

Rigidity and Tolerance in point processes: Gaussian zeroes and Ginibre eigenvalues

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Abstract

Let Π be a translation invariant point process on the complex plane $\mathbb{C}$ and let $\mathcal{D} \subset \mathbb{C}$ be a bounded open set. We ask what does the point configuration Π_{out} obtained by taking the points of Π outside $\mathcal{D}$ tell us about the point configuration Π_{in} of Π inside $\mathcal{D}$? We show that for the Ginibre ensemble, Π_{out} determines the number of points in Π_{in} . For the translation-invariant zero process of a planar Gaussian Analytic Function, we show that Π_{out} determines the number as well as the centre of mass of the points in Π_{in} . Further, in both models we prove that the outside says “nothing more” about the inside, in the sense that the conditional distribution of the inside points, given the outside, is mutually absolutely continuous with respect to the Lebesgue measure on its supporting submanifold.

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1 Introduction

A point process Π on $\mathbb{C}$ is a random locally finite point configuration on the two dimensional Euclidean plane. We will consider simple point processes, namely, those in which no two points are at the same location. A simple point process can, equivalently, be looked upon as a random discrete measure $[\Pi] = \sum_{z \in \Pi} \delta_z$. The probability distribution of the point process Π is a probability measure $\mathbb{P}[\Pi]$ on the Polish space of locally finite point sets on the plane. The relevant topology on the space of locally finite point sets is that of weak convergence of measures, when we identify a point configuration with the counting measure it induces on $\mathbb{C}$. This explains the equivalence of the two ways of looking at a point process. Point processes on the plane have been studied extensively. For a lucid exposition on general point processes one can look at [DaV97]. The group of translations of $\mathbb{C}$ acts in a natural way on the space of locally finite point configurations on $\mathbb{C}$. Namely, the translation by z , denoted by T_z , takes the point configuration Λ to the configuration $T_z(\Lambda) := \{x + z \mid x \in \Lambda\}$. A point process Π is said to be translation invariant if Π and $T_z(\Pi)$ have the same distribution for all $z \in \mathbb{C}$. The (first) intensity measure ρ_1 of a point process on $\mathbb{C}$ is the measure on $\mathbb{C}$ given by $\rho_1(B) = \mathbb{E}[\Pi \cap B]$ for every Borel set $B \subset \mathbb{C}$. Higher order intensity measures of a point process can also be defined in a similar manner, see e.g. Definition 1.2.2 in [HKPV10]. In this work we will focus primarily on translation invariant point processes on $\mathbb{C}$ whose intensities are absolutely continuous with respect to the Lebesgue measure.

Let $\mathcal{D}$ be a bounded open set in $\mathbb{C}$. Let $\mathcal{S}$ denote the Polish space of locally finite point configurations on $\mathbb{C}$. The decomposition $\mathbb{C} = \mathcal{D} \cup \mathcal{D}^c$ induces a factorization $\mathcal{S} = \mathcal{S}_{\text{in}} \times \mathcal{S}_{\text{out}}$, where $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ are respectively the spaces of finite point configurations on $\mathcal{D}$ and locally finite point configurations on $\mathcal{D}^c$. This immediately leads to the natural decomposition $\Upsilon = (\Upsilon_{\text{in}}, \Upsilon_{\text{out}})$ for any $\Upsilon \in \mathcal{S}$, and consequently a decomposition of the point process Π as $\Pi = (\Pi_{\text{in}}, \Pi_{\text{out}})$.

In this paper, we ask the following question: if we know the configuration Π_{out} , what can we conclude about Π_{in} ? We consider with regard to this question the main natural examples of translation invariant point processes on the plane, and provide a complete answer in each case.

The (homogeneous) Poisson point process is the most natural example of a translation-invariant point process on the plane. One of its characterising properties is that the point configurations in two disjoint measurable sets are independent of each other. Therefore, our question is a triviality for the Poisson process: the points outside $\mathcal{D}$ do not provide any information about the points inside $\mathcal{D}$.

The two main natural examples of translation invariant point processes in the plane that have non-trivial spatial correlations are the Ginibre ensemble and the zeroes of the planar Gaussian Analytic Function. We refer the reader to [HKPV10] for a detailed study of these ensembles. The Ginibre ensemble (in the finite dimensional setting) was introduced in the physics literature by Ginibre [Gin65] as a model based on non-Hermitian random matrices; the infinite Ginibre ensemble is obtained as a weak limit of these finite dimensional point processes. Like the Poisson process, it is translation-invariant and ergodic under rigid motions of the plane. In fact, it is a determinantal point process with the determinantal kernel $K(z, w) = \sum_{j=0}^{\infty} \frac{(z\bar{w})^j}{j!} = e^{z\bar{w}}$ and the

background measure $\frac{1}{\pi} e^{-|z|^2} d\mathcal{L}(z)$. Here $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{C}$. Krishnapur ([Kr06], Theorem 3.0.5) has shown that the Ginibre ensemble is the unique determinantal point process on $\mathbb{C}$ that is isometry invariant, has a sesqui-holomorphic kernel (i.e., the determinantal kernel is

holomorphic in the first variable and anti-holomorphic in the second), and is normalized to have unit intensity.

The standard planar Gaussian Analytic Function (abbreviated henceforth as GAF) is the random entire function defined by the (random) Taylor series $f(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k$ where ξ_k -s are i.i.d. standard complex Gaussians. We are interested in the point configuration on $\mathbb{C}$ given by the zeroes of this GAF. The GAF zero process is translation invariant and ergodic, and exhibits local repulsion. It has been studied intensively by several authors including Nazarov, Sodin, Tsirelson, and others (see, e.g., [FH99], [STs1-04], [STs2-06], [STs3-05], [NSV07], [NS10]). Zelditch, Shiffman, and others have studied their counterparts from a geometric perspective (see, e.g., [ShZe08], [ShZeZ-08], [ShZe10], [ShZeZh11], [ZeiZe10], [Ze13]).

Sodin [S00] has shown that in the class of Gaussian analytic functions (i.e., analytic functions whose finite dimensional marginals are Gaussian distributions), the standard planar GAF is the only one to have a translation invariant zero-set (up to scaling and multiplication by a deterministic entire function with no zeroes).

For further details on the Gaussian zero and Ginibre ensembles, we refer the reader to Section 3.

Given a pair of random variables (X, Y) which has a joint distribution on a product of Polish spaces $\mathcal{S}_1 \times \mathcal{S}_2$, we can define the *regular conditional distribution* γ of Y given X by the family of probability measures $\gamma(s_1, \cdot)$ parametrized by the elements $s_1 \in \mathcal{S}_1$ such that for any Borel sets $A \subset \mathcal{S}_1$ and $B \subset \mathcal{S}_2$ we have

$$\mathbb{P}(X \in A, Y \in B) = \int_A \gamma(s_1, B) d\mathbb{P}[X](s_1)$$

where $\mathbb{P}[X]$ denotes the marginal distribution of X . For details on regular conditional distributions, see, e.g., [Pa00] (Chapter 5, Section 8) or [Du10] (Section 5.1.3).

Recall that $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ are Polish spaces. Hence, by the general theory of conditional measures, there exists a *regular conditional distribution* ϱ of Π_{in} given Π_{out} . Clearly, ϱ can be seen as the distribution of a point process on $\mathcal{D}$ which depends on Υ_{out} .

Let $\underline{\zeta}$ be the vector whose coordinates are the points of Π_{in} taken in uniform random order (the length of this vector is a random variable). We will denote the conditional distribution of $\underline{\zeta}$ given Π_{out} by ρ . Formally, it is a family of probability measures $\rho(\Upsilon_{\text{out}}, \cdot)$ on $\bigcup_{m=0}^{\infty} \mathcal{D}^m$ parametrized by $\Upsilon_{\text{out}} \in \mathcal{S}_{\text{out}}$.

There is a simple relationship between $\rho(\Upsilon_{\text{out}}, \cdot)$ and $\varrho(\Upsilon_{\text{out}}, \cdot)$. Consider the natural map ϕ from $\bigcup_{m=0}^{\infty} \mathcal{D}^m$ to $\mathcal{S}_{\text{in}}$ which makes a point configuration Υ_{in} of size m from a vector $\underline{\zeta}$ in $\mathcal{D}^m$ by forgetting the order of the coordinates of $\underline{\zeta}$. It is easy to see that $\mathbb{P}[\Pi_{\text{out}}]$ -a.s. we have $\phi_* \rho(\Upsilon_{\text{out}}, \cdot) = \varrho(\Upsilon_{\text{out}}, \cdot)$. The main reason for using ρ is that it allows us to easily compare our conditional measures with other standard measures, which can be thought of as appropriate reference measures, e.g. in the setting of Theorems 1.2 and 1.4.

For two measures μ_1 and μ_2 defined on the same measure space Ω with $\mu_1 \ll \mu_2$ (meaning μ_1 is absolutely continuous with respect to μ_2), we will denote by $\frac{d\mu_1}{d\mu_2}(\omega)$ the Radon Nikodym derivative of μ_1 with respect to μ_2 evaluated at $\omega \in \Omega$. By $\mu_1 \equiv \mu_2$, we mean that the measures μ_1 and μ_2 are mutually absolutely continuous, that is $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$.

In Theorems 1.1-1.4 we denote the Ginibre ensemble by $\mathcal{G}$ and the GAF zero ensemble by $\mathcal{Z}$. As before, $\mathcal{D}$ is a bounded open set in $\mathbb{C}$.

In the case of the Ginibre ensemble, we prove that a.s. the points outside $\mathcal{D}$ determine the number of points inside $\mathcal{D}$, and “nothing more”.

Theorem 1.1. *For the Ginibre ensemble, there is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s.*

$$\text{Number of points in } \mathcal{G}_{\text{in}} = N(\mathcal{G}_{\text{out}}) .$$

Since a.s. the length of $\underline{\zeta}$ equals $N(\mathcal{G}_{\text{out}})$, we can assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ is supported on $\mathcal{D}^{N(\Upsilon_{\text{out}})}$.

Theorem 1.2. *For the Ginibre ensemble, $\mathbb{P}$ -a.s. the measure $\rho(\mathcal{G}_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}$ on $\mathcal{D}^{N(\mathcal{G}_{\text{out}})}$ are mutually absolutely continuous.*

In the case of the GAF zero process, we prove that the points outside $\mathcal{D}$ determine the number as well as the centre of mass (or equivalently, the sum) of the points inside $\mathcal{D}$, and “nothing more”.

Theorem 1.3. *For the GAF zero ensemble,*

(i) *There is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s.*

$$\text{Number of points in } \mathcal{Z}_{\text{in}} = N(\mathcal{Z}_{\text{out}}) .$$

(ii) *There is a measurable function $S : \mathcal{S}_{\text{out}} \rightarrow \mathbb{C}$ such that a.s.*

$$\text{Sum of the points in } \mathcal{Z}_{\text{in}} = S(\mathcal{Z}_{\text{out}}) .$$

For a possible value Υ_{out} of $\mathcal{Z}_{\text{out}}$, define the set of admissible vectors of inside points (obtained by considering all possible orderings of such inside point configurations)

$$\Sigma_{S(\Upsilon_{\text{out}})} := \{ \underline{\zeta} \in \mathcal{D}^{N(\Upsilon_{\text{out}})} : \sum_{j=1}^{N(\Upsilon_{\text{out}})} \zeta_j = S(\Upsilon_{\text{out}}) \}$$

where $\underline{\zeta} = (\zeta_1, \dots, \zeta_{N(\Upsilon_{\text{out}})})$.

Since a.s. the length of $\underline{\zeta}$ equals $N(\Upsilon_{\text{out}})$, we can assume that each measure $\rho(\Upsilon_{\text{out}}, \cdot)$ gives us the distribution of a random vector in $\mathcal{D}^{N(\Upsilon_{\text{out}})}$ supported on $\Sigma_{S(\Upsilon_{\text{out}})}$.

Theorem 1.4. *For the GAF zero ensemble, $\mathbb{P}$ -a.s. the measure $\rho(\mathcal{Z}_{\text{out}}, \cdot)$ and the Lebesgue measure $\mathcal{L}_{\Sigma}$ on $\Sigma_{S(\mathcal{Z}_{\text{out}})}$ are mutually absolutely continuous.*

We view the conservation laws as the “**rigidity**” properties of the respective point processes. The absolute continuity (with respect to the Lebesgue measure on the conserved submanifold) of the conditional distribution of the vector of inside points can be viewed as “**tolerance**”. The heuristic is that due to such mutual absolute continuity, the inside points can form (almost) any configuration on this conserved submanifold.

Notice that the question whether $\mathbb{P}[\Pi_{\mathcal{D}}] \ll \mathbb{P}[\Pi]$ for a given $\mathcal{D}$ can be phrased in terms of the conditional distribution ϱ , and the same holds for the main questions addressed in this paper.

Therefore, in more general terms, we are interested in the support and the regularity properties of ϱ .

For a finite point process of fixed size n , the phenomenon of outside points determining the number of inside points is a triviality. However, the behaviour of infinite point processes can be quite different, even when they arise as distributional limits of finite point processes. For example, let us take the disk of area n centred at the origin and consider the point process Π_n given by n uniform points inside it. Let $\mathcal{D}$ be the unit disk centred at the origin. The process Π_n clearly has the property that the point configuration outside $\mathcal{D}$ determines the number of points inside $\mathcal{D}$. However, the distributional limit of Π_n -s, as $n \rightarrow \infty$, is the Poisson point process, which has no such property. Hence, the rigidity of numbers for the Ginibre or GAF zero ensembles cannot be understood as a limit of the simple rigidity of numbers for finite ensembles. The reasons behind this phenomenon, therefore, lie in the stronger spatial correlations of these ensembles.

In addition to answering our central question mentioned in the beginning, Theorems 1.1-1.4 also provide information on the relative strength of spatial correlations in the Ginibre and the GAF zero ensembles. While a simple visual inspection suffices to (heuristically) distinguish a sample of the Poisson process from that of either the Ginibre or the GAF zero process (of the same intensity), the latter two are hard to set apart between themselves. It is therefore an interesting question to devise mathematical statistics that distinguish them. The qualitative idea is that the spatial correlation is much stronger in the GAF zero process than in the Ginibre ensemble. There can be several possible approaches to quantify this heuristic observation. One important feature to look at, for instance, is the rate of decay of the hole probabilities. However, it turns out that both the Ginibre ensemble and the Gaussian zero process behave similarly in this respect. For more details, one can refer to [HKPV10]. Our results demonstrate that the GAF zeroes have greater spatial dependence, in the sense that the point configuration in the exterior of a open set dictates more about the one in its interior.

In [STs1-04], Sodin and Tsirelson compared the GAF zero process (CAZP in their terminology) with various models of perturbed lattices. They noticed that the lattice process

$$\{\sqrt{3\pi}(k + li) + ce^{2\pi im/3}\eta_{k,l} : k, l \in \mathbb{Z}, m = 0, 1, 2\},$$

(where $\eta_{k,l}$ are i.i.d. standard complex Gaussians, $c \in (0, \infty)$ is a parameter and i denotes the imaginary unit) achieves “asymptotic similarity” with the GAF zero process (in the sense that the variances of scaled linear statistics have similar asymptotic behaviour to those for the GAF zeroes). Sodin and Tsirelson further observed that the above perturbed lattice model satisfied two conservation laws: one pertaining to the “mass” and another pertaining to the “centre of mass”. They predicted similar conservation laws for the GAF zero process, although the sense in which such laws would hold was left open to interpretation. Theorem 1.3 establishes two conservation laws, one of which preserves the “mass” (i.e., the number of points), and the other one preserves the “centre of mass”. Moreover, Theorem 1.4 says that these are the only conservation laws for the GAF zero process. We further note that among the perturbed lattice models in [STs1-04], the one that achieves “asymptotic similarity” with the Ginibre ensemble is the process

$$\{\sqrt{\pi}(k + li) + c\eta_{k,l} : k, l \in \mathbb{Z}\}$$

where $\eta_{k,l}$ and c are as before. In this model, we have one conserved quantity (namely, the “mass”). In Theorems 1.1 and 1.2, we obtain a conservation law for the “mass” (i.e., the number of points) in the Ginibre ensemble, and further, show that there are no other conserved quantities.

En route to proving the main theorems mentioned above, we obtain results that are interesting in their own right. For example we prove that the harmonic sum $(\sum_{z \in \Pi} \frac{1}{z})$, for the Ginibre ensemble as well as for the GAF zero process, is a.s. finite (in a precise technical sense specified in Propositions 8.6 and 10.13 and the remarks thereafter). In fact, we show that this sum has a finite first moment for both processes. It is not hard to see that the corresponding sum for the Poisson process does not converge in any reasonable sense. Even for the Ginibre or the GAF zero ensembles, the corresponding sum does not converge absolutely. The underlying reason for the conditional convergence is the mutual cancellation arising from the higher degree of symmetry exhibited by a typical point configuration in the Ginibre or the GAF zero process. This is yet another manifestation of the fact that the Gaussian zeros or the Ginibre eigenvalues exhibit a much more regular arrangement (which indicates greater rigidity) than, say, the Poisson process.

In a more precise sense, we can consider finite dimensional approximations $\mathcal{G}_n$ and $\mathcal{Z}_n$ to $\mathcal{G}$ and $\mathcal{Z}$ respectively (for details see Sections 3.1 and 3.2) and define the finite sums $\alpha_k(n) = (\sum_{z \in \Pi} 1/z^k)$ when $\Pi = \mathcal{G}_n$ or $\mathcal{Z}_n$. Our results, as in Proposition 8.9 and Proposition 10.16, establish that both for the Ginibre and the GAF zeroes, these sums converge in probability as $n \rightarrow \infty$. The limit α_k can be justifiably taken to be an analogue of the sum $(\sum_{z \in \Pi} 1/z^k)$ for the respective limiting process $\mathcal{G}$ or $\mathcal{Z}$.

