1378 & 2756 & \ldots & & \\
 & & & 81 & 162 & 324 & 648 & 1296 & \ldots & & \\
 & & 27 & 54 & 108 & 216 & 432 & 864 & \ldots & & \\
 & 9 & 18 & 36 & 72 & 144 & 288 & 576 & \ldots & & \\
 3 & 6 & 12 & 24 & 48 & 96 & 192 & 384 & \ldots & & \\
1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & \ldots &
\end{array}
$$

To determine the Minkowski dimension we need to consider the truncated sectors

$$\Delta_N^i := \{p^{n-m}q^m \in \Delta \ : \ ip^{n-m}q^m \leq N\}, \qquad N \in \mathbb{N}.$$

The right-hand-side boundary of this sector is approximately determined by a grid approximation of a line with slope

$$\gamma = \frac{\log p}{\log p - \log q} < 0.$$

This "broken" line is determined by the Sturmian sequence associated to γ, as follows. Given an irrational slope γ we consider the line γx. It will intersect the integer grid consecutively (starting at the origin) in horizontal and vertical segments. It is classical that if $|\gamma| > 1$, then for some integer n_γ depending on γ only, a sequence of n_γ or $n_\gamma + 1$ vertical intersections will be followed by a single horizontal intersection. If $|\gamma| < 1$ then a sequence of n_γ or $n_\gamma + 1$ horizontal directions will be followed by a single vertical intersection. The set of boundary "squares" $\partial_r^i(N)$ of the configuration in the truncated triangle are the integer grid squares that have an intersection with the line $\gamma x + N$. These squares are given by

$$\partial_r^i(N) = \{(n,m) \ : \ ip^{n-m}q^m \leq N < ip^{n-m+1}q^m\}.$$

Denote by $n(i, N)$ the maximum of those integers n for which one can find a pair $(n, m) \in \partial_r^i(N)$. The numbers $m(i, N)$ are defined similarly.

If $p^2 > q$, i.e. $|\gamma| > 1$ (for instance when $p = 2$ and $q = 3$), then they have the structure

$$(n_1, m_1), (n_1, m_1 + 1) \cdots , (n_1, m_1 + k_1), (n_2, m_1 + k_1), \cdots , (n_2, m_2), \cdots ,$$

where $n_1 = n(i, N) + 1$, $m_1 = 0$, and for $j \geq 2$, $n_j = n_{j-1} - 1$, $m_j = m_{j-1} + k_{j-1}$ (with $k_{j-1} = n_\gamma$ or $n_\gamma + 1$), until $m_j = n_j = m(i, N) + 1$. The case where $p^2 < q$ (i.e. $|\gamma| < 1$) is symmetric.

So we have exactly $(n(i, N) + 1) - (m(i, N) + 1) = n(i, N) - m(i, N)$ "steps" to the left and $m(i, N) + 1$ steps up. All together we have $n(i, N) + 1$ boundary "squares". The "squares" $(n_j - 1, m_j + k_j) \in \partial_r^i(N)$ are called *pivotal* if $(n_j, m_j + k_j) \in \partial_r^i(N)$. There are exactly $n(i, N) - m(i, N)$ such squares.

For instance, when $p = 2$, $q = 3$, each truncated section is a "triangle" of right slope $- \log 2 / (\log 3 - \log 2)$. When $N = 27$ and $N = 243$, the truncated triangles are

$$
\begin{array}{ccccccccc}
 & & & & & & 243 & & \\
 & & & & & 81 & 162 & & \\
 & & 27 & & & 27 & 54 & 108 & 216 \\
 & 9 & 18 & & 9 & 18 & 36 & \mathbf{72} & 144 \\
3 & 6 & 12 & 24 & 3 & 6 & 12 & \mathbf{24} & \mathbf{48} & 96 & 192 \\
1 & 2 & 4 & 8 & 16 & 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128
\end{array}
$$

Truncated triangles Δ_N^i for $i = 1$, $N = 27$ and $N = 243$. In bold characters, the three integers $(24, 2 \times 24, 3 \times 24)$ located in a "corner".