We also provide a new proof of a reconstruction theorem for the planar GAF from its zeroes, which essentially says that the GAF zeroes determine a.s. the GAF itself, up to a factor of modulus 1. In what follows, α_k will denote the random variable introduced above for GAF zeroes, P_k will be the k -th Newton polynomial (for details, see Section 12.1). Define

$$a_k = (-1)^k P_k(\alpha_1, \dots, \alpha_k)$$

and

$$\chi = \lim_{k \rightarrow \infty} k^{1/2} \left(\sum_{j=0}^{k-1} j! |a_j|^2 \right)^{-1/2}.$$

The existence of the limit will be proved in the course of proving Theorem 1.5. We state the reconstruction theorem as:

Theorem 1.5. *Consider the random analytic function $g(z) = \sum_{k=0}^{\infty} \chi a_k z^k$, which is measurable with respect to the GAF zeroes. There is a random variable ζ with uniform distribution on $\mathbb{S}^1$ and independent of the GAF zeroes, such that a.s. we have $f(z) = \zeta g(z)$, where f is the standard planar Gaussian analytic function.*

It is interesting to compare Theorem 1.5 with the Hadamard Factorization Theorem (see, e.g., [Rud87] Chapter 15) for reconstructing analytic functions from their zeroes, where we exploit the fact that the standard planar GAF is a.s. of order 2. In the case of the Hadamard factorization, the key problem is that there is a (random) analytic function of the form $\exp(p)$ (where p is a polynomial of degree at most 2) that occurs as a factor in front of the canonical product formed from the Gaussian zeroes, and a priori there is no concrete information about this function. E.g., it can, in principle, depend on the Gaussian zeroes. However, in Theorem 1.5 there is a concrete description of the factor ζ and also the precise dependence of g on the GAF zeroes.

The description in Theorem 1.5 is optimal in the sense that the factor ζ cannot be done away with. This can be seen from the fact that if θ is a random variable that is uniform in $\mathbb{S}^1$ and

independent of the ξ_i -s (the ξ_i -s are i.i.d. standard complex Gaussians), then the random analytic functions θf and f are both distributed as planar GAF-s and have the same zeroes; hence from the zeroes of f we can hope to recover the coefficients of f only up to such a phase θ . Here we recall the definition of the planar GAF $f(z) := \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k$.

A reconstruction theorem for the planar GAF from its zeroes has been obtained by Ben Hough in [Bh05] (Section 3, Lemma 9), using methods which closely parallel the proof of a similar reconstruction result for the zeroes of the Gaussian analytic function on the hyperbolic plane (Theorem 6 in [PV05]). Our approach here is, however, very different from [Bh05] and [PV05], and the reconstruction result follows in a brief and simple manner from the estimates we already obtain in Section 10.2. We thereby obtain a new expression for the GAF in terms of its zeroes, and en route we construct analogues of the classic Vieta formulas relating the zero set of the GAF to its coefficients. It would be interesting to compare the expression that we obtain to that obtained in [Bh05]; this seems to lead to new and intriguing identities satisfied a.s. by the zeroes of the planar GAF (for instance, just by comparing the value of $|f(0)|$).

Theorems 1.2 and 1.4 naturally lead to the question whether something more detailed or more quantitative can be said about the conditional measures in question. E.g., can we conclude something more explicit about the probability density function which is shown to exist in Theorems 1.2 and 1.4? This question is addressed in [G12-II], where it is shown that the conditional densities are, in fact, comparable to squared Vandermonde densities. In particular, this shows that even under spatial conditioning, the Gaussian zeroes and the Ginibre eigenvalues exhibit mutual repulsion.

Theorems 1.1 and 1.3 can be seen as statements on the conservation of moments. To be more precise, the number of points in a domain $\mathcal{D}$ can be thought of as the zeroth moment of the points in $\mathcal{D}$, and their sum can be thought of as their first moment. Thus, Theorems 1.1 and 1.2 correspond to the statement that the points outside $\mathcal{D}$ determine the zeroth moment of the inside points for the Ginibre ensemble, and nothing more. In the same spirit, Theorems 1.3 and 1.4 correspond to the statement that the points outside $\mathcal{D}$ determine the zeroth and the first moments of the inside points for the GAF zero ensemble, and nothing more. It is natural to ask whether there exist point processes where the points outside $\mathcal{D}$ determine the first k moments (and nothing more) of the points in $\mathcal{D}$, for $k \geq 3$. In the forthcoming paper [GK], a sequence of such point processes is constructed, which correspond to the zeroes of a natural family of random analytic functions with Gaussian coefficients. In the same paper, it is also established that a determinantal point process is rigid only if its determinantal kernel, considered as an integral operator, is a projection.

The central question of this paper is partially motivated by the concepts of insertion and deletion tolerance of a point process. Consider a point process Π . Fix a bounded open set $\mathcal{D}$, and add to Π a random point uniformly distributed in $\mathcal{D}$. Let us call this perturbed point process $\Pi_{\mathcal{D}}$. The point process Π is said to be **insertion tolerant** if $\mathbb{P}[\Pi_{\mathcal{D}}] \ll \mathbb{P}[\Pi]$ for all bounded open sets $\mathcal{D}$. **Deletion tolerance** is similarly defined by deleting a point inside $\mathcal{D}$ (if one exists) uniformly at random. Insertion and deletion tolerance have been investigated by Burton and Keane ([BK89]) in the context of percolation, by Holroyd and Peres ([HP05]) for studying invariant allocation on the plane, and by Hecklen and Lyons ([HLy03]) in the setting of random spanning forests. They have also been studied as topics of their own importance by Holroyd and Soo ([HS10]).

Questions on rigidity and tolerance of point processes can be studied in other settings, for instance, in dimensions other than 2 and in geometries other than the Euclidean setting. For the continuum sine kernel process on $\mathbb{R}$ (which arises as the universal bulk limit of Wigner random matrices), rigidity of the number of points in an interval is proved in [G12-I]. In the same work, rigidity for

the number of points is also established for a wide class of stationary determinantal point processes on $\mathbf{Z}$. For i.i.d. Gaussian perturbations of $\mathbf{Z}^d$, deletion intolerance has been proved in [HS10] for $d = 1, 2$. In the forthcoming paper [PS14], it is shown that for i.i.d. Gaussian perturbations (with mean 0 and variance σ^2) of $\mathbf{Z}^d$ for $d \geq 3$, there is a critical value $\sigma_c(d)$ such that the point process is deletion intolerant for $\sigma < \sigma_c(d)$ and deletion tolerant for $\sigma > \sigma_c(d)$. In other geometries, it has been shown in [HS10] that for the Gaussian analytic function on the hyperbolic disk, the point process of zeroes is both insertion and deletion tolerant.

The rigidity and tolerance properties of point processes can be exploited to study various other phenomena on random point sets. For example, in [GKP12], we study applications of rigidity and tolerance to continuum percolation on the Ginibre and GAF zero ensembles. In [G12-I], we explore the relationship between rigidity phenomena in point processes and the completeness properties of certain systems of random exponential functions.

2 Plan of the paper

In Section 3, we begin with a detailed description of the models that we study, and also provide an abstract framework in which other models having similar characteristics can be investigated. Further, we show in Section 5 that for proving the main Theorems 1.1 - 1.4, it suffices to establish them in the case where $\mathcal{D}$ is a disk.

In order to study rigidity phenomena, we devise a unified approach in Section 6, where Theorem 6.1 gives general criteria for a function (of the inside points) to be rigid with respect to a point process. We complete the proofs of Theorems 1.1 and 1.3 by establishing the relevant criteria for the Ginibre and the GAF zero processes.

In Section 7, we study tolerance properties in the general setup introduced in Section 3.3. Theorem 7.2 lays down conditions under which certain tolerance behaviour of a point process can be established. Proving Theorems 1.2 and 1.4, therefore, amounts to showing that the relevant conditions hold for our models. However, unlike the rigidity phenomena, this requires substantially more work, and is carried out in two stages for each process. First, we obtain some estimates for the point processes $\mathcal{G}_n$ and $\mathcal{Z}_n$, which are finite approximations to $\mathcal{G}$ and $\mathcal{Z}$ respectively (see Sections 3.1 and 3.2 for definitions). For the Ginibre ensemble, this is done in Section 8, and for the GAF zeroes this is done in Section 10. Finally, we apply these estimates to deduce that the relevant conditions for tolerance behaviour hold for our models; this is carried out for the Ginibre ensemble in Section 9 and for the GAF zeroes in Section 11.

3 Models and Setup

3.1 The Ginibre Ensemble

Let us consider an $n \times n$ matrix $X^n, n \geq 1$ whose entries are i.i.d. standard complex Gaussians. The vector of its eigenvalues, in uniform random order, has the joint density (with respect to the Lebesgue measure on $\mathbb{C}^n$) given by

$$p(z_1, \dots, z_n) = \frac{1}{\pi^n \prod_{k=1}^n k!} e^{-\sum_{k=1}^n |z_k|^2} \prod_{i < j} |z_i - z_j|^2.$$

Recall that a determinantal point process on the Euclidean space $\mathbb{R}^d$ with kernel K and background measure μ is a point process on $\mathbb{R}^d$ whose k -point intensity functions with respect to the measure $\mu^{\otimes k}$ are given by

$$\rho_k(x_1, \dots, x_k) = \det \left[(K(x_i, x_j))_{i,j=1}^k \right].$$

Typically, K has to be such that the integral operator defined by K is a non-negative trace class contraction mapping $L^2(\mu)$ to itself. For a detailed study of determinantal point processes, we refer the reader to [HKPV10], [Bo11] and [Sos00]. More details on random matrices can be found in [DG09] and [AGZ09]. A simple calculation involving Vandermonde determinants shows that the eigenvalues of X^n (considered as a random point configuration) form a determinantal point process on $\mathbb{C}$. Its kernel is given by $K_n(z, w) = \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!}$ with respect to the background measure $d\gamma(z) = \frac{1}{\pi} e^{-|z|^2} d\mathcal{L}(z)$ where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{C}$. This point process is the Ginibre ensemble (of dimension n), which we will denote by $\mathcal{G}_n$. As $n \rightarrow \infty$, these point processes converge, in distribution, to a determinantal point process given by the kernel $K(z, w) = e^{z\bar{w}} = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!}$ with respect to the same background measure γ . This limiting point process is the infinite Ginibre ensemble $\mathcal{G}$. It is known that $\mathcal{G}$ is ergodic under the natural action of the translations of the plane, see e.g. [Sos00] Section 3 and Theorem 7 therein.

For more details on the Ginibre ensemble, we refer the reader to [HKPV10] Chapter 6.

3.2 The GAF zero process

Let $\{\xi_k\}_{k=0}^{\infty}$ be a sequence of i.i.d. standard complex Gaussians. Define

$$f_n(z) = \sum_{k=0}^n \xi_k \frac{z^k}{\sqrt{k!}} \quad (\text{for } n \geq 0), \quad f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}.$$

These are Gaussian processes on $\mathbb{C}$ with covariance kernels given by $K_n(z, w) = \sum_{k=0}^n \frac{(z\bar{w})^k}{k!}$ and $K(z, w) = \sum_{k=0}^{\infty} \frac{(z\bar{w})^k}{k!} = e^{z\bar{w}}$ respectively. A.s. f_n and f are entire functions and the functions f_n converge to f (in the sense of the uniform convergence of holomorphic functions on compact sets). It is not hard to see (e.g., via Rouché's theorem) that this implies that the corresponding point processes of zeroes, denoted by $\mathcal{Z}_n$, converge a.s. to the zero process $\mathcal{Z}$ of the GAF (in the sense of locally finite point configurations converging on compact sets). It is known that $\mathcal{Z}$ is ergodic under the natural action of the translations of the plane, see e.g. [HKPV10] Chapter 2 and Proposition 2.3.7 therein.

For more details on the GAF zero process, we refer the reader to [HKPV10] Chapter 2.

3.3 The General Setup

Fix a Euclidean space $\mathcal{E}$ equipped with a non-negative regular Borel measure μ . Let $\mathcal{S}$ denote the space of countable locally finite point configurations on $\mathcal{E}$. Endow $\mathcal{S}$ with its canonical topology, namely, the topology on $\mathcal{S}$ is that of weak convergence of measures, under the identification of a locally finite point configuration with the counting measure it induces on $\mathbb{C}$ (which gives $\mathcal{S}$ a canonical Borel σ -algebra). Equivalently, the topology of $\mathcal{S}$ is that of the convergence (in the

Hausdorff sense) of finite configurations obtained by restriction to open balls in $\mathcal{E}$. It is known that $\mathcal{S}$, with this topology, is a Polish space. Fix a bounded open set $\mathcal{D} \subset \mathcal{E}$. Corresponding to the decomposition $\mathcal{E} = \mathcal{D} \cup \mathcal{D}^c$, we have $\mathcal{S} = \mathcal{S}_{\text{in}} \times \mathcal{S}_{\text{out}}$, where $\mathcal{S}_{\text{in}}$ and $\mathcal{S}_{\text{out}}$ denote the spaces of finite point configurations on $\mathcal{D}$ and locally finite point configurations on $\mathcal{D}^c$ respectively.

Let Ξ be a measure space equipped with a probability measure $\mathbb{P}$. For a random variable $Z : \Xi \rightarrow \mathcal{X}$ (where $\mathcal{X}$ is a Polish space), we define the push forward $Z_*\mathbb{P}$ of the measure $\mathbb{P}$ by $Z_*\mathbb{P}(A) = \mathbb{P}(Z^{-1}(A))$ where A is a Borel set in $\mathcal{X}$. Also, for a point process $Z' : \Xi \rightarrow \mathcal{S}$, we can define point processes $Z'_{\text{in}} : \Xi \rightarrow \mathcal{S}_{\text{in}}$ and $Z'_{\text{out}} : \Xi \rightarrow \mathcal{S}_{\text{out}}$ by restricting the random configuration Z' to $\mathcal{D}$ and $\mathcal{D}^c$ respectively.

Let $X, X^n : \Xi \rightarrow \mathcal{S}$ be random variables such that $\mathbb{P}$ -a.s., we have $X^n \rightarrow X$ (in the topology of $\mathcal{S}$). We demand that the point processes X, X^n have their first intensity measures absolutely continuous with respect to μ . We can identify X_{in} (by taking the points in uniform random order) with the random vector $\underline{\zeta}$ which lives in $\bigcup_{m=0}^{\infty} \mathcal{D}^m$. The analogous quantity for X^n will be denoted by $\underline{\zeta}^n$.

For our models we can take $\mathcal{E}$ to be $\mathbb{C}$, μ to be the Lebesgue measure, and $\mathcal{D}$ to be a bounded open set.

In the case of the Ginibre ensemble, we can define the processes $\mathcal{G}_n$ and $\mathcal{G}$ on the same underlying probability space so that a.s. we have $\mathcal{G}_n \subset \mathcal{G}_{n+1} \subset \mathcal{G}$ for all $n \geq 1$. For a reference, see [Go10], in particular Theorem 3 therein. A bit of explanation is in order as to how we obtain this assertion. The $\mathcal{G}_n$ -s and $\mathcal{G}$ are determinantal point processes, whose kernels satisfy the Loewner ordering condition in Theorem 3, [Go10]. More precisely, if K_n is the kernel corresponding to $\mathcal{G}_n$ and K is the kernel corresponding to $\mathcal{G}$, then we have

$$K_n \preceq K_{n+1} \preceq K, \quad (1)$$

where $\preceq$ is the Loewner order. By $L_1 \preceq L_2$ for two operators L_1 and L_2 acting on a Hilbert space H , we mean that $L_1 - L_2$ is a non-negative definite operator acting on H . We extend this notion in the obvious way to the case when L_1 and L_2 are instead the kernels of integral operators acting on H . The inequality (1) is true because the K_n -s correspond to projections onto increasingly large subspaces of the Fock-Bargmann space (increasing with n). Thus, Theorem 3 in [Go10] implies that for every n , the process $\mathcal{G}_n$ is stochastically dominated by $\mathcal{G}_{n+1}$, which in turn is stochastically dominated by $\mathcal{G}$. Applying Strassen's Theorem (see, e.g., [Str65]) for point processes, we get the desired coupling. We take $(\Xi, \mathbb{P})$ to be this underlying probability space (where all the processes are coupled together), $X^n = \mathcal{G}_n$ and $X = \mathcal{G}$.

In the case of the Gaussian zero process, we take $(\Xi, \mathbb{P})$ to be a measure space on which we have countably many standard complex Gaussian random variables denoted by $\{\xi_k\}_{k=0}^{\infty}$. Then X^n is the zero set of the polynomial $f_n(z) = \sum_{k=0}^n \xi_k \frac{z^k}{\sqrt{k!}}$, and X is the zero set of the entire function $f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}$. The fact that $X^n \rightarrow X$ $\mathbb{P}$ -a.s. follows from Rouché's theorem.

4 A broad outline of the proof strategy

We devise a general approach to prove rigidity and tolerance of point processes. In our specific models of interest, we execute this approach, which principally involves obtaining precise estimates on certain quantities related to the finite dimensional approximations of our models.

The proof of the rigidity theorems involves statistics with small variance. Let A be a statistic (of the point process X living on a space $\mathcal{E}$) whose rigidity we want to prove, in the context of a domain $\mathcal{D} \subset \mathcal{E}$. Generally, $A = A(X_{\text{in}})$ will be a function of the points X_{in} inside the domain $\mathcal{D}$. Suppose we can construct a family of statistics H_ε of the point process X such that $\text{Var}[H_\varepsilon] \rightarrow 0$ as the parameter $\varepsilon \rightarrow 0$. Suppose also that H_ε can be written as

$$H_\varepsilon(X) = A(X_{\text{in}}) + \tilde{H}_\varepsilon(X_{\text{out}}),$$

where $\tilde{H}_\varepsilon$ is a statistic which depends only on the points X_{out} outside $\mathcal{D}$. Since $\text{Var}[H_\varepsilon] \rightarrow 0$ as $\varepsilon \rightarrow 0$, for small ε we have, roughly speaking, $H_\varepsilon \approx \mathbb{E}[H_\varepsilon]$ (in the sense that the difference between the two quantities is small with high probability).

But $\mathbb{E}[H_\varepsilon]$ can be computed, in principle, using the one and multi point intensity measures of the point process X . This means that

$$A(X_{\text{in}}) + \tilde{H}_\varepsilon(X_{\text{out}}) \approx \mathbb{E}[H_\varepsilon],$$

which is known for each ε . The quantity $\tilde{H}_\varepsilon(X_{\text{out}})$ can also be computed exactly, because the point configuration X_{out} is known. This suggests that for each ε , we get an approximation to $A(X_{\text{in}})$ (in the form of $\mathbb{E}[H_\varepsilon] - \tilde{H}_\varepsilon(X_{\text{out}})$), which a.s. converges to its actual value as $\varepsilon \rightarrow 0$. Typically, the statistic H_ε would be a linear statistic, meaning that there is a function $\Phi_\varepsilon : \mathcal{E} \rightarrow \mathbb{C}$ such that $H_\varepsilon(X) = \int \Phi_\varepsilon d[X]$. In this case, we have $\mathbb{E}[H_\varepsilon] = \int \Phi_\varepsilon d\rho_1(x)$, where ρ_1 is the first intensity measure of the point process. Often (though not always), the family of functions Φ_ε can be taken to be the scalings of a single function Φ (chosen to have some specific properties); e.g. $\Phi_\varepsilon(x) = \Phi(\varepsilon x)$.