Using this representation, one observes that the conditions on the three digits $(x_\ell, x_{p\ell}, x_{q\ell})$ in the sets X_F from (38) are then expressed in terms of three consecutive terms located in a "corner", since the integers (n, m) corresponding to $(\ell, p\ell, q\ell)$ always have the same relative locations inside the truncated sections Δ_N^i (see the example above).

Let us return to our main example: $S = \langle 2, 3 \rangle$, $\gamma(S) = 3$ with

$$
\begin{aligned}
X_{2,3} &= \left\{ (x_k)_{k=1}^\infty \in \Sigma_2 : \ x_k x_{2k} x_{3k} = 0 \ \forall\, k \in \mathbb{N} \right\}. \\
\Omega &= \left\{ (\omega_k)_{k=1}^\infty \in \Sigma_2 : \ \omega_i \omega_j \omega_k = 0 \ \text{if} \ 2\ell_i = \ell_j, \ 3\ell_i = \ell_k \right\}.
\end{aligned}
$$

The table below lists the first elements of S, denoted ℓ_k, and the corresponding $A_k = |\mathrm{Pref}_k(\Omega)|$, needed to compute the Minkowski dimension, see Theorem 2.1.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ℓ_k	1	2	3	4	6	8	9	12	16	18	24	27	32	36	48
A_k	2	4	7	14	25	50	90	160	320	584	1039	1861	3722	6772	12050

k	16	17	18	19	20	21	22	23	24
ℓ_k	54	64	72	81	96	108	128	144	162
A_k	21909	43818	79784	143028	254433	461161	922322	1679220	3055130

k	25	26	27	28	29	30	31
ℓ_k	192	216	243	256	288	324	384
A_k	5434757	9855663	17665509	35331018	64326200	116676724	207555865

The algorithm to determine $\mathrm{Pref}_k(\Omega)$ is based on the "triangle" representation. Consider an integer k and ℓ_k, and then the corresponding truncated triangle $\Delta^1_{\ell_k}$ (for instance for $k = 12$ (equivalently, for $N = 27$) as in the figure above). A word in $\mathrm{Pref}_k(\Omega)$ can be represented as an array of zeros and ones, whose entries are located at the grid vertices in the truncated triangle. Moreover, the condition $x_k x_{2k} x_{3k} = 0$ means that there are no three consecutive ones in a "corner".

```
        27                              1                       1
      9   18          ⟶           0   1                   0   1
    3   6   12   24             0   1   1   1           0   1   0   1
  1   2   4   8   16         1   0   1   0   1       1   0   1   0   1
```

Truncated triangle with $k = 12$, one bad and one good configuration.

In the algorithm, we first generate all possible first two lines of 0-1 bits (which yields 2^9 configurations when $k = 12$) and then keep only those which are admissible (the key point is the simplicity of the selection procedure, i.e. no three ones in a corner). Then we generate the third line, and keep only the suitable configurations. And so on. This argument yields the values of A_k in a very reasonable time, up to $k = 31$.

Actually it is a very interesting combinatorial problem in itself to determine the number of admissible configurations A_k, even in a simpler geometrical context (for instance in a $N \times N$-square, with the forbidden "corner"). There is numerical evidence that A_k has a power law of k, but to confirm this would certainly require further investigations.

It is not hard to show that $7/4 \leq A_{k+1}/A_k \leq 2$. Using this and the data in the table, we obtain for the Minkowski dimension (Theorem 2.1)

$$0.9573399 \leq \dim_M(X_{2,3}) \leq 0.9623350.$$

To estimate the Hausdorff dimension we can use Theorem 2.5. We get explicit rigorous estimates by going to a fixed level n, and either assuming that there are no restrictions further on, whence $t(u) = 2^{\ell_n(\ell_{n+1}^{-1}+\cdots)}$, $|u| = n$, to get an upper bound, or to assume that all the digits that follow are 0's, whence $t(u) = 1$, to get a lower bound. Then all the values of $t(u)$, with $|u| \leq n - 1$, are obtained recursively using (12) and (11). We did the calculation with $n = 25$ and obtained

$$0.9246585 < \dim_H(X_{2,3}) < 0.9405728$$

(the convergence is slow, but we think that the upper bound is closer to the truth).

7.1. **Further examples.** The difficulty with the set $X_{2,3}$ and the function $F(x, y, z) = x_1 y_1 z_1$ comes essentially from the fact that, fixing the bits at the frontier of the truncated triangles is not enough to deduce the values of all digits inside the triangle (there is some long-range dependence between the bits). For some suitable functions F and the corresponding sets X_F, this is not the case, and the situation is easier.

Definition. *We call a function F deterministic if for all $i, j \in \{0, 1\}$ there is a unique solution $k \in \{0, 1\}$ for one of the following implicit equations*

$$i)\ F(i, j, k) = 0, \quad ii)\ F(i, k, j) = 0, \quad iii)\ F(k, i, j) = 0.$$

The existence of the solution simply means that the constraint on the configurations is well posed and the uniqueness is simply the solvability of the implicit function equation. Now we can formulate the following "rigidity" theorem.

Theorem 7.1. *Let $p < q \in \mathbb{N}$, $(p, q) = 1$, and let $F \colon \Sigma_2 \times \Sigma_2 \times \Sigma_2 \to \mathbb{R}$ be a "deterministic" function $F(x, y, z)$ depending only on the first coordinates of $x, y, z \in \Sigma_2$. Then recalling the definition (38) of X_F, one has*

$$(39) \qquad\qquad \dim_H(X_F) = \dim_M(X_F) = \frac{q-1}{q}.$$

Proof. A moment's thought makes it clear that for a function fulfilling i) any configuration in Δ_N^i is uniquely determined by choosing the $n(i,N)+1$ boundary values $(n,m) = (0,0),(1,0),\cdots,(n(i,N),0)$ freely. For a function fulfilling ii), any configuration in Δ_N^i is uniquely determined by the free choice of the upper boundary values $(n,m) = (0,0),(1,1),(2,2),\cdots,(m(i,N),m(i,N))$ and the pivotal "squares". For a function fulfilling iii), any configuration is uniquely determined by the free choice of the values in $\partial_r^i(N)$. In all cases, there are exactly $2^{n(i,N)+1}$ configurations in Δ_N^i.

The common dimension can easily be computed, we explain it for a function F fulfilling i). Given an admissible sequence $x \in X_F$, observe that we can choose all the values x_i arbitrarily as long as i is not divisible by q. The values at positions qj, $j \geq 1$, are then completely determined. Hence, exactly $\frac{q-1}{q}$ of the positions can be arbitrary and the rest is determined. This immediately implies, by standard argument, that

$$\dim_M(X_F) = \frac{q-1}{q}.$$

The Minkowski and the Hausdorff dimension coincide as can be derived from part (ii) of Theorem 2.2.   □

Example 7.2. *The function $F(x,y,z) = (2x_1 - 1)(2y_1 - 1)(2z_1 - 1) - 1$ is "deterministic", hence (39) holds. This case is reminiscent of the well-studied Ledrappier shift, i.e. each of the triples $(x_\ell, x_{p\ell}, x_{q\ell})$ has an even number of 0's, or equivalently*

$$x_\ell + x_{p\ell} + x_{q\ell} = 1 \quad (mod \ 2).$$

Example 7.3. *For $F(x,y,z) = (y - xz)^2$ (which fulfills ii) but not i) or iii)) and $F(x,y,z) = (x - yz)^2$ (which fulfills iii) but not i) or ii)), (39) is satisfied.*

Example 7.4. *Consider $F(x,y,z) = (x - y)^2 + (x - z)^2$. This function is not "deterministic". In this case, each triangle Δ_N^i has exactly 2 configurations: all 0's or all 1's. Hence the total number of cylinders of length N equals the number of non-empty triangles Δ_N^i. Since $(i,S) = 1$ this number equals the number of i's that are neither divisible by p nor by q. We have exactly $p + q - 1$ residue classes modulo pq that are divisible by p or q. Therefore the numbers i with $(i,S) = 1$ can be divided into $pq - p - q + 1$ arithmetic sequences of step length pq. Hence*

$$\dim_M(X_F) = 1 - \frac{p+q-1}{pq}.$$

In this case, the Hausdorff dimension of X_F coincides with its Minkowski dimension, using the argument (ii) of Theorem 2.2: each prefix (with our interpretation, each finite triangle) has only one possible continuation.

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