Our approach to tolerance proceeds via finite dimensional approximations X^n to the (infinite size) point process X (which are coupled together). The idea of tolerance pertains to proving the absolute continuity of certain conditional measures with respect to a standard reference measure. Let us consider a scenario where we want to show that a certain measure μ is mutually absolutely continuous with respect to the Lebesgue measure on an open set E (in our problem μ will usually correspond to the conditional distribution of the vector of points corresponding to X_{in} given X_{out}). This would mean that μ has a density f with respect to the Lebesgue measure on E (which we denote by λ), that is positive almost everywhere. Let us further simplify to the case when we need to prove that the density f is, in fact, bounded away from 0 and ∞ . While we might not know a priori that μ even has a density, it would suffice to prove (in this case) that for any small ball $B \subset E$, we have $\mu(B)/\lambda(B)$ is bounded between two constants, independent of B .

If we want to do this when μ is a conditional measure coming from a point process X of infinite size, then obtaining explicit estimates on $\mu(B)$ directly from a description of X can be an intractable problem. However, for a point process of finite size, we can often write down concrete expressions for the conditional measures, resulting from the fact that often there is a closed form expression for the joint density of the points. Then, we can estimate the ratio $\mu_n(B)/\lambda(B)$, where μ_n is the conditional distribution of the (vector of points corresponding to) X_{in}^n , given X_{out}^n . If we can establish upper and lower bounds on $\mu_n(B)/\lambda(B)$ which are uniform in n , then we can try to take limits and claim that similar bounds hold for $\mu(B)/\lambda(B)$. But as n changes, the conditioning variable (i.e., X_{out}^n) also changes, in a way that is determined by the coupling of X^n -s and X (ensuring that $X_{\text{out}}^n \rightarrow X_{\text{out}}$). Thus, the convergence of the (random) measures μ_n to μ (random because of their dependence on X_{out}^n and X_{out}) poses a significant technical challenge. In the event that appropriate probabilistic estimates can be obtained from the finite dimensional approximations, this problem can be tackled and the limit pushed through; this is the content of Theorem 7.2.

We approach the question of tolerance in a modular fashion. In Section 7, we work in the general setup of coupled point processes and state Theorem 7.2. Then in separate sections, we obtain the necessary finite dimensional estimates for the models of our interest, and show that these indeed satisfy the conditions laid out in Theorem 7.2. This enables us to carry out the approximation procedure outlined above.

5 Reduction from a general $\mathcal{D}$ to a disk

In this Section we intend to prove that in order to obtain Theorems 1.1-1.4, it suffices to consider the case where $\mathcal{D}$ is an open disk centred at the origin. We will demonstrate the proof for Theorems 1.3 and 1.4, the arguments for Theorems 1.1 and 1.2 are on similar lines.

Let $\mathcal{D}$ be a bounded open set in $\mathbb{C}$. By the translation invariance of the GAF zero ensemble, we take the origin to be in the interior of $\mathcal{D}$. Let $\mathcal{D}_0$ be a disk (centred at the origin) which contains $\overline{\mathcal{D}}$ in its interior (where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$).

Suppose we know the point configuration $\mathcal{Z}_{\text{out}}$ to be equal to Υ_{out} . Further, suppose that we can show that the point configuration $\Upsilon_{\text{out}}^{\mathcal{D}_0}$ outside $\mathcal{D}_0$ determines the number N_0 and the sum S_0 of the points inside $\mathcal{D}_0$ a.s. Since we also know the number and the sum of the points inside $\mathcal{D}_0 \setminus \mathcal{D}$, we can determine the number N as well as the sum S of the points in $\mathcal{D}$. This proves the rigidity theorem for the GAF zero ensemble (Theorem 1.3) for a general $\mathcal{D}$.

Now suppose we have the tolerance Theorem 1.4 for a disk. To obtain Theorem 1.4 for $\mathcal{D}$, we appeal to the tolerance Theorem 1.4 for the disk $\mathcal{D}_0$. Define

$$\Sigma := \left\{ (\lambda_1, \dots, \lambda_N) : \sum_{j=1}^N \lambda_j = S, \lambda_j \in \mathcal{D} \right\}$$

and

$$\Sigma_0 := \left\{ (\lambda_1, \dots, \lambda_{N_0}) : \sum_{j=1}^{N_0} \lambda_j = S_0, \lambda_j \in \mathcal{D}_0 \right\}.$$

The conditional distribution of the vector of points inside $\mathcal{D}_0$, given $\Upsilon_{\text{out}}^{\mathcal{D}_0}$, lives on Σ_0 , in fact it has a density f_0 which is positive a.e. with respect to Lebesgue measure on Σ_0 . Let there be k points in $\mathcal{D}_0 \setminus \mathcal{D}$ and let their sum be s , clearly we have $N = N_0 - k$ and $S = S_0 - s$. We parametrize Σ by the last $N - 1$ coordinates. Note that the set $U := \{(\lambda_2, \dots, \lambda_N) : (S - \sum_{j=2}^N \lambda_j, \lambda_2, \dots, \lambda_N) \in \Sigma\}$ is an open subset of $\mathcal{D}^{N-1}$. Further, we define the set $V := \{(\lambda_1, \dots, \lambda_k) : \lambda_i \in \mathcal{D}_0 \setminus \mathcal{D}, \sum_{i=1}^k \lambda_i = s\}$.

Let the points in $\mathcal{D}_0 \setminus \mathcal{D}$, taken in uniform random order, form the vector $\underline{z} = (z_1, \dots, z_k)$. Then we can condition the vector of points in $\mathcal{D}_0$ to have its last k coordinates equal to $\underline{z}$, to obtain the following formula for the conditional density of the vector of points in $\mathcal{D}$ at $(\zeta_1, \dots, \zeta_N) \in \Sigma$ (with respect to the Lebesgue measure on Σ):

$$f(\zeta_1, \zeta_2, \dots, \zeta_N) = \frac{f_0(\zeta_1, \zeta_2, \dots, \zeta_N, z_1, \dots, z_k)}{\int_U f_0(s - (\sum_{j=2}^N w_j), w_2, \dots, w_N, z_1, \dots, z_k) dw_2 \cdots dw_N}. \quad (2)$$

It is clear that for a.e. $\underline{z} \in V$, we have f is strictly positive a.e. with respect to Lebesgue measure on Σ , because the same is true of f_0 on Σ_0 .

6 The Rigidity Phenomenon

We begin by giving a precise definition of rigidity. Recall the general setup in Section 3.3.

Definition 1. A measurable function $f_{\text{in}} : \mathcal{S}_{\text{in}} \rightarrow \mathbb{C}$ is said to be **rigid** with respect to the point process X on $\mathcal{S}$ if there is a measurable function $f_{\text{out}} : \mathcal{S}_{\text{out}} \rightarrow \mathbb{C}$ such that a.s. we have $f_{\text{in}}(X_{\text{in}}) = f_{\text{out}}(X_{\text{out}})$.

In this section, we prove that the number of points in $\mathcal{D}$ in the case of the Ginibre ensemble and the number as well as the sum of the points in $\mathcal{D}$ for the GAF zero process are rigid. In fact, we will state some general conditions that ensure such rigid behaviour, and then show that the Ginibre and the GAF satisfy the relevant conditions.

From here on in this section, we will consider point processes on $\mathbb{C}$. We will use linear statistics of point processes as the main tool that will enable us to obtain rigidity results for such point processes.

Definition 2. Let Π be a point process on $\mathbb{C}$ and let φ be a compactly supported continuous function on $\mathbb{C}$. The **linear statistic** corresponding to φ is the random variable $\int \varphi d[\Pi] = \sum_{z \in \Pi} \varphi(z)$.

By a C_c^k function on $\mathbb{C}$, we mean a compactly supported C^k function on $\mathbb{C}$.

We can now state:

Theorem 6.1. Let Π be a point process on $\mathbb{C}$ whose first intensity is absolutely continuous with respect to Lebesgue measure, and let $\mathcal{D}$ be a bounded open set in $\mathbb{C}$. Let φ be a continuous function on $\mathbb{C}$. Suppose that for any $1 > \varepsilon > 0$, we have a C_c^2 function Φ^ε such that $\Phi^\varepsilon = \varphi$ on $\mathcal{D}$, and $\text{Var}(\int_{\mathbb{C}} \Phi^\varepsilon d[\Pi]) < \varepsilon$. Then the function $T : \mathcal{S}_{\text{in}} \rightarrow \mathbb{C}$ given by $T(\Upsilon_{\text{in}}) = \int_{\mathcal{D}} \varphi d[\Upsilon_{\text{in}}] = \sum_{z \in \Upsilon \cap \mathcal{D}} \varphi(z)$ is rigid with respect to Π .

Proof. Consider the sequence of C_c^2 functions $\Phi^{2^{-n}}, n \geq 1$. Note that $\mathbb{E} \left[\int_{\mathbb{C}} \Phi^{2^{-n}} d[\Pi] \right] = \int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L}$ where $\rho_1(z)$ is the one point intensity function of Π . It follows from Chebyshev's inequality that

$$\mathbb{P} \left(\left| \int_{\mathbb{C}} \Phi^{2^{-n}} d[\Pi] - \mathbb{E} \left[\int_{\mathbb{C}} \Phi^{2^{-n}} d[\Pi] \right] \right| > 2^{-n/4} \right) \leq 2^{-n/2}.$$

The Borel Cantelli lemma implies that with probability 1, as $n \rightarrow \infty$ we have

$$\left| \int_{\mathbb{C}} \Phi^{2^{-n}} d[\Pi] - \mathbb{E} \left[\int_{\mathbb{C}} \Phi^{2^{-n}} d[\Pi] \right] \right| \rightarrow 0.$$

But

$$\int_{\mathbb{C}} \Phi^{2^{-n}} d[\Pi] = \int_{\mathcal{D}} \Phi^{2^{-n}} d[\Pi] + \int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\Pi].$$

Thus we have, as $n \rightarrow \infty$

$$\left| \int_{\mathcal{D}} \Phi^{2^{-n}} d[\Pi] + \int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\Pi] - \int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L} \right| \rightarrow 0. \quad (3)$$

If we know Π_{out} , we can compute $\int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\Pi]$ exactly, also ρ_1 is known explicitly; in case of a translation invariant point process Π it is, in fact, a constant $c(\Pi)$. Hence, from the limit in (3), a.s. we can obtain $\int_{\mathcal{D}} \Phi^{2^{-n}} d[\Pi] = \int_{\mathcal{D}} \varphi d[\Pi]$ as the limit

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{C}} \Phi^{2^{-n}} \rho_1 d\mathcal{L} - \int_{\mathcal{D}^c} \Phi^{2^{-n}} d[\Pi] \right).$$

■

We now use Theorem 6.1 to establish Theorems 1.1 and 1.3.

Proof of Theorems 1.1 and 1.3. We have already seen in Section 5 that it suffices to take $\mathcal{D}$ to be a disk centred at the origin. We intend to construct functions Φ^ε as in Theorem 6.1.

Let r_0 be the radius of $\mathcal{D}$. Fix $\varepsilon > 0$.

We begin with the continuous function $\tilde{\Psi}$ on $\mathbb{R}_+ \cup \{0\}$ such that

$$\tilde{\Psi}(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq 2r_0, \\ -\frac{\varepsilon}{2} \log r + \frac{\varepsilon}{2} \log 2r_0 + 1 & \text{for } 2r_0 \leq r \leq 2r_0 \exp(2/\varepsilon), \\ 0 & \text{for } r \geq 2r_0 \exp(2/\varepsilon). \end{cases}$$

Notice that $\tilde{\Psi}'(r) = -\varepsilon/2r$ and $\tilde{\Psi}''(r) = \varepsilon/2r^2$ for $2r_0 \leq r \leq 2r_0 \exp(2/\varepsilon)$. We can then smooth the function $\tilde{\Psi}$ at $2r_0$ and $2r_0 \exp(2/\varepsilon)$ such that the resulting function Ψ_1 is C^2 on the positive reals, takes the values 1 for $r \leq r_0$ and 0 for $r \geq 2r_0 e^{2/\varepsilon} + 1$, and satisfies the inequalities $|\Psi_1'(r)| \leq \varepsilon/r$ and $|\Psi_1''(r)| \leq \varepsilon/r^2$ for all $r > 0$. Finally, we define the radial C_c^2 function Ψ on $\mathbb{C}$ as $\Psi(z) = \Psi_1(|z|)$.

For $\mathcal{G}$, we know (see [RV07] Theorem 11) that there exists a constant $C_1 > 0$ such that for every radial C_c^2 function Ψ we have

$$\text{Var} \left(\int_{\mathbb{C}} \Psi d[\mathcal{G}] \right) \leq C_1 \int_{\mathbb{C}} |\nabla \Psi(z)|^2 d\mathcal{L}(z).$$

But from the definition of Ψ it is clear that $\int_{\mathbb{C}} |\nabla \Psi(z)|^2 d\mathcal{L}(z) \leq C_2 \varepsilon$. We apply Theorem 6.1 with $\varphi \equiv 1$, and choose $\Phi^{C_1 C_2 \varepsilon} = \Psi$ as defined above; recall that $\Psi \equiv 1$ on $\mathcal{D}$.

For $\mathcal{Z}$, we know (see [NS11] Theorem 1.1) that there exists a constant $C_3 > 0$ such that every C_c^2 function ϑ satisfies

$$\text{Var} \left(\int_{\mathbb{C}} \vartheta d[\mathcal{Z}] \right) \leq C_3 \int_{\mathbb{C}} |\Delta \vartheta(z)|^2 d\mathcal{L}(z).$$

For the rigidity of the number of points of $\mathcal{Z}$ in $\mathcal{D}$ we make exactly the same choice as we did for $\mathcal{G}$, and note that $\int_{\mathbb{C}} |\Delta \Psi(z)|^2 d\mathcal{L}(z) \leq C_4 \varepsilon^2$. For the rigidity of the sum of points of $\mathcal{Z}$ in $\mathcal{D}$ we intend to apply Theorem 6.1 with $\varphi(z) = z$. We consider the function $\theta(z) = z\Psi(z)$, Ψ as before. Observe that $\Delta \theta(z) = 4 \frac{\partial \Psi}{\partial \bar{z}}(z) + z \Delta \Psi(z)$. Using this, for $\varepsilon < 1$, we get $\int_{\mathbb{C}} |\Delta(z\Psi(z))|^2 d\mathcal{L}(z) \leq C_5 \varepsilon$. It remains to note that for $z \in \mathcal{D}$ we have $\Psi(z) = 1$ and $\theta(z) = z$. ■

7 Tolerance: Limits of Conditional Measures

The tolerance properties are established for both models by obtaining explicit bounds on conditional probability measures for finite approximations (finite matrices in case of Ginibre and polynomials for GAF) and then passing to the limit. In this Section, we state and prove some general conditions (in the context of the abstract setup considered in Section 3.3), which will enable us to make the transition from the finite ensembles to the infinite one.

We start with the following general proposition:

Proposition 7.1. *Let Γ be a second countable topological space. Let Σ be a countable basis of open sets in Γ and let $\mathfrak{A} := \{\bigcup_{i=1}^k \sigma_i : \sigma_i \in \Sigma, k \geq 1\}$. Let $c > 0$. To verify that two non-negative Borel measures μ_1 and μ_2 on Γ satisfy $\mu_1(B) \leq c\mu_2(B)$ for all Borel sets B in Γ where μ_2 is known to be a regular Borel measure, it suffices to verify the inequality for all sets in $\mathfrak{A}$.*

Proof. Any open set $U \subset \Gamma$ is a countable union $\bigcup_{i=1}^{\infty} \sigma_i, \sigma_i \in \Sigma$ because the sets in Σ form a basis for the topology on Γ . If we have $\mu_1(\bigcup_{i=1}^n \sigma_i) \leq c\mu_2(\bigcup_{i=1}^n \sigma_i)$, then we can let $n \rightarrow \infty$ to obtain $\mu_1(U) \leq c\mu_2(U)$. Once we have the inequality for all open sets U , we can extend it to all Borel sets as follows. Let B be a Borel set and U be an open set containing B . Then we have

$$\mu_1(B) \leq \mu_1(U) \leq c\mu_2(U).$$

But for a regular Borel measure μ and for any Borel set $B \subset \Gamma$, we have $\mu(B) = \inf_{B \subset U} \mu(U)$ where the infimum is taken over all open sets U containing B . Therefore, taking infimum over all open sets U containing B on the right hand side of the last display, we obtain

$$\mu_1(B) \leq c\mu_2(B),$$

as desired. ■

In this Section, we will work in the setup of Section 3.3, specifying $\mathcal{D}$ to be an open ball, and requiring that the first intensity of our point process X is absolutely continuous with respect to the Lebesgue measure on $\mathcal{E}$. We further assume that X exhibits rigidity of the number of points. In other words,

Definition 3. *There is a measurable function $N : \mathcal{S}_{\text{out}} \rightarrow \mathbb{N} \cup \{0\}$ such that a.s. we have*

$$\text{Number of points in } X_{\text{in}} = N(X_{\text{out}}).$$

In such a situation, we can identify X_{in} (by taking the points in uniform random order) with a random vector ζ taking values in $\mathcal{D}^{N(X_{\text{out}})}$. Studying the conditional distribution $\varrho(X_{\text{out}}, \cdot)$ of X_{in} given X_{out} is then the same as studying the conditional distribution of this random vector given X_{out} . We will denote the latter distribution by $\rho(X_{\text{out}}, \cdot)$. Note that it is supported on $\mathcal{D}^{N(X_{\text{out}})}$ (see Section 1 for details).

For $m > 0$, let $\mathfrak{W}_{\text{in}}^m$ denote the countable basis for the topology on $\mathcal{D}^m$ formed by open balls contained in $\mathcal{D}^m$ and having rational centres and rational radii. We define the collection of sets $\mathfrak{A}^m := \{\bigcup_{i=1}^k A_i : A_i \in \mathfrak{W}_{\text{in}}^m, k \geq 1\}$.

Fix an integer $n \geq 0$, a closed annulus $B \subset \mathcal{D}^c$ whose centre is at the origin and which has a rational inradius (or inradius equal to the radius of $\mathcal{D}$) and a rational outradius, and a collection of n disjoint open balls B_i with rational radii and centres having rational coordinates such that $\{B_i \cap \mathcal{D}^c\}_{i=1}^n \subset B$. Let $\Phi(n, B, B_1, \dots, B_n)$ be the Borel subset of $\mathcal{S}_{\text{out}}$ defined as follows:

$$\Phi(n, B, B_1, \dots, B_n) = \{\Upsilon \in \mathcal{S}_{\text{out}} : |\Upsilon \cap B| = n, |\Upsilon \cap B_i| = 1\}.$$

Then the countable collection $\Sigma_{\text{out}} = \{\Phi(n, B, B_1, \dots, B_n) : n, B, B_i \text{ as above}\}$ is a basis for the topology of $\mathcal{S}_{\text{out}}$. This follows from the characterization of the topology (on the space of locally finite point configurations) as that of the convergence (in the Hausdorff sense) of the restrictions to balls. Define the collection of sets $\mathcal{B} := \{\bigcup_{i=1}^k \Phi_i : \Phi_i \in \Sigma_{\text{out}}, k \geq 1\}$.

We will denote by Ω^m the event that $|X_{\text{in}}| = m$, and by Ω_n^m we will denote the event $|X_{\text{in}}^n| = m$; we recall here that X^n is an approximation of X .

Definition 4. Let P and Q be two sets, and let $\alpha(p, q)$ and $\beta(p, q)$ be non-negative functions defined on $P \times Q$. We write $\alpha(p, q) \asymp_q \beta(p, q)$ if there exist positive functions $k_1(q), k_2(q)$ defined on Q such that

$$k_1(q)\alpha(p, q) \leq \beta(p, q) \leq k_2(q)\alpha(p, q) \text{ for all } p, q.$$

The main point is that k_1, k_2 in the above inequalities are uniform on P .

We will also use the notation introduced in Section 3.3. We will define an “exhausting” sequence of events as:

Definition 5. A sequence of events $\{\Omega(j)\}_{j \geq 1}$ is said to exhaust another event Ω if $\Omega(j) \subset \Omega(j+1) \subset \Omega$ for all j and $\mathbb{P}(\Omega \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$.

Let $\mathcal{M}(\mathcal{D}^m)$ denote the space of all probability measures on $\mathcal{D}^m$. Let $\mathfrak{B}(\mathcal{D}^m)$ denote the Borel sigma-algebra on $\mathcal{D}^m$. For a mapping $Y : S \rightarrow T$, by $Y^{-1}(B)$ (where $B \subset T$) we mean the set $\{\omega \in S : Y(\omega) \in B\}$. We recall the notion of two measures μ_1 and μ_2 being mutually absolutely continuous, denoted $\mu_1 \equiv \mu_2$, from Section 1. Finally, recall the definition of $\mathfrak{A}^m$ and $\mathcal{B}$ from the beginning of this section.

Now we are ready to state the following important technical reduction:

Theorem 7.2. Fix $m \geq 0$ such that $\mathbb{P}(\Omega^m) > 0$. Suppose that:

(a) There is a map $\nu : \mathcal{S}_{\text{out}} \times \mathfrak{B}(\mathcal{D}^m) \rightarrow [0, 1]$ such that for each $\Upsilon_{\text{out}} \in \mathcal{S}_{\text{out}}$ we have $\nu(\Upsilon_{\text{out}}, \cdot) \in \mathcal{M}(\mathcal{D}^m)$ and for each Borel set $A \subset \mathcal{D}^m$, the map $\Upsilon_{\text{out}} \rightarrow \nu(\Upsilon_{\text{out}}, A)$ is measurable.

(b) For each fixed j we have a sequence $\{n_k\}_{k \geq 1}$ (which might depend on j) and corresponding events $\Omega_{n_k}(j)$ such that:

(i) $\Omega_{n_k}(j) \subset \Omega_{n_k}^m$.

(ii) $\Omega(j) := \varinjlim_{k \rightarrow \infty} \Omega_{n_k}(j)$ exhaust Ω^m as $j \uparrow \infty$.

(iii) For all $A \in \mathfrak{A}^m$ and $B \in \mathcal{B}$ we have

$$\mathbb{P}[(X_{\text{in}}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \asymp_j \int_{(X_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi) + \vartheta(k; j, A, B). \quad (\text{C3})$$

where $\lim_{k \rightarrow \infty} \vartheta(k; j, A, B) = 0$ for each fixed j, A and B .

Then a.s. on the event Ω^m we have

$$\rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot). \quad (\text{C4})$$

We defer the proof of Theorem 7.2 to Section 13.

We conclude this section with the following simple observations:

Remark 7.1. If Theorem 7.2 holds for all $m \geq 0$, then we can conclude that (C4) holds a.e. $\xi \in \Xi$.

Remark 7.2. The relation (C3) is equivalent to the following statement: for each fixed j , there exist positive quantities C_j, c_j such that for any $A \in \mathfrak{A}^m$, $B \in \mathcal{B}$ we have:

$$\overline{\lim}_{k \rightarrow \infty} \mathbb{P}[(X_{\text{in}}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \leq C_j \overline{\lim}_{k \rightarrow \infty} \int_{(X_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi)$$

and

$$\underline{\lim}_{k \rightarrow \infty} \mathbb{P}[(X_{\text{in}}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \geq c_j \underline{\lim}_{k \rightarrow \infty} \int_{(X_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi).$$

We emphasize that C_j, c_j in the above pair of inequalities do not depend on A or B .

Remark 7.3. In Theorem 7.2, one should think of $\{n_k\}_k$ as the index of an approximating subsequence of point processes $\{X^{n_k}\}_k$. The parameter j , on the other hand, enumerates an exhaustion of the relevant part Ω^m of the sample space (as a toy example, one can think of the subsets $\{|Y| \leq j\}$ for some random variable Y , which exhaust the entire sample space as $j \rightarrow \infty$).

8 Tolerance of the Ginibre Ensemble

In this section we obtain several estimates necessary to prove Theorem 1.2 in the case where $\mathcal{D}$ is a fixed disk centred at the origin. Fix an integer $m \geq 0$.

8.1 Estimates for $\mathcal{G}_n$

Let $\delta \in (0, 1)$. We consider the event $\Omega_n^{m, \delta}$ which entails that $\mathcal{G}_n$ has exactly m points inside $\mathcal{D}$, and there is a δ separation between $\partial\mathcal{D}$ and $(\mathcal{G}_n)_{\text{out}}$; the analogous event for $\mathcal{G}$ will be denoted by $\Omega^{m, \delta}$. Notice that $\Omega^{m, \delta}$ has positive probability (which is bounded away from 0 uniformly in δ), so by the convergence of $\mathcal{G}_n$ -s to $\mathcal{G}$, we obtain that $\Omega_n^{m, \delta}$ -s have positive probability that is uniformly bounded away from 0 for large enough n .

A brief explanation is in order for the last assertion regarding $\mathbb{P}(\Omega^{m, \delta}) > 0$. First, note that the events $\Omega^{m, \delta} \uparrow \Omega^m$, so it suffices to prove $\mathbb{P}(\Omega^m) > 0$. But $\mathcal{G}$ being a determinantal point process with the kernel discussed in Section 3, the number of points of $\mathcal{G}$ in $\mathcal{D}$ is a sum of independent Bernoulli random variables, each with success probability strictly between 0 and 1. This implies that for each $m \geq 0$, we have $\mathbb{P}(\Omega^m) > 0$.

We denote the points of $\mathcal{G}_n$ inside $\mathcal{D}$ (in uniform random order) by $\underline{\zeta} = (\zeta_1, \dots, \zeta_m)$ and those outside $\mathcal{D}$ (in uniform random order) by $\underline{\omega} = (\omega_1, \omega_2, \dots, \omega_{n-m})$. Following the notation introduced in Section 1, for a vector $(\underline{\zeta}, \underline{\omega})$ as above, we denote $\Upsilon_{\text{in}} = \{\zeta_i\}_{i=1}^m$, $\Upsilon_{\text{out}} = \{\omega_j\}_{j=1}^{n-m}$, and $\Upsilon = \Upsilon_{\text{in}} \cup \Upsilon_{\text{out}}$. For a vector $\underline{\gamma} = (\gamma_1, \dots, \gamma_N)$ of N points in $\mathbb{C}$, we denote by $\Delta(\underline{\gamma})$ the Vandermonde determinant $\prod_{i < j} (\gamma_i - \gamma_j)$. For two vectors $\underline{\gamma}_1, \underline{\gamma}_2$ we set $\Delta(\underline{\gamma}_1, \underline{\gamma}_2) = \Delta(\underline{\gamma})$ where $\underline{\gamma}$ denotes the concatenated vector $(\underline{\gamma}_1, \underline{\gamma}_2)$.

Then the conditional distribution $\rho(\Upsilon_{\text{out}}, \underline{\zeta})$ of $\underline{\zeta}$ given Υ_{out} has the density

$$\rho_{\underline{\omega}}^n(\underline{\zeta}) = C(\underline{\omega}) |\Delta(\underline{\zeta}, \underline{\omega})|^2 \exp \left(- \sum_{k=1}^m |\zeta_k|^2 \right) \quad (4)$$

with respect to the Lebesgue measure on $\mathcal{D}^m$, where $C(\underline{\omega})$ is the normalizing constant (which depends on $\underline{\omega}$). Let $(\underline{\zeta}, \underline{\omega})$ and $(\underline{\zeta}', \underline{\omega})$ correspond to two configurations such that the event $\Omega_n^{m, \delta}$ occurs in both cases. Then the ratio of the conditional densities at these two points is given by

$$\frac{\rho_{\underline{\omega}}^n(\underline{\zeta}')}{\rho_{\underline{\omega}}^n(\underline{\zeta})} = \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2 \exp(-\sum_{k=1}^m |\zeta'_k|^2)}{|\Delta(\underline{\zeta}, \underline{\omega})|^2 \exp(-\sum_{k=1}^m |\zeta_k|^2)}. \quad (5)$$

Our goal in this section is to obtain upper and lower bounds on the ratio of conditional densities $\frac{\rho_{\underline{\omega}}^n(\underline{\zeta}')}{\rho_{\underline{\omega}}^n(\underline{\zeta})}$. We will take up, one after the other, the issues of bounding the quantities appearing in the expression (4) for this ratio.

Clearly, $\exp \left(- \sum_{k=1}^m |\zeta'_k|^2 \right) / \exp \left(- \sum_{k=1}^m |\zeta_k|^2 \right)$ is bounded above and below by constants which are functions of m and $\mathcal{D}$.

To study the ratio of the Vandermonde determinants, we define

$$\Gamma(\underline{\zeta}, \underline{\omega}) = \prod_{1 \leq i \leq m, 1 \leq j \leq n-m} (\zeta_i - \omega_j).$$

Then we have

$$\frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} = \frac{|\Delta(\underline{\zeta}')|^2 |\Gamma(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta})|^2 |\Gamma(\underline{\zeta}, \underline{\omega})|^2}. \quad (6)$$

In order to bound $\frac{|\Gamma(\underline{\zeta}', \underline{\omega})|^2}{|\Gamma(\underline{\zeta}, \underline{\omega})|^2}$ from above and below uniformly in $\underline{\zeta}, \underline{\zeta}' \in \mathcal{D}^m$, it suffices to bound

$\frac{|\Gamma(\underline{\zeta}, \underline{\omega})|}{|\Gamma(\underline{0}, \underline{\omega})|}$ from above and below uniformly in $\underline{\zeta} \in \mathcal{D}^m$. Here $\underline{0}$ is the vector of all 0-s in $\mathcal{D}^m$. We observe that

$$\frac{|\Gamma(\underline{\zeta}, \underline{\omega})|}{|\Gamma(\underline{0}, \underline{\omega})|} = \prod_{i=1}^m \left(\prod_{j=1}^{n-m} \left| \frac{\zeta_i - \omega_j}{\omega_j} \right| \right).$$

Since m is fixed, it suffices to bound $\prod_{j=1}^{n-m} \left| \frac{\zeta_0 - \omega_j}{\omega_j} \right|$ from above and below uniformly in $\zeta_0 \in \mathcal{D}$. To this end we prove

Proposition 8.1. *Let $\zeta_0 \in \mathcal{D}$ and $\omega_1, \dots, \omega_{n-m}$ be such that $\text{dist}(\omega_j, \mathcal{D}) > \delta$ for all j . Then we have*

$$\left| \log \left(\prod_{j=1}^{n-m} \left| \frac{\zeta_0 - \omega_j}{\omega_j} \right| \right) \right| \leq K_1(\mathcal{D}) \left| \sum_{j=1}^{n-m} \frac{1}{\omega_j} \right| + K_2(\mathcal{D}) \left| \sum_{j=1}^{n-m} \frac{1}{\omega_j^2} \right| + K_3(\mathcal{D}, \delta) \left(\sum_{j=1}^{n-m} \frac{1}{|\omega_j|^3} \right).$$

Here $K_1(\mathcal{D})$, $K_2(\mathcal{D})$ and $K_3(\mathcal{D}, \delta)$ are constants depending on $\mathcal{D}$ and δ .

Proof. We begin with $\log \left| \frac{\zeta_0 - \omega_j}{\omega_j} \right| = \log \left| 1 - \frac{\zeta_0}{\omega_j} \right|$. Due to the δ -separation between $\partial\mathcal{D}$ and ω , the ratio $\theta_j := \frac{\zeta_0}{\omega_j}$ satisfies $|\theta_j| \leq \frac{r}{r+\delta} < 1$. Let $\log$ be the branch of the complex logarithm given by the power series development $\log(1 - z) = -\left(\sum_{k=1}^{\infty} \frac{z^k}{k}\right)$ for $|z| < 1$. Then we have $\log|1 - \theta_j| = \Re \log(1 - \theta_j) = -\Re(\theta_j) - \frac{1}{2}\Re(\theta_j^2) + h(\theta_j)$ where $|h(\theta_j)| \leq K'_3(\mathcal{D}, \delta)|\theta_j|^3$, where K'_3 is a constant depending on $\mathcal{D}$ and δ . Hence,

$$\left| \log \left(\prod_{j=1}^{n-m} \left| \frac{\zeta_0 - \omega_j}{\omega_j} \right| \right) \right| = \left| \sum_{j=1}^{n-m} \log \left(\left| \frac{\zeta_0 - \omega_j}{\omega_j} \right| \right) \right| = \left| -\Re \left(\sum_{j=1}^{n-m} \theta_j \right) - \frac{1}{2}\Re \left(\sum_{j=1}^{n-m} \theta_j^2 \right) + \sum_{j=1}^{n-m} h(\theta_j) \right|.$$

Recall that $\theta_j = \frac{\zeta_0}{\omega_j}$ and $|\zeta_0| \leq r$ and $|h(\theta_j)| \leq K'_3(\mathcal{D}, \delta)|\theta_j|^3$. The triangle inequality applied to the above gives us the statement of the Proposition with $K_1(\mathcal{D}) = r$, $K_2(\mathcal{D}) = \frac{1}{2}r^2$ and $K_3(\mathcal{D}, \delta) = K'_3(\mathcal{D}, \delta)r^3$. $\blacksquare$

As a result, we have

Proposition 8.2. *Conditionally on $\Omega_n^{m, \delta}$, there exists a constant $K(\mathcal{D}, \delta) > 0$ such that*

$$\exp \left(-4mK(\mathcal{D}, \delta)\mathbf{X}_n \right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{\rho_{\underline{\omega}}^n(\underline{\zeta}')}{\rho_{\underline{\omega}}^n(\underline{\zeta})} \leq \exp \left(4mK(\mathcal{D}, \delta)\mathbf{X}_n \right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2},$$

where $\mathbf{X}_n = \left| \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega} \right| + \left| \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega^2} \right| + \left(\sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^3} \right)$ and $\mathbb{E}[\mathbf{X}_n] \leq c_1(\mathcal{D}, m) < \infty$.

Proof. Clearly, it suffices to bound $|\Delta(\underline{\zeta}', \underline{\omega})|^2 / |\Delta(\underline{\zeta}, \underline{\omega})|^2$ from above and below.

From Proposition 8.1, on $\Omega_n^{m, \delta}$ we have $\left| \log \left(\prod_{j=1}^{n-m} \left| \frac{\zeta_0 - \omega_j}{\omega_j} \right| \right) \right| \leq K(\mathcal{D}, \delta)\mathbf{X}_n$ for any $\zeta_0 \in \mathcal{D}$, where $K(\mathcal{D}, \delta) = \max\{K_1, K_2, K_3\}$. Considering this for each $\zeta_i, i = 1, \dots, m$, exponentiating and then taking product over $i = 1, \dots, m$, we get

$$\exp \left(-mK(\mathcal{D}, \delta)\mathbf{X}_n \right) \leq \frac{|\Gamma(\underline{\zeta}, \underline{\omega})|}{|\Gamma(\underline{0}, \underline{\omega})|} \leq \exp \left(mK(\mathcal{D}, \delta)\mathbf{X}_n \right).$$

The same estimate holds for ζ' . Since $\frac{|\Gamma(\underline{\zeta}', \underline{\omega})|}{|\Gamma(\underline{\zeta}, \underline{\omega})|} = \frac{|\Gamma(\underline{\zeta}', \underline{\omega})|}{|\Gamma(\underline{0}, \underline{\omega})|} \bigg/ \frac{|\Gamma(\underline{\zeta}, \underline{\omega})|}{|\Gamma(\underline{0}, \underline{\omega})|}$, we have

$$\exp \left(-2mK(\mathcal{D}, \delta)\mathbf{X}_n \right) \leq \frac{|\Gamma(\underline{\zeta}', \underline{\omega})|}{|\Gamma(\underline{\zeta}, \underline{\omega})|} \leq \exp \left(2mK(\mathcal{D}, \delta)\mathbf{X}_n \right).$$

In view of (5) and (6), this leads to the desired bound $\frac{\rho_{\omega}^n(\zeta')}{\rho_{\omega}^n(\zeta)}$. In Proposition 8.6, we will see that each of the three random sums defining $\mathbf{X}_n$ has finite expectation. Moreover, those expectations are bounded uniformly (in n) by quantities depending only on $\mathcal{D}$ and m . This yields the statement $\mathbb{E}[\mathbf{X}_n] \leq c_1(\mathcal{D}, m) < \infty$. ■

Corollary 8.3. *Given $M > 0$, we can replace $\mathbf{X}_n$ in Proposition 8.2 by a uniform bound M except on an event of probability less than $c_1(m, \mathcal{D})/M$.*

8.2 Estimates for Inverse Powers

Our aim in this section is to estimate the sums of inverse powers of the points in $\mathcal{G}$ and $\mathcal{G}_n$ outside a disk containing the origin. To this end, we first discuss certain estimates on the variance of linear statistics, which are uniform in n . Let $B(0; r)$ denote the disk of radius r centred at the origin.

Proposition 8.4. *Let φ be a compactly supported Lipschitz function, supported inside the disk $B(0; r)$ with Lipschitz constant $\kappa(\varphi)$. Let $\varphi_R(z) := \varphi(z/R)$. Then $\text{Var}(\int \varphi_R(z) d[\mathcal{G}_n](z)) \leq C(\varphi)$, where $C(\varphi)$ is a quantity that is independent of n , and depends on φ only through $\kappa(\varphi)$. The same conclusion holds for $\mathcal{G}$ in place of $\mathcal{G}_n$.*

To prove Proposition 8.4, we will make use of a general fact about determinantal point processes:

Lemma 8.5. *Let Π be a determinantal point process with Hermitian kernel K . Let K be a reproducing kernel with respect to its background measure γ , which means $K(x, y) = \int K(x, z)K(z, y) d\gamma(z)$ for all x, y . Let φ, ψ be compactly supported continuous functions. Then we have*

$$\text{Cov} \left[\int \varphi d[\Pi], \int \psi d[\Pi] \right] = \frac{1}{2} \iint (\varphi(z) - \varphi(w)) \overline{(\psi(z) - \psi(w))} |K(z, w)|^2 d\gamma(z) d\gamma(w).$$

Proof. Denote by ρ_1 and ρ_2 respectively the one and two-point correlation functions of Π . Then we can write

$$\begin{aligned} \text{Cov} \left[\int \varphi d[\Pi], \int \psi d[\Pi] \right] &= \int \varphi(x) \overline{\psi(y)} \rho_2(x, y) d\gamma(x) d\gamma(y) + \int \varphi(x) \overline{\psi(x)} \rho_1(x) d\gamma(x) \\ &\quad - \left(\int \varphi(x) \rho_1(x) d\gamma(x) \right) \overline{\left(\int \psi(y) \rho_1(y) d\gamma(y) \right)}. \end{aligned}$$

But $\rho_1(x) = K(x, x)$ and $\rho_2(x, y) = K(x, x)K(y, y) - |K(x, y)|^2$. Using this, the expression for the variance in the display above reduces to

$$\iint \varphi(x) \overline{\psi(x)} K(x, x) d\gamma(x) - \iint \varphi(x) \overline{\psi(y)} |K(x, y)|^2 d\gamma(x) d\gamma(y). \quad (7)$$

But since K is the integral kernel corresponding to a projection operator and $K(y, x) = \overline{K(x, y)}$, we have $K(x, x) = \iint |K(x, y)|^2 d\gamma(y)$. Using this, the expression for the variance in (7) reduces to

$$\begin{aligned} &\iint \left(\varphi(x) \overline{\psi(x)} - \varphi(x) \overline{\psi(y)} \right) |K(x, y)|^2 d\gamma(x) d\gamma(y) \\ &= \frac{1}{2} \iint (\varphi(x) - \varphi(y)) \overline{(\psi(x) - \psi(y))} |K(x, y)|^2 d\gamma(x) d\gamma(y), \end{aligned}$$

as desired. In the last step, we have used symmetry in x and y . ■

Proof of Proposition 8.4. We give the proof when $r = 1$, from here the general case is obtained by scaling, because any function φ supported on $B(0; r)$ is equal to the scaling Φ_r of some function Φ supported on $B(0; 1)$ and having the same continuity and differentiability properties as φ . Notice that $(\Phi_r)_R = \Phi_{rR}$, so the result for φ can be deduced from the result for Φ . In what follows we deal with $\mathcal{G}_n$, the result for $\mathcal{G}$ follows, for instance, from taking limits as $n \rightarrow \infty$ for the result for $\mathcal{G}_n$.

Using Lemma 8.5, we have

$$\text{Var} \left(\int \varphi_R(z) d[\mathcal{G}_n](z) \right) = \frac{1}{2} \iint |\varphi_R(z) - \varphi_R(w)|^2 |K_n(z, w)|^2 d\gamma(z) d\gamma(w)$$

where γ is the standard complex Gaussian measure. Now,

$$|\varphi_R(z) - \varphi_R(w)|^2 = |\varphi(z/R) - \varphi(w/R)|^2 \leq \frac{1}{R^2} \kappa(\varphi)^2 |z - w|^2.$$

Therefore, it suffices to bound the integral $\int_{A(R)} |z - w|^2 |K_n(z, w)|^2 d\gamma(z) d\gamma(w)$ on the set

$$A(R) := \{(z, w) : \min\{|z|, |w|\} \leq R\}$$

because outside $A(R)$, we have $\varphi_R(z) = \varphi_R(w) = 0$. We begin with

$$\begin{aligned} \int_{A(R)} |z - w|^2 |K_n(z, w)|^2 d\gamma(z) d\gamma(w) &\leq \int_{A_1(R)} |z - w|^2 |K_n(z, w)|^2 d\gamma(z) d\gamma(w) \\ &\quad + \int_{A_2(R)} |z - w|^2 |K_n(z, w)|^2 d\gamma(z) d\gamma(w) \end{aligned}$$

where

$$A_1(R) = \{|z| \leq 2R, |w| \leq 2R\} \text{ and } A_2(R) = \{|z| \leq R, |w| \geq 2R\} \cup \{|w| \leq R, |z| \geq 2R\}.$$

We first address the case of $A_2(R)$. By symmetry, it suffices to bound the integral over the region $\{|z| \leq R, |w| \geq 2R\}$. In this region, $||z| - |w|| \geq R$, and $|K_n(z, w)|^2 e^{-|z|^2 - |w|^2} \leq e^{-||z| - |w||^2}$. The integral is bounded from above by

$$\frac{1}{\pi^2} \int_{|z| \leq R} \left(\int_{|w| \geq 2R} (|z| + |w|)^2 e^{-(|z| - |w|)^2} d\mathcal{L}(w) \right) d\mathcal{L}(z).$$

It is not hard to see that the inner integral is $O(R^2 e^{-R^2})$. Integrating over $|z| \leq R$ gives another factor of R^2 , so the total contribution is $o(1)$ as $R \rightarrow \infty$.

For the integral over $A_1(R)$, we proceed as follows:

$$\begin{aligned} &\int_{A_1(R)} |z - w|^2 |K_n(z, w)|^2 d\gamma(z) d\gamma(w) \\ &= \frac{1}{\pi^2} \int \left(|z|^2 - z\bar{w} - \bar{z}w + |w|^2 \right) \left(\sum_{j=0}^{n-1} (z\bar{w})^j / j! \right) \left(\sum_{j=0}^n (\bar{z}w)^j / j! \right) e^{-|z|^2 - |w|^2} d\mathcal{L}(z) d\mathcal{L}(w). \end{aligned}$$

Now, we integrate the $|z - w|^2$ part term by term. Due to radial symmetry, only some specific terms from $|K_n(z, w)|^2$ contribute. For example, when we integrate the $|z|^2$ term in $|z - w|^2$, only

the $\frac{(z\bar{w})^j}{j!} \frac{(\bar{z}w)^j}{j!}$, $0 \leq j \leq n-1$ terms in the expanded expression for $|K_n(z, w)|^2$ contribute. When we integrate $z\bar{w}$, only the $\frac{(z\bar{w})^j}{j!} \frac{(\bar{z}w)^{j+1}}{(j+1)!}$, $0 \leq j \leq n-2$ terms provide non-zero contributions. Due to symmetry between z and w , it is enough to bound the contribution from $(|z|^2 - z\bar{w})$ by $O(R^2)$.

• $|z|^2$ term:

Let us introduce the change of variables by $x = |z|^2$ and $y = |w|^2$. Then the contribution in the above integral coming from

$$\frac{(z\bar{w})^j}{j!} \frac{(\bar{z}w)^j}{j!}$$

can be written as

$$\int_0^{4R^2} \int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} \frac{y^j}{j!} e^{-y} dx dy.$$

So, the total contribution due to all such terms, ranging from $j = 0, \dots, n-1$ is

$$\sum_{j=0}^{n-1} \left(\int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} dx \right) \left(\int_0^{4R^2} \frac{y^j}{j!} e^{-y} dy \right).$$

• $z\bar{w}$ term:

As above, the contribution coming from the

$$\frac{(z\bar{w})^j}{j!} \frac{(\bar{z}w)^{j+1}}{(j+1)!}$$

is given by

$$\int_0^{4R^2} \int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} \frac{y^{j+1}}{(j+1)!} e^{-y} dx dy.$$

Therefore the total contribution from $0 \leq j \leq n-2$ is

$$\sum_{j=0}^{n-2} \left(\int_0^{4R^2} \frac{x^{j+1}}{j!} e^{-x} dx \right) \left(\int_0^{4R^2} \frac{y^{j+1}}{(j+1)!} e^{-y} dy \right).$$

We interpret $\frac{x^j}{j!} e^{-x} dx$ as a gamma density, the corresponding random variable being denoted by Γ_{j+1} .

The contribution due to the $|z|^2$ term is $\sum_{j=0}^{n-1} \mathbb{E}[\Gamma_{j+1} \mathbb{1}_{(\Gamma_{j+1} \leq 4R^2)}] \mathbb{P}[\Gamma_{j+1} \leq 4R^2]$ and that due to the $z\bar{w}$ term is $\sum_{j=0}^{n-2} \mathbb{E}[\Gamma_{j+1} \mathbb{1}_{(\Gamma_{j+1} \leq 4R^2)}] \mathbb{P}[\Gamma_{j+2} \leq 4R^2]$.

The difference between the above two terms can be written as:

$$\mathbb{E}[\Gamma_n \mathbb{1}_{(\Gamma_n \leq 4R^2)}] \mathbb{P}[\Gamma_n \leq 4R^2] + \sum_{j=1}^{n-1} \mathbb{E}[\Gamma_j \mathbb{1}_{(\Gamma_j \leq 4R^2)}] \left(\mathbb{P}[\Gamma_j \leq 4R^2] - \mathbb{P}[\Gamma_{j+1} \leq 4R^2] \right). \quad (8)$$

All the expectations in the above are $\leq 4R^2$, and $\mathbb{P}[\Gamma_j \leq 4R^2] \geq \mathbb{P}[\Gamma_{j+1} \leq 4R^2]$ because Γ_{j+1} stochastically dominates Γ_j (this can be seen, e.g., by considering a sequence $\{X_i\}_{i=1}^\infty$ of i.i.d. exponentials with mean 1 and using a natural coupling of the Γ_j -s by setting $\Gamma_j = X_1 + \dots + X_j$). Therefore the absolute value of (8), by a telescopic sum, is $\leq 4R^2 \mathbb{P}[\Gamma_1 \leq 4R^2] \leq 4R^2$. Combining all of these, we see that $\text{Var} \left(\int \varphi_R(z) d[\mathcal{G}_n](z) \right)$ is bounded by $(1/R^2) \kappa(\varphi)^2 8R^2 = C(\varphi)$. It is clear that $C(\varphi)$ depends on φ only through $\kappa(\varphi)$. $\blacksquare$

Let r_0 be the radius of $\mathcal{D}$. Let φ be a non-negative radial C_c^∞ function supported on complex numbers with radius in $[r_0, 3r_0]$ such that $\varphi = 1$ on $[\frac{3r_0}{2}, 2r_0]$ and $\varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r)$, for $0 \leq r \leq \frac{1}{2}r_0$. In other words, φ is a test function supported on the annulus between r_0 and $3r_0$ and its “ascent” to 1 is twice as fast as its “descent”. Let $\tilde{\varphi}$ be another radial test function with the same support as φ , satisfying $\tilde{\varphi}(r_0 + xr_0) = 1$ for $0 \leq x \leq \frac{1}{2}$ and $\tilde{\varphi} = \varphi$ otherwise. Recall that for a test function ψ and $L > 0$ we denote by ψ_L the scaled function $\psi_L(z) = \psi(z/L)$.

Observe that the functions $\tilde{\varphi}$ and φ_{2^j} for $j \geq 1$ form a partition of unity on $\mathcal{D}^c$.

For the rest of the paper, we introduce the following notation for the dilations of a set:

Notation. For a set $\Lambda \subset \mathbb{C}$ and $R > 0$, we denote $R \cdot \Lambda = \{Rz : z \in \Lambda\}$.

Proposition 8.6. Let r_0 be the radius of $\mathcal{D}$. Let φ and $\tilde{\varphi}$ be defined as above.

(i) The random variables

$$S_l(n) := \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{G}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} d[\mathcal{G}_n](z) = \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \quad (\text{for } l \geq 1)$$

and

$$\tilde{S}_l(n) := \int \frac{\tilde{\varphi}(z)}{|z|^l} d[\mathcal{G}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l} d[\mathcal{G}_n](z) = \sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \quad (\text{for } l \geq 3)$$

have finite first moments which, for every fixed l , are bounded above by quantities which do not depend on n .

(ii) There exists $k_0 = k_0(\varphi) \geq 1$, not depending on n or l , such that for $k \geq k_0$ the “tails” of $S_l(n)$ and $\tilde{S}_l(n)$ beyond the disk $2^k \cdot \mathcal{D}$, given by

$$\tau_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} d[\mathcal{G}_n](z) \quad (\text{for } l \geq 1) \quad \text{and} \quad \tilde{\tau}_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l} d[\mathcal{G}_n](z) \quad (\text{for } l \geq 3)$$

satisfy the estimates

$$\mathbb{E} \left[\left| \tau_l^n(2^k) \right| \right] \leq C_1(\varphi, l) / 2^{kl} \quad \text{and} \quad \mathbb{E} \left[\left| \tilde{\tau}_l^n(2^k) \right| \right] \leq C_2(\varphi, l) / 2^{k(l-2)}.$$

All of the above remain true when $\mathcal{G}_n$ is replaced by $\mathcal{G}$, for which we use the notations S_l , $\tilde{S}_l$, $\tau_l(R)$ and $\tilde{\tau}_l(R)$ to denote the quantities corresponding to $S_l(n)$, $\tilde{S}_l(n)$, $\tau_l^n(R)$ and $\tilde{\tau}_l^n(R)$ respectively.

Remark 8.1. For $\mathcal{G}$, by the sum $\left(\sum_{\omega \in \mathcal{G} \cap \mathcal{D}^c} \frac{1}{\omega^l} \right)$ we denote the quantity

$$S_l = \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{G}](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} d[\mathcal{G}](z)$$

due to the obvious analogy with $\mathcal{G}_n$, where the corresponding sum $S_l(n)$ is indeed equal to $\left(\sum_{\omega \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \right)$ with its usual meaning.

Proof. Recall that the functions $\tilde{\varphi}$ and φ_{2^j} for $j \geq 1$ form a partition of unity on $\mathcal{D}^c$, hence we have the identities appearing in part (i).

Set $\psi_k^n(l) = \int \frac{\varphi_{2^k}(z)}{z^l} d[\mathcal{G}_n](z)$ for $k \geq 1$, and $\psi_0^n(l) = \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{G}_n](z)$. When $l \geq 3$ we also define $\gamma_k^n(l) = \int \frac{\varphi_{2^k}(z)}{|z|^l} d[\mathcal{G}_n](z)$ for $k \geq 1$, and $\gamma_0^n(l) = \int \frac{\tilde{\varphi}(z)}{|z|^l} d[\mathcal{G}_n](z)$. Let $\Psi_k(l)$ and $\Gamma_k(l)$ denote the analogous quantities defined with respect to $\mathcal{G}$ instead of $\mathcal{G}_n$.

We begin with the observation that for $k \geq 1$ we have $\mathbb{E}[\psi_k^n(l)] = 0$ (owing to the rotational symmetry of the distributions $\mathcal{G}_n$ -s). This implies that

$$\mathbb{E}[|\psi_k^n(l)|] \leq (\mathbb{E}[|\psi_k^n(l)|^2])^{1/2} = \sqrt{\text{Var}[\psi_k^n(l)]}.$$

We then apply Proposition 8.4 to the function $\varphi(z)/z^l$ and $R = 2^k$ to obtain $\mathbb{E}[|\psi_k^n(l)|] \leq C(\varphi, l)/(2^k)^l$. We also note that

$$\mathbb{E}[|\psi_0^n(l)|] \leq \frac{1}{\pi} \int_{3 \cdot \mathcal{D} \setminus \mathcal{D}} \frac{K_n(z, z) e^{-|z|^2}}{|z|^l} d\mathcal{L}(z) \leq \frac{1}{\pi} \int_{3 \cdot \mathcal{D} \setminus \mathcal{D}} \frac{1}{|z|^l} d\mathcal{L}(z) = c(l).$$

This implies that for $l \geq 1$

$$\mathbb{E}[|S_l(n)|] \leq \sum_{k=0}^{\infty} \mathbb{E}[|\psi_k^n(l)|] < \infty.$$

The desired bound for $\mathbb{E}[\tilde{S}_l(n)]$ follows from a direct computation of the expectation using the first intensity, and noting that the first intensity of $\mathcal{G}_n$ (with respect to Lebesgue measure) is $K_n(z, z) e^{-|z|^2} \leq 1$ for all z .

The estimates for τ and $\tilde{\tau}$ follow by using the above argument for the sums $\sum_{j=k}^{\infty} \psi_j^n(l)$ and $\sum_{j=k}^{\infty} \gamma_j^n(l)$. ■

Corollary 8.7. For $R = 2^k$ for $k \geq k_0$ (as in Proposition 8.6), we have $\mathbb{P}[|\tau_l^n(R)| > R^{-l/2}] \leq c_1(\varphi, l) R^{-l/2}$ and $\mathbb{P}[|\tilde{\tau}_l^n(R)| > R^{-(l-2)/2}] \leq c_2(\varphi, l) R^{-(l-2)/2}$, and these estimates remain true when $\mathcal{G}_n$ is replaced with $\mathcal{G}$.

Proof. We use the estimates on the expectation of $|\tau_l^n(R)|$ and $|\tilde{\tau}_l^n(R)|$ from Proposition 8.6 and apply Markov's inequality. ■

With notations as above, we have

Proposition 8.8. *For each $l \geq 1$ we have $S_l(n) \rightarrow S_l$ in probability, and for each $l \geq 3$ we have $\tilde{S}_l(n) \rightarrow \tilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all l .*

Proof. Fix $\delta > 0$. Given $\varepsilon > 0$, we choose $R = 2^k$ large enough such that $c_1 R^{-l/2} < \varepsilon/4$ (for $l \geq 1$) and $c_2 R^{-(l-2)/2} < \varepsilon/4$ for $l \geq 3$ (as in Corollary 8.7), as well as $R^{-l/2} < \delta/4$ for $l = 1, 2$ and $R^{-(l-2)/2} < \delta/4$ for $l \geq 3$. We now recall notations from the beginning of the proof of Proposition 8.6. By definition, we have $S_l(n) = \sum_{j=0}^k \psi_j^n(l) + \tau_l^n(2^k)$. On the disk of radius R , we have $\mathcal{G}_n \rightarrow \mathcal{G}$ a.s. Now choose n large enough so that we have $|\sum_{j=0}^k \psi_j^n(l) - \sum_{j=0}^k \Psi_j(l)| < \delta/3$ except on an event of probability $\varepsilon/2$. By choice of R , we have $|\tau_l^n(2^k)| < \delta/3$ and $|\tau_l(2^k)| < \delta/3$ except on an event of probability $< \varepsilon/2$. Combining all these, we have $\mathbb{P}(|S_l(n) - S_l| > \delta) \leq \varepsilon$, proving that $S_l(n) \rightarrow S_l$ in probability.

For each l , given any sequence we can find a subsequence along which this convergence is a.s. A diagonal argument now gives us a subsequence for which a.s. convergence holds simultaneously for all l .

When $l \geq 3$, the argument for $\tilde{S}_l$ is similar. ■

Define $S_l(\mathcal{D}, n) = \sum_{z \in \mathcal{G}_n \cap \mathcal{D}} 1/z^l$ and $S_l(\mathcal{D}) = \sum_{z \in \mathcal{G} \cap \mathcal{D}} 1/z^l$. Set $\alpha_l(n) = S_l(\mathcal{D}, n) + S_l(n)$ and $\alpha_l = S_l(\mathcal{D}) + S_l$. Observe that $\alpha_l(n) = \sum_{z \in \mathcal{G}_n} 1/z^l$. Notice that a.s. there is no point at the origin, so each of these random variables is well defined. With these definitions, we have:

Proposition 8.9. *For each l , $\alpha_l(n) \rightarrow \alpha_l$ in probability as $n \rightarrow \infty$. Hence, there is a subsequence such that $\alpha_l(n) \rightarrow \alpha_l$ a.s. when $n \rightarrow \infty$ along this subsequence, simultaneously for all l .*

Proof. Since the finite point configurations given by $\mathcal{G}_n|_{\mathcal{D}} \rightarrow \mathcal{G}|_{\mathcal{D}}$ a.s. and a.s. there is no point at the origin, therefore $S_l(\mathcal{D}, n) \rightarrow S_l(\mathcal{D})$ a.s. This, combined with Proposition 8.8, gives us the desired result. The last assertion follows essentially from the fact that the convergence $\mathcal{G}_n|_{\mathcal{D}} \rightarrow \mathcal{G}|_{\mathcal{D}}$ (for a fixed realization of the random variables in question) is weak convergence of counting measures. Since the point configurations in a given realization of the $\mathcal{G}_n$ -s are bounded away from 0 (by an amount which depends on the realization), therefore the convergence of S_l -s can be obtained by integrating the counting measures against the function $\frac{1}{z^l}$ (mollified near the origin and away from the point configurations to make it a continuous function). ■

9 Limiting Procedure for the Ginibre Ensemble

The aim of this section is to use the estimates derived in Section 8 to verify the conditions laid out in Theorem 7.2, so that the limiting procedure outlined in Section 7 can be executed. This will lead us to a proof of Theorem 1.2 for a disk $\mathcal{D}$. We have already seen in Section 5 that this implies Theorem 1.2 for general $\mathcal{D}$.

Proof of Theorem 1.2 for a disk. We will appeal to Theorem 7.2 with $\mathcal{D}$ an open disk. We already know from Theorem 1.1 that the number of points in $\mathcal{D}$ is rigid. In terms of the notation used in Section 7, we set $X = \mathcal{G}$ and $X^n = \mathcal{G}_n$.

Fix an integer $m \geq 0$. Consider the event that m points of $\mathcal{G}$ are inside $\mathcal{D}$. The result in Theorem 1.2 is trivial for $m = 0$. Hence, we can assume that $m > 0$. For $\omega \in \mathcal{S}_{\text{out}}$, our candidate for $\nu(\omega, \cdot)$ (refer to Theorem 7.2) is the probability measure $Z^{-1}|\Delta(\underline{\zeta})|^2 d\mathcal{L}(\underline{\zeta})$ on $\mathcal{D}^m$, where Z is the normalizing constant. Heuristically, this is in tune with the fact that the conditional measure is supported on $\mathcal{D}^m$ and with the estimate in Proposition 8.2 on conditional densities. It is also a mathematical expression of the notion that ω determines only the number of points in $\mathcal{D}$, and “nothing more”. Notice that ν is a constant when considered as a function mapping $\mathcal{S}_{\text{out}}$ to $\mathcal{M}(\mathcal{D}^m)$, in the sense that ν does not depend on the outside configuration. Since a.s. $\nu(X_{\text{out}}, \cdot)$ is mutually absolutely continuous with respect to the Lebesgue measure on $\mathcal{D}^m$, Theorem 7.2 would imply that the same holds true for the conditional distribution $\rho(X_{\text{out}}, \cdot)$ of the points inside $\mathcal{D}$ (treated as a vector in the uniform random order).

Now we construct, for each j , the sequences $\{n_k(j)\}_{k \geq 1}$ and the events $\Omega_{n_k}(j)$. We proceed as follows. From Proposition 8.8, we get a subsequence n_k such that a.s. $S_1(n_k) \rightarrow S_1$, $S_2(n_k) \rightarrow S_2$ and $\tilde{S}_3(n_k) \rightarrow \tilde{S}_3$. This is going to be our subsequence $n_k(j)$ for all j .

Let $M_j \uparrow \infty$ be a sequence of positive numbers such that none of them is an atom of the distributions of $|S_1|$, $|S_2|$ and $|\tilde{S}_3|$ (in case such atom at all exists).

We will first **define the events** $\Omega_{n_k}(j)$ by the following conditions:

- (i) There are exactly m points of $\mathcal{G}_{n_k}$ inside $\mathcal{D}$.
- (ii) $\text{Dist}(\mathcal{D}, \mathcal{G}_{n_k} \cap \mathcal{D}^c) \geq 1/M_j$ (so $\Omega_{n_k}(j) \subset \Omega_{n_k}^{m, \delta}$ with $\delta = 1/M_j$, defined at the beginning of section 8.1).
- (iii) $|S_1(n_k)| < M_j, |S_2(n_k)| < M_j, |\tilde{S}_3(n_k)| < M_j$.

Recall that for an event $\mathcal{A}$ and a random variable U on the same probability space, we say that $\mathcal{A}$ is measurable with respect to U if $\mathcal{A}$ is measurable with respect to the sigma-algebra generated by U . Clearly, each $\Omega_{n_k}(j)$ is measurable with respect to $(\mathcal{G}_{n_k})_{\text{out}}$. On the event $\Omega_{n_k}^m$ (which entails that $|(\mathcal{G}_{n_k})_{\text{in}}| = m$; refer Section 7), the points in $(\mathcal{G}_{n_k})_{\text{out}}$, considered in uniform random order, yield a vector $\underline{\omega}$ in $(\mathcal{D}^c)^{n_k-m}$. Denoting the conditional distribution of $\underline{\zeta}$ given $\underline{\omega}$ to be $\rho_{\underline{\omega}}^{n_k}(\underline{\zeta})$ we recall from Proposition 8.2 that on the event $\Omega_{n_k}^{m, \delta}$ we have

$$\exp\left(-4mK(\mathcal{D}, \delta)\mathbf{X}_{n_k}\right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{\rho_{\underline{\omega}}^{n_k}(\underline{\zeta}')}{\rho_{\underline{\omega}}^{n_k}(\underline{\zeta})} \leq \exp\left(4mK(\mathcal{D}, \delta)\mathbf{X}_{n_k}\right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}.$$

Therefore, the bounds on $S_1(n_k)$, $S_2(n_k)$ and $\tilde{S}_3(n_k)$ as in condition (iii) above imply that on $\Omega_{n_k}(j)$ we have, with $\delta = 1/M_j$,

$$\exp\left(-12mK(\mathcal{D}, \delta)M_j\right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{\rho_{\underline{\omega}}^{n_k}(\underline{\zeta}')}{\rho_{\underline{\omega}}^{n_k}(\underline{\zeta})} \leq \exp\left(12mK(\mathcal{D}, \delta)M_j\right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}.$$

Let $A \in \mathfrak{A}^m$ and $B \in \mathfrak{B}$. Considering the last inequality on the right hand side, and cross multiplying, we get

$$\rho_{\underline{\omega}}^{n_k}(\underline{\zeta}') |\Delta(\underline{\zeta})|^2 \leq \rho_{\underline{\omega}}^{n_k}(\underline{\zeta}) \exp \left(12mK(\mathcal{D}, \delta)M_j \right) |\Delta(\underline{\zeta}')|^2.$$

Observe that $\Omega_{n_k}(j)$ is measurable with respect to $(\mathcal{G}_{n_k})_{\text{out}}$, hence $\underline{\omega}$ determines whether the above inequality holds. In particular, it holds for all $\underline{\zeta}, \underline{\zeta}'$ for any $\underline{\omega}$ is such that $\Omega_{n_k}(j)$ occurs. We now integrate the above inequalities, first with respect to the Lebesgue measure in the variable $\underline{\zeta}' \in A$, yielding

$$|\Delta(\underline{\zeta})|^2 \int_A \rho_{\underline{\omega}}^{n_k}(\underline{\zeta}') d\mathcal{L}(\underline{\zeta}') \leq \exp \left(12mK(\mathcal{D}, \delta)M_j \right) \rho_{\underline{\omega}}^{n_k}(\underline{\zeta}) \int_A |\Delta(\underline{\zeta}')|^2 d\mathcal{L}(\underline{\zeta}').$$

Then we integrate with respect to the Lebesgue measure in the variable $\underline{\zeta} \in \mathcal{D}^m$ to obtain

$$\int_{\mathcal{D}^m} |\Delta(\underline{\zeta})|^2 d\mathcal{L}(\underline{\zeta}) \int_A \rho_{\underline{\omega}}^{n_k}(\underline{\zeta}') d\mathcal{L}(\underline{\zeta}') \leq \exp \left(12mK(\mathcal{D}, \delta)M_j \right) \int_A |\Delta(\underline{\zeta}')|^2 d\mathcal{L}(\underline{\zeta}').$$

Finally, recalling again that $\Omega_{n_k}(j)$ is measurable with respect to $(\mathcal{G}_{n_k})_{\text{out}}$, we integrate with respect to the distribution of $\underline{\omega}$ on the event

$$\{\underline{\omega} \in (\mathcal{D}^c)^{n_k-m} : (\mathcal{G}_{\text{out}}^{n_k})^{-1}(\{\omega_1, \dots, \omega_{n_k-m}\}) \in (\mathcal{G}_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)\}$$

to obtain

$$\begin{aligned} & \left(\int_{\mathcal{D}^m} |\Delta(\underline{\zeta})|^2 d\mathcal{L}(\underline{\zeta}) \right) \mathbb{P}[(\mathcal{G}_{\text{in}}^{n_k} \in A) \cap (\mathcal{G}_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \\ & \leq \exp \left(12mK(\mathcal{D}, \delta)M_j \right) \int_{(\mathcal{G}_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)} \left(\int_A |\Delta(\underline{\zeta}')|^2 d\mathcal{L}(\underline{\zeta}') \right) d\mathbb{P}(\underline{\omega}). \end{aligned}$$

We can carry out the same procedure with the inequality on the left hand side. Combined with our definition of ν , it can be seen that together they give us condition (C3) in Theorem 7.2.

All that remains now is to show that events $\Omega(j) := \varliminf_{k \rightarrow \infty} \Omega_{n_k}(j)$ exhausts Ω^m .

Since the M_j -s are increasing, we automatically have $\Omega(j) \subset \Omega(j+1)$. Moreover, recall that Ω^m is the event that there are m points of $\mathcal{G}$ inside $\mathcal{D}$. Each $\Omega_{n_k}(j)$ satisfies this condition for $\mathcal{G}_{n_k}$. And finally, $\mathcal{G}_{n_k} \rightarrow \mathcal{G}$ a.s. on $\mathcal{D}$. These three facts together imply that $\Omega(j) \subset \Omega^m$ for each j . It only remains to check that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$. This is the goal of the next proposition, which will complete the proof of Theorem 1.2 for a disk. $\blacksquare$

Proposition 9.1. *Let $\Omega(j)$ be as defined above. Then there exists $\Omega^{\text{corr}}(j)$ measurable with respect to $\mathcal{G}_{\text{out}}$ such that $\mathbb{P}(\Omega(j) \Delta \Omega^{\text{corr}}(j)) = 0$. Further, $\mathbb{P}(\Omega^m \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$.*

Proof. We first show the existence of $\Omega^{\text{corr}}(j)$ with the desired properties. Accordingly, until the beginning of the last paragraph of this proof, j is fixed; in particular j does not depend on the parameter ε to be introduced below.

Let $\varepsilon > 0$. We will construct an event A_ε (which depends on j) such that A_ε is measurable with respect to $\mathcal{G}_{\text{out}}$ and there is a bad set $\Omega_{\text{bad}}^\varepsilon$ (which also depends on j) of probability $< \varepsilon$ such that $\Omega(j) \setminus \Omega_{\text{bad}}^\varepsilon = A_\varepsilon \setminus \Omega_{\text{bad}}^\varepsilon$. As a result, $\mathbb{P}(\Omega(j) \Delta A_\varepsilon) < \varepsilon$. Then $\Omega^{\text{corr}}(j) = \varliminf_{n \rightarrow \infty} A_{2^{-n}}$ will give us the desired event. It is easy to check that $\mathbb{P}(\Omega(j) \Delta \Omega^{\text{corr}}(j)) = 0$. Let $\delta = \delta(\varepsilon) < 1$ be a small number, depending on ε , to be chosen later.

We define the event A_ε by the following conditions:

- (i) $N(\mathcal{G}_{\text{out}}) = m$, where N is as in Theorem 1.1.
- (ii) There is a $\frac{1}{M_j}$ separation between $\partial\mathcal{D}$ and the points ω of $\mathcal{G}$ outside $\mathcal{D}$.
- (iii) $|S_1| < M_j - \delta, |S_2| < M_j - \delta, |\tilde{S}_3| < M_j - \delta$.

It is clear from the definition of A_ε that it is measurable with respect to $\mathcal{G}_{\text{out}}$.

We can compare the definition of the event A_ε with a similar description for the event $\Omega(j)$. Since $\Omega(j)$ is the liminf of the events $\Omega_{n_k}(j)$ (as $k \rightarrow \infty$), $\Omega(j)$ occurs if and only if each of the conditions (i), (ii) and (iii) in the definition of $\Omega_{n_k}(j)$ are true for $\mathcal{G}_{n_k}$ for all large enough k . The reader may notice the close similarity of the defining conditions of $\Omega_{n_k}(j)$ with those of A_ε . In order to show that $\Omega(j) \setminus \Omega_{\text{bad}}^\varepsilon = A_\varepsilon \setminus \Omega_{\text{bad}}^\varepsilon$, we need to establish that on $(\Omega_{\text{bad}}^\varepsilon)^c$, condition (i) in the definition of $\Omega_{n_k}(j)$ being true for all large enough k is equivalent to condition (i) in the definition of A_ε being true, and similarly with conditions (ii) and (iii) respectively in the two definitions.

By Proposition 8.8, $S_i(n_k) \rightarrow S_i, i = 1, 2$ and $\tilde{S}_3(n_k) \rightarrow \tilde{S}_3$ a.s. along our chosen subsequence $\{n_k\}_k$. By Egorov's Theorem, there is a bad event Ω^1 of probability $< \varepsilon/4$ such that outside Ω^1 , this convergence is uniform. Therefore, on $(\Omega^1)^c$ there is a large k_0 such that $|S_i(n_k) - S_i| < \delta, i = 1, 2$ and $|\tilde{S}_3(n_k) - \tilde{S}_3| < \delta$ for all $k \geq k_0$.

By the a.s. convergence of $\mathcal{G}_{n_k}$ -s to $\mathcal{G}$ on the compact set $2 \cdot \overline{\mathcal{D}}$, there is a large k_1 and a small event Ω^2 of probability $< \varepsilon/4$ outside which condition (i) in the definition of $\Omega_{n_k}(j)$ are simultaneously true or simultaneously false (simultaneity in k) for all $\mathcal{G}_{n_k}, k \geq k_1$. By the coupling of $\mathcal{G}_{n_k}$ -s and $\mathcal{G}$ (which entails that $\mathcal{G}_n \subset \mathcal{G}_{n+1} \subset \mathcal{G}$ for all n), condition (i) in the definition of $\Omega_{n_k}(j)$ being simultaneously true for $\mathcal{G}_{n_k}, k \geq k_1$ would imply that the condition (i) in the definition of A_ε is also true. The same statement holds with “true” replaced by “false”. Thus, on the event $(\Omega^2)^c$, exactly one of the following two possibilities hold :

- (a) condition (i) defining $\Omega_{n_k}(j)$ is true for all $k \geq k_1$ and condition (i) defining A_ε is true,
- (b) condition (i) defining $\Omega_{n_k}(j)$ is false for all $k \geq k_1$ and condition (i) defining A_ε is false.

Similarly, the a.s. convergence of $\mathcal{G}_{n_k}$ -s to $\mathcal{G}$ on the compact set $2 \cdot \overline{\mathcal{D}}$ guarantees that there is a large k_2 and a small event Ω^3 of probability $< \varepsilon/4$ such that on $(\Omega^3)^c$, exactly one of the two possibilities hold:

- (a) condition (ii) defining $\Omega_{n_k}(j)$ is true for all $k \geq k_1$ and condition (ii) defining A_ε is true,
- (b) condition (ii) defining $\Omega_{n_k}(j)$ is false for all $k \geq k_1$ and condition (ii) defining A_ε is false.

Since M_j is not an atom of the distribution of $|S_1|, |S_2|$ or $|\tilde{S}_3|$, there is an event Ω^4 (of probability $< \varepsilon/4$) outside which each $|S_i|$ or $|\tilde{S}_3|$ is either $> M_j + 2\delta$ or $< M_j - 2\delta$ ($\delta < 1$ is chosen based on ε so that this condition is satisfied).

Define $\Omega_{\text{bad}}^\varepsilon = \Omega^1 \cup \Omega^2 \cup \Omega^3 \cup \Omega^4$; clearly $\mathbb{P}(\Omega_{\text{bad}}^\varepsilon) < \varepsilon$. We note that on $(\Omega_{\text{bad}}^\varepsilon)^c$, the condition (i) in the definition of $\Omega_{n_k}(j)$ being true for all k large enough is equivalent to condition (i) in the definition of A_ε being true. Similarly, on $(\Omega_{\text{bad}}^\varepsilon)^c$, the condition (ii) in the definition of $\Omega_{n_k}(j)$ being true for all k large enough is equivalent to condition (ii) in the definition of A_ε being true.

In order to show that $\Omega(j) \setminus \Omega_{\text{bad}}^\varepsilon = A_\varepsilon \setminus \Omega_{\text{bad}}^\varepsilon$ we are left to verify the only one remaining point: on $(\Omega_{\text{bad}}^\varepsilon)^c$, the condition (iii) in the definition of $\Omega_{n_k}(j)$ being true for all large enough k is equivalent to the condition (iii) in the definition of A_ε being true.

Suppose the event $\Omega(j) \setminus \Omega_{\text{bad}}^\varepsilon$ occurs. Then we have $|S_i(n_k)| < M_j$ (with $i = 1, 2$) and $|\tilde{S}_3(n_k)| < M_j$, for all large enough k . Hence $|S_i(n_k) - S_i| < \delta$ implies $|S_i| < M_j + \delta$ for $i = 1, 2$, similarly $|\tilde{S}_3(n_k) - \tilde{S}_3| < \delta$ implies $|\tilde{S}_3| < M_j + \delta$. But we are on $(\Omega^4)^c$, so for $i = 1, 2$, we have $|S_i| \in (M_j - 2\delta, M_j + 2\delta)^c$, hence $|S_i| < M_j + \delta$ implies $|S_i| < M_j - 2\delta$. Similarly, we get $|\tilde{S}_3| < M_j - 2\delta$. Hence, the event A_ε also occurs.

Conversely, suppose the event $A_\varepsilon \setminus \Omega_{\text{bad}}^\varepsilon$ occurs. Then for $i = 1, 2$ we have $|S_i| < M_j - \delta$. But $|S_i(n_k) - S_i| < \delta$ for all large enough k , because we are on $(\Omega^1)^c$. This implies $|S_i(n_k)| < M_j$ for all large enough k . Similarly, $|\tilde{S}_3(n_k)| < M_j$ for all large enough k . These facts together mean that the event $\Omega(j)$ also occurs.

Hence $\Omega(j) \setminus \Omega_{\text{bad}}^\varepsilon = A_\varepsilon \setminus \Omega_{\text{bad}}^\varepsilon$. As a result, we have $(\Omega(j) \Delta A_\varepsilon) \subset \Omega_{\text{bad}}^\varepsilon$, and $\mathbb{P}(\Omega(j) \Delta A_\varepsilon) < \varepsilon$.

To show that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$, we define events $A'(j) \subset A_\varepsilon$ above by replacing δ by 1 in the condition (iii) in the definition of A_ε . Clearly, $A'(j) \subset A_\varepsilon$ for each $0 < \varepsilon < 1$, and therefore $A'(j) \subset \lim_{n \rightarrow \infty} A_{2^{-n}} = \Omega(j)^{\text{corr}}$. It is also clear that $\mathbb{P}(\Omega^m \setminus A'(j)) \rightarrow 0$ as $j \rightarrow \infty$, because S_1, S_2 and $\tilde{S}_3$ are well defined random variables (with no mass at ∞) and the separation of $1/M_j$ between D and the points of $\mathcal{G}_{\text{out}}$ converges to 0. These two facts imply that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$. ■

This completes the proof of Theorem 1.2 in the case of $\mathcal{D}$ being a disk.

10 Tolerance of the GAF zeros for a disk

In this section we obtain the estimates necessary to prove Theorem 1.4 in the case where $\mathcal{D}$ is a disk. By translation invariance of $\mathcal{Z}$, we can take $\mathcal{D}$ to be centred at the origin.

A key goal in this section is to quantitatively compare the conditional densities of the (vector of) inside zeroes (for finite degree polynomials) at two possible locations, along the lines of Proposition 10.10. Let $(\underline{\zeta}, \underline{\omega})$ be a randomly chosen vector of points corresponding to the ensemble $\mathcal{Z}_n$, with $\underline{\zeta}$ corresponding to the zeroes inside $\mathcal{D}$ and $\underline{\omega}$ corresponding to those outside $\mathcal{D}$. Let $(\underline{\zeta}', \underline{\omega})$ be another possible vector of zeroes such that $\sum_i \zeta'_i = \sum_i \zeta_i$. We want to compare the conditional densities at these two locations. In Subsection 10.1 we introduce the conditional densities $\rho_{\underline{\omega}, s}^n(\underline{\zeta})$. We observe that the ratio $\rho_{\underline{\omega}, s}^n(\underline{\zeta}') / \rho_{\underline{\omega}, s}^n(\underline{\zeta})$ of conditional densities (at two different locations $\underline{\zeta}$ and $\underline{\zeta}'$) splits neatly into a ratio of Vandermonde determinants and a ratio of an explicit function of the elementary symmetric polynomials of the zeroes, as in (10). In Subsection 10.1.1, we deal with the ratio of the Vandermonde determinants, on similar lines to the Ginibre case. In Subsection 10.2, we prove estimates on inverse power sums of the zeroes, which are necessary to complete the study of the Vandermonde ratio.

The procedure to estimate the ratio involving the elementary symmetric functions is more elaborate. As is evident from (10), it is of relevance to compare the quantities $\sigma_k(\underline{\zeta}', \underline{\omega})$ and $\sigma_k(\underline{\zeta}, \underline{\omega})$, where σ_k denotes the elementary symmetric polynomial of order k . Since the vectors of zeroes differ only in the co-ordinates corresponding to the zeroes inside $\mathcal{D}$, we begin with an equation that splits the difference between $\sigma_k(\underline{\zeta}', \underline{\omega})$ and $\sigma_k(\underline{\zeta}, \underline{\omega})$ into quantities involving only the inside zeroes and quantities involving only the outside zeroes; see (12).

Since we need to estimate the ratio of certain functions of the σ_k -s, the next step is to apply the above technique to express these quantities in terms of $\sigma_k(\underline{\omega})$ and $\sigma_k(\underline{\zeta}, \underline{\omega})$; see (13), (14) and (15). While $\sigma_k(\underline{\zeta}, \underline{\omega})$ is a convenient quantity to deal with probabilistically (because it is essentially a ratio of two independent Gaussians), the same cannot be said of $\sigma_k(\underline{\omega})$. Hence, the next step is to express $\sigma_k(\underline{\omega})$ in terms of a series in $\sigma_j(\underline{\zeta}, \underline{\omega})$ (for $j \leq k$). This is done in Proposition 10.4, exploiting a certain recursive structure in the algebraic relationship between $\sigma_k(\underline{\zeta}, \underline{\omega})$ and $\sigma_k(\underline{\omega})$. In Proposition 10.6, we quantitatively establish that in this series expansion, it is essentially the leading term that dominates.

All that remains now is to put the pieces together, and obtain the desired estimates by controlling the leading term discussed above, along with bounds on the error accrued in the process. This is done in Proposition 10.7. Finally, all the above ingredients are combined to formally state a quantitative comparison result between the conditional densities at the two locations in Proposition 10.10.

10.1 Polynomial Approximations

We focus on the event $\Omega_n^{m,\delta}$ which entails that f_n has exactly m zeroes inside $\mathcal{D}$, and there is a δ separation between $\partial\mathcal{D}$ and the outside zeroes. The corresponding event for the GAF zero process has positive probability, so by the distributional convergence $\mathcal{Z}_n \rightarrow \mathcal{Z}$, we have that $\Omega_n^{m,\delta}$ has positive probability (which is bounded away from 0 as $n \rightarrow \infty$).

Let us denote the zeroes of f_n (in uniform random order) inside $\mathcal{D}$ by $\underline{\zeta} = (\zeta_1, \dots, \zeta_m)$ and those outside $\mathcal{D}$ by $\underline{\omega} = (\omega_1, \omega_2, \dots, \omega_{n-m})$. Let $s = \sum_{j=1}^m \zeta_j$. and $\Sigma_S := \{(\zeta_1, \dots, \zeta_m) \in \mathcal{D}^m : \sum_{j=1}^m \zeta_j = s\}$. Then the conditional density $\rho_{\underline{\omega}, s}^n(\underline{\zeta})$ of $\underline{\zeta}$ given $\underline{\omega}, s$ is of the form (see, e.g., [FH99])

$$\rho_{\underline{\omega}, s}^n(\underline{\zeta}) = C(\underline{\omega}, s) |\Delta(\zeta_1, \dots, \zeta_m, \omega_1, \dots, \omega_{n-m})|^2 \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)^{-n-1} \quad (9)$$

where $C(\underline{\omega}, s)$ is the normalizing factor, and for a vector $\underline{v} = (v_1, \dots, v_N)$ we define the elementary symmetric polynomial of order k as

$$\sigma_k(\underline{v}) = \sum_{1 \leq i_1 < \dots < i_k \leq N} v_{i_1} \cdots v_{i_k}$$

and for two vectors $\underline{u}$ and $\underline{v}$, $\sigma_k(\underline{u}, \underline{v})$ is defined to be $\sigma_k(\underline{w})$ where the vector $\underline{w}$ is obtained by concatenating the vectors $\underline{u}$ and $\underline{v}$. As a matter of convention, for any vector $\underline{v}$ we set $\sigma_k(\underline{v}) = 0$ for all $k \leq 0$ or $\geq \text{Length of } \underline{v}$.

Throughout this Subsection 10.1, the zeroes will be those of f_n with n fixed. Define the quantity

$$D(\underline{\zeta}, \underline{\omega}) = \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2 \right).$$

Let $(\underline{\zeta}, \underline{\omega})$ and $(\underline{\zeta}', \underline{\omega})$ be two vectors of points (under $\mathcal{Z}_n$). Then the ratio of the conditional densities at these two vectors is given by

$$\frac{\rho_{\underline{\omega}, s}^n(\underline{\zeta}')}{\rho_{\underline{\omega}, s}^n(\underline{\zeta})} = \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)^{n+1} / \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}', \underline{\omega})}{\sqrt{\binom{n}{k} k!}} \right|^2 \right)^{n+1} = \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \left(\frac{D(\underline{\zeta}', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})} \right)^{-(n+1)}. \quad (10)$$

The expression (10) has two distinct components: the ratio of two Vandermonde determinants and the ratio of certain expressions involving the elementary symmetric functions of the zeroes. We will consider these two components in two separate sections.

10.1.1 Ratio of Vandermondes

Here we consider the quantity $|\Delta(\underline{\zeta}', \underline{\omega})|^2 / |\Delta(\underline{\zeta}, \underline{\omega})|^2$. We proceed exactly as in the case of the Ginibre ensemble. We refer the reader to section 8.1. The estimates here are valid for all pairs $(\underline{\zeta}, \underline{\zeta}') \in \mathcal{D}^m \times \mathcal{D}^m$.

Proposition 10.1. *On the event $\Omega_n^{m, \delta}$, there exist quantities $K(\mathcal{D}, \delta) > 0$ and $\mathbf{X}_n(\underline{\omega}) > 0$ such that for any $(\underline{\zeta}, \underline{\zeta}') \in \mathcal{D}^m \times \mathcal{D}^m$ we have*

$$\exp \left(-2mK(\mathcal{D}, \delta) \mathbf{X}_n(\underline{\omega}) \right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2} \leq \frac{|\Delta(\underline{\zeta}', \underline{\omega})|^2}{|\Delta(\underline{\zeta}, \underline{\omega})|^2} \leq \exp \left(2mK(\mathcal{D}, \delta) \mathbf{X}_n(\underline{\omega}) \right) \frac{|\Delta(\underline{\zeta}')|^2}{|\Delta(\underline{\zeta})|^2}$$

where $\mathbf{X}_n(\underline{\omega}) = \left| \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j} \right| + \left| \sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{\omega_j^2} \right| + \left(\sum_{\omega_j \in \mathcal{G}_n \cap \mathcal{D}^c} \frac{1}{|\omega_j|^3} \right)$ and $\mathbb{E}[\mathbf{X}_n(\underline{\omega})] \leq c_1(\mathcal{D}, m) < \infty$.

Remark 10.1. *The estimates on $\left| \sum \frac{1}{\omega_j} \right|$, $\left| \sum \frac{1}{\omega_j^2} \right|$ and $\left(\sum \frac{1}{|\omega_j|^3} \right)$ which are necessary for Proposition 10.1 are proved in section 10.2.*

Corollary 10.2. *Given $M > 0$, we can replace the $\mathbf{X}_n(\underline{\omega})$ in Proposition 10.1 by a bound M except on an event of probability less than $c(m, \mathcal{D})/M$.*

10.1.2 Ratio of Symmetric Functions

In this section we will restrict $\underline{\zeta}$ and $\underline{\zeta}'$ to lie in the same constant-sum hyperplane. For $s \in \mathbb{C}$, define

$$\Sigma_s := \{ \underline{\zeta} \in \mathcal{D}^m : \sum_{i=1}^m \zeta_i' = s \}.$$

We want to bound the ratio $(D(\underline{\zeta}', \underline{\omega})/D(\underline{\zeta}, \underline{\omega}))^{n+1}$ from above and below. Our main goal is:

Proposition 10.3. *Suppose $(\underline{\zeta}, \underline{\omega})$ is randomly generated from $\mathcal{Z}_n$, let $s = \sum_{i=1}^m \zeta_i$, and define Σ_s as above. Given $M > 0$ large enough, $\exists n_0$ such that for all $n \geq n_0$ the following is true: with probability $\geq 1 - C/M$ we have, on the event $\Omega_n^{m, \delta}$, the inequality*

$$e^{-2K(m, \mathcal{D})M \log M} \leq \left(D(\underline{\zeta}', \underline{\omega}) / D(\underline{\zeta}, \underline{\omega}) \right)^{n+1} \leq e^{2K(m, \mathcal{D})M \log M}$$

for all $\underline{\zeta}' \in \Sigma_s$.

We will first prove several auxiliary propositions.

We begin with the observation

$$\sigma_k(\underline{\zeta}, \underline{\omega}) = \sum_{i=0}^m \sigma_i(\underline{\zeta}) \sigma_{k-i}(\underline{\omega}). \quad (11)$$

Note that $\sigma_0(\underline{\zeta}) = 1$, $\sigma_1(\underline{\zeta}) = s = \sigma_1(\underline{\zeta}')$ and $|\sigma_i(\underline{\zeta})| < \binom{m}{i} r_0^i$ for all $i \leq m$, where r_0 is the radius of $\mathcal{D}$. Since both $\underline{\zeta}$ and $\underline{\zeta}' \in \Sigma_s$, we have

$$\sigma_k(\underline{\zeta}', \underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) + \sum_{i=2}^m [\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})] \sigma_{k-i}(\underline{\omega}). \quad (12)$$

Taking modulus squared on both sides and using the identity

$$\left| \sum_j a_j \right|^2 = \sum_j |a_j|^2 + 2 \sum_{j \neq k} \Re(a_j \overline{a_k}),$$

we have $|\sigma_k(\underline{\zeta}', \underline{\omega})|^2 =$

$$\begin{aligned} & |\sigma_k(\underline{\zeta}, \underline{\omega})|^2 + \sum_{i=2}^m |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|^2 |\sigma_{k-i}(\underline{\omega})|^2 + \sum_{i=2}^m 2 \Re \left((\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})) \overline{\sigma_k(\underline{\zeta}, \underline{\omega})} \sigma_{k-i}(\underline{\omega}) \right) \\ & + \sum_{i,j=2}^m \sum_{i \neq j} 2 \Re \left(\overline{(\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta}))} (\sigma_j(\underline{\zeta}') - \sigma_j(\underline{\zeta})) \overline{\sigma_{k-i}(\underline{\omega})} \sigma_{k-j}(\underline{\omega}) \right). \end{aligned}$$

Dividing throughout by $\left(\sqrt{\binom{n}{k}} k! \right)^2$ and then summing the above over $k = 0, \dots, n$ we get

$$\begin{aligned} \sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}', \underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 &= \sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 + \sum_{i=2}^m |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|^2 \left(\sum_{k=0}^n \left| \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 \right) \\ &+ \sum_{i=2}^m 2 \Re \left((\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})) \left(\sum_{k=0}^n \frac{\overline{\sigma_k(\underline{\zeta}, \underline{\omega})}}{\sqrt{\binom{n}{k}} k!} \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right) \right) \\ &+ \sum_{i,j=2}^m \sum_{i \neq j} 2 \Re \left(\overline{(\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta}))} (\sigma_j(\underline{\zeta}') - \sigma_j(\underline{\zeta})) \left(\sum_{k=0}^n \frac{\overline{\sigma_{k-i}(\underline{\omega})}}{\sqrt{\binom{n}{k}} k!} \frac{\sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right) \right). \end{aligned}$$

Applying the triangle inequality to the above, we obtain

$$D(\underline{\zeta}, \underline{\omega}) - A(\underline{\zeta}, \underline{\zeta}', \underline{\omega}) \leq D(\underline{\zeta}', \underline{\omega}) \leq D(\underline{\zeta}, \underline{\omega}) + A(\underline{\zeta}, \underline{\zeta}', \underline{\omega}) \quad (13)$$

where we recall the fact that $D(\underline{\zeta}, \underline{\omega}) = \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 \right)$ and

$$\begin{aligned} A(\underline{\zeta}, \underline{\zeta}', \underline{\omega}) := & \sum_{i=2}^m |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|^2 \left(\sum_{k=0}^n \left| \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 \right) + \sum_{i=2}^m 2 |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})| \left| \sum_{k=0}^n \frac{\overline{\sigma_k(\underline{\zeta}, \underline{\omega})}}{\sqrt{\binom{n}{k}} k!} \frac{\sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right| \\ & + \sum_{i \neq j=2}^m 2 |\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})| |\sigma_j(\underline{\zeta}') - \sigma_j(\underline{\zeta})| \left| \sum_{k=0}^n \frac{\overline{\sigma_{k-i}(\underline{\omega})}}{\sqrt{\binom{n}{k}} k!} \frac{\sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|. \end{aligned}$$

Divide throughout by $D(\underline{\zeta}, \underline{\omega})$ in the above inequalities, and observe that

- (a) i and j vary between 2 and m (which is fixed and finite),
- (b) $|\sigma_i(\underline{\zeta}') - \sigma_i(\underline{\zeta})|$ is bounded for each i as discussed immediately before (12).

In view of these facts, we conclude that in order to bound $(D(\underline{\zeta}', \underline{\omega})/D(\underline{\zeta}, \underline{\omega}))^{n+1}$ between two constants, it suffices to show that the quantities

$$D_i^{(1)}(\underline{\zeta}, \underline{\omega}) := \left| \sum_{k=0}^n \frac{\overline{\sigma_k(\underline{\zeta}, \underline{\omega})} \sigma_{k-i}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right| / \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 \right) \quad (14)$$

and

$$D_{i,j}^{(2)}(\underline{\zeta}, \underline{\omega}) := \left| \sum_{k=0}^n \frac{\overline{\sigma_{k-i}(\underline{\omega})} \sigma_{k-j}(\underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right| / \left(\sum_{k=0}^n \left| \frac{\sigma_k(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{k}} k!} \right|^2 \right) \quad (15)$$

for $m \geq i, j \geq 2$ are bounded above by M/n with high probability (depending on M) where $M > 0$ is a constant.

Observe, by Vieta's formula, that $\sigma_k(\underline{\zeta}, \underline{\omega})/\sqrt{\binom{n}{k}} k! = (-1)^k \xi_{n-k}/\xi_n$, where the ξ_j -s are standard complex Gaussians. On the other hand, we do not yet have good control over $\sigma_k(\underline{\omega})$. In order for us to obtain such control, the idea is to obtain a convenient expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_i(\underline{\zeta}, \underline{\omega})$. To this end, we formulate:

Proposition 10.4. *On the event $\Omega_n^{m,\delta}$ we have, for $0 \leq k \leq n - m$,*

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) + \sum_{r=1}^k g_r \sigma_{k-r}(\underline{\zeta}, \underline{\omega})$$

where a.s. the random variables g_r are $O(K(\mathcal{D}, m)^r)$ as $r \rightarrow \infty$, for a deterministic quantity $K(\mathcal{D}, m)$ and the constant in O being deterministic and not depending on n and δ .

Proof. We begin with the observation that

$$\sigma_k(\underline{\zeta}, \underline{\omega}) = \sum_{r=0}^m \sigma_r(\underline{\zeta}) \sigma_{k-r}(\underline{\omega}). \quad (16)$$

From this, we have

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) - \sum_{r=1}^m \sigma_r(\underline{\zeta}) \sigma_{k-r}(\underline{\omega}). \quad (17)$$

Notice that the coefficients $\sigma_r(\underline{\zeta})$ in the above expansion are independent of k . This suggests a recursive pattern in the expression of $\sigma_k(\underline{\omega})$ in terms of $\sigma_j(\underline{\zeta}, \underline{\omega})$ with $j \leq k$. Proceeding inductively in (17) we can similarly expand each of the lower order $\sigma_{k-r}(\underline{\omega})$ in terms of $\sigma_j(\underline{\zeta}, \underline{\omega})$ and obtain an expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_j(\underline{\zeta}, \underline{\omega})$, $j = 1, \dots, k$:

$$\sigma_k(\underline{\omega}) = \sigma_k(\underline{\zeta}, \underline{\omega}) + \sum_{r=1}^k g_r \sigma_{k-r}(\underline{\zeta}, \underline{\omega}). \quad (18)$$

Consequently, the coefficient of $\sigma_k(\underline{\zeta}, \underline{\omega})$ is 1 and the rest of the coefficients are polynomials in $\sigma_j(\underline{\zeta}), j = 1, \dots, m$. Due to the inductive structure, it is easy to see that g_r -s satisfy a recurrence relation:

$$\begin{pmatrix} g_i \\ g_{i-1} \\ \dots \\ g_{i-m+1} \end{pmatrix} = A \begin{pmatrix} g_{i-1} \\ g_{i-2} \\ \dots \\ g_{i-m} \end{pmatrix} \quad (19)$$

where

$$A := \begin{pmatrix} -\sigma_1(\underline{\zeta}) & -\sigma_2(\underline{\zeta}) & \dots & -\sigma_m(\underline{\zeta}) \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with the boundary conditions $g_i = -\sigma_i(\underline{\zeta})$ for $i = 0, \dots, m-1$ and $g_i = 0$ for $i < 0$.

We observe that the eigenvalues of A are precisely the negatives of the inside zeroes $\zeta_1, \dots, \zeta_m$. Due to the recursive structure in (19), we note that g_k is an element of A^k applied to the vector $(-\sigma_{m-1}(\underline{\zeta}), \dots, -\sigma_0(\underline{\zeta}))$. This implies that g_k is a linear combination of the entries of A^k (with the coefficients in the linear combination being independent of k , but depending on $\mathcal{D}$ and m).

From Proposition 10.5, we see that if the eigenvalues of a matrix A have modulus $< \rho'$, then the entries of the matrix A^k are $o(\rho'^k)$. Now, the eigenvalues of A are ζ_i -s and for each i we have $|\zeta_i| \leq \text{radius}(\mathcal{D}) = K(\mathcal{D})$. Combining the last two paragraphs, we have $g_r = O((K(\mathcal{D}) + 1)^r)$. Crucially, the constant in the O above does not depend on n and δ (although it can depend on m). ■

We complete this discussion with the following result on the growth of matrix entries:

Proposition 10.5. *Let A be a $d \times d$ matrix. Let $\rho' > \rho(A) := \max\{|\lambda| : \lambda \text{ an eigenvalue of } A\}$. Then $(A^r)_{ij} = o(\rho'^r)$ as $r \rightarrow \infty$, for all i, j .*

Proof. This follows from the well known result (Gelfand's spectral radius theorem): $\rho(A) = \lim_{r \rightarrow \infty} \|A^r\|^{1/r}$ where $\|\cdot\|$ denotes the Frobenius norm of the matrix, and the fact that $\max_{1 \leq i, j \leq d} |B_{ij}| \leq \|B\|$ for any $d \times d$ matrix B . ■

The aim of the next proposition is to show that in the expansion of $\sigma_k(\underline{\omega})$ in terms of $\sigma_r(\underline{\zeta}, \underline{\omega})$ with $r \leq k$, it is essentially the leading term $\sigma_k(\underline{\zeta}, \underline{\omega})$ that matters. For clarity, we will switch to a different indexing of the σ -s and involve the Gaussian coefficients of the polynomial more directly. Recall that $\frac{\sigma_{n-r}(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n}{n-r}(n-r)!}} = (-1)^{n-r} \frac{\xi_r}{\xi_n}$. Divide both sides of the expansion proved in Proposition 10.4 by $\sqrt{\binom{n}{k}k!}$ and note that $\sqrt{\binom{n}{k}k!} = \sqrt{\binom{n}{k-r}(k-r)!} \sqrt{(n-k+1) \cdots (n-k+r)}$.

Notation. We denote the falling factorial (of order k) of an integer x by

$$(x)_k := x(x-1) \cdots (x-k+1).$$

For $k = 0$ we set $(x)_0 = 1$.

Switching variables to $l = n - k$, we can rewrite the expansion in Proposition 10.4 as

$$\frac{\sigma_{n-l}(\underline{\omega})}{\sqrt{\binom{n}{n-l}(n-l)!}} = (-1)^{n-l} \frac{\xi_l}{\xi_n} + \sum_{r=1}^{n-l} (-1)^{n-l-r} g_r \frac{\xi_{l+r}}{\xi_n} \frac{1}{\sqrt{(l+r)_r}}$$

for $l \geq m$. Define for any integer $l \geq 1$

$$\eta_l^{(n)} = \sum_{r=1}^{n-l} (-1)^{n-l-r} g_r \xi_{l+r} \frac{1}{\sqrt{(l+r)_r}}. \quad (20)$$

As a result, for $l \geq m$ we have

$$\frac{\sigma_{n-l}(\underline{\omega})}{\sqrt{\binom{n}{n-l}(n-l)!}} = \frac{1}{\xi_n} \left[(-1)^{n-l} \xi_l + \eta_l^{(n)} \right]. \quad (21)$$

Proposition 10.6. *Let $\eta_l^{(n)}$ be as in (20). Then there exist positive random variables η_l (independent of n) such that a.s. $|\eta_l^{(n)}| \leq \eta_l$, and for fixed $l_0 \in \mathbb{N}$ and large enough $M > 0$ we have*

$$(i) \mathbb{P} \left[\eta_l > \frac{M}{l^{1/8}} \text{ for some } l \geq 1 \right] \leq e^{-c_1 M^2}$$

$$(ii) \mathbb{P} \left[\eta_l > \frac{M}{l^{1/8}} \text{ for some } l \geq l_0 \right] \leq e^{-c_2 M^2 l_0^{1/4}}$$

where c_1, c_2 are constants that depend on the domain $\mathcal{D}$ and on m .

Proof. We begin by recalling from Proposition 10.4 that a.s. $|g_r| \leq K^r$ for some $K = K(\mathcal{D}, m)$. Moreover, $\sqrt{(l+p)} \geq \sqrt{2} l^{1/4} p^{1/4}$. Hence,

$$|\eta_l^{(n)}| \leq \sum_{r=1}^{n-l} \frac{K^r}{2^{r/2} l^{r/4} (r!)^{1/4}} |\xi_{l+r}|.$$

Let B be such that $\sup_{r \geq 1} \frac{K^r}{2^{r/2} (r!)^{1/8}} \leq B$. Then $|\eta_l^{(n)}| \leq \sum_{r=1}^{n-l} \frac{B |\xi_{l+r}|}{l^{r/4} (r!)^{1/8}}$.

Define $\eta_l = \sum_{r=1}^{\infty} \frac{B |\xi_{l+r}|}{l^{r/4} (r!)^{1/8}}$. Clearly, $|\eta_l^{(n)}| \leq \eta_l$. Now, $|\xi_{l+r}|^2$ is an exponential random variable with mean 1, we have

$$\mathbb{P} \left(|\xi_{l+r}| \geq \frac{M l^{r/8} (r!)^{1/16}}{B} \right) = \exp \left(-\frac{M^2 l^{r/4} (r!)^{1/8}}{B^2} \right). \quad (22)$$

Moreover,

$$\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \exp \left(-\frac{M^2 l^{r/4} (r!)^{1/8}}{B^2} \right) \leq C_1 \exp(-M^2/B^2).$$

So we can have $|\xi_{l+r}| \leq \frac{M l^{r/8} (r!)^{1/16}}{B}$ for all $l \geq 1, r \geq 0$, except on an event of probability $\leq C_1 \exp(-M^2/B^2)$. The last expression can be made arbitrarily small by fixing M to be large

enough. Note that for large enough M , we have $C \exp(-M^2/B^2) \leq \exp(-cM^2)$. On the complement of this small event, we have for all $l \geq 1$

$$\eta_l \leq \sum_{r=1}^{\infty} \frac{M}{l^{r/8}(r!)^{1/16}} \leq \sum_{r=1}^{\infty} \frac{M}{l^{r/8}(r!)^{1/16}} \leq C \frac{M}{l^{1/8}}.$$

We then absorb this factor of C into M by changing the constant c_1 appearing in the exponent of the tail estimate (i) we want to prove.

For proving (ii) we proceed exactly as in the case of (i) above and sum (22) over $r \geq 1$ and $l \geq l_0$. $\blacksquare$

We are now ready to prove bounds on the quantities $D_{i,j}^{(1)}(\zeta, \omega)$ and $D_{i,j}^{(2)}(\zeta, \omega)$ in (14) and (15). To this end, we first express them in terms of ξ -s, using $\sigma_{n-l}(\underline{\zeta}, \underline{\omega}) / \sqrt{\binom{n}{n-l}(n-l)!} = (-1)^{n-l} \xi_l / \xi_n$ and (21). For $2 \leq i, j \leq m$ these expressions, (after canceling out $|\xi_n|^2$ from numerator and denominator) reduce to :

$$D_i^{(1)}(\zeta, \omega) = \left| \sum_{l=0}^{n-i} \frac{\overline{(-1)^{n-l} \xi_l} \left((-1)^{n-l-i} \xi_{l+i} + \eta_{l+i}^{(n)} \right)}{\sqrt{(l+i)_i}} \right| / \left(\sum_{l=0}^n |\xi_l|^2 \right) \quad (23)$$

and

$$D_{i,j}^{(2)}(\zeta, \omega) = \left| \sum_{l=0}^{n-i \wedge n-j} \frac{\overline{\left((-1)^{n-l-i} \xi_{l+i} + \eta_{l+i}^{(n)} \right)} \left((-1)^{n-l-j} \xi_{l+j} + \eta_{l+j}^{(n)} \right)}{\sqrt{(l+i)_i} \sqrt{(l+j)_j}} \right| / \left(\sum_{l=0}^n |\xi_l|^2 \right). \quad (24)$$

Define

$$\begin{aligned} \mathbf{E}_n &= \sum_{l=0}^n |\xi_l|^2, & \mathbf{L}_{ij}^{(n)} &= \sum_{l=0}^{n-i \wedge n-j} \frac{(-1)^{i+j} \overline{\xi_{l+i}} \xi_{l+j}}{\sqrt{(l+i)_i} \sqrt{(l+j)_j}}, \\ \mathbf{M}_{ij}^{(n)} &= \sum_{l=0}^{n-i \wedge n-j} \frac{|\xi_{l+i}| |\eta_{l+j}|}{\sqrt{(l+i)_i} \sqrt{(l+j)_j}}, & \mathbf{N}_{ij}^{(n)} &= \sum_{l=0}^{n-i \wedge n-j} \frac{\eta_{l+i} \eta_{l+j}}{\sqrt{(l+i)_i} \sqrt{(l+j)_j}}. \end{aligned}$$

Recall that for $i = 0$ the product $(l+i)_i$ in the denominator is replaced by 1. Expanding out the products and applying the triangle inequality in (23) and (24), we observe that in order to obtain upper bounds for $D_i^{(1)}(\zeta, \omega)$ and $D_{i,j}^{(2)}(\zeta, \omega)$, it suffices to estimate from above the following quantities (remember that $|\eta_l^{(n)}| \leq \eta_l$ for all l):

$$\frac{\left(\left| \mathbf{L}_{0j}^{(n)} \right| + \mathbf{M}_{0j}^{(n)} \right)}{\mathbf{E}_n}, \quad \frac{\left(\left| \mathbf{L}_{ij}^{(n)} \right| + \mathbf{M}_{ij}^{(n)} + \mathbf{M}_{ji}^{(n)} + \mathbf{N}_{ij}^{(n)} \right)}{\mathbf{E}_n} \text{ for } 2 \leq i, j \leq m. \quad (25)$$

To be more specific, by a simple application of the triangle inequality, $D_i^{(1)}(\zeta, \omega)$ is bounded above by $\left(\left| \mathbf{L}_{0j}^{(n)} \right| + \mathbf{M}_{0j}^{(n)} \right) / \mathbf{E}_n$ and $D_{i,j}^{(2)}(\zeta, \omega)$ is bounded above by $\left(\left| \mathbf{L}_{ij}^{(n)} \right| + \mathbf{M}_{ij}^{(n)} + \mathbf{M}_{ji}^{(n)} + \mathbf{N}_{ij}^{(n)} \right) / \mathbf{E}_n$.

Let

$$\mathbf{Y}_n = \sum_{i=2}^m |\mathbf{L}_{0i}^{(n)}| + \sum_{i=2}^m \mathbf{M}_{0i}^{(n)} + \sum_{i,j \geq 2}^m |\mathbf{L}_{ij}^{(n)}| + \sum_{i,j \geq 2}^m \mathbf{M}_{ij}^{(n)} + \sum_{i,j \geq 2}^m \mathbf{M}_{ji}^{(n)} + \sum_{i,j \geq 2}^m \mathbf{N}_{ij}^{(n)}.$$

Recall (13) and that for fixed m and $\mathcal{D}$, we have that $\sigma_i(\underline{\zeta})$ are uniformly bounded. Putting all these together, we deduce that: for some constant $K(m, \mathcal{D})$ we have

$$1 - K(m, \mathcal{D}) \frac{\mathbf{Y}_n}{\mathbf{E}_n} \leq \frac{D(\underline{\zeta}', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})} \leq 1 + K(m, \mathcal{D}) \frac{\mathbf{Y}_n}{\mathbf{E}_n}. \quad (26)$$

Regarding $\mathbf{Y}_n$ and $\mathbf{E}_n$, we have the following estimates:

Proposition 10.7. *Given $M > 0$ we have:*

- (i) $\mathbb{P}[\mathbf{Y}_n \geq M \log M] \leq c(m, \mathcal{D})/M$,
- (ii) *There exists n_0 such that for $n \geq n_0$ we have $\mathbb{P}[\frac{n}{2} \leq |\mathbf{E}_n| \leq 2n] \geq 1 - \frac{1}{M}$.*

Proof. Part (i):

We will show that if X is one of $\mathbf{L}_{ij}^{(n)}$, $\mathbf{M}_{ij}^{(n)}$ or $\mathbf{N}_{ij}^{(n)}$, then it satisfies $\mathbb{P}[|X| > M \log M] < c'(m, \mathcal{D})/M$. This would imply $\mathbb{P}[\mathbf{Y}_n > M \log M] < c_1(m, \mathcal{D})/M$ albeit for a different constant $c(m, \mathcal{D})$. This is easily seen as follows. The number of summands in the definition of $\mathbf{Y}_n$ is a polynomial in m , let us call it $p(m)$. Now, for each summand X of $\mathbf{Y}_n$, we have

$$\mathbb{P}\left[|X| > \frac{M \log M}{p(m)}\right] \leq \mathbb{P}\left[|X| > \frac{M}{p(m)} \log \frac{M}{p(m)}\right] \leq c'(m, \mathcal{D})p(m)/M.$$

If $\mathbf{Y}_n > M \log M$ and there are $p(m)$ summands, at least one of the summands must be $> M \log M/p(m)$. By a union bound over the summands, this implies $\mathbb{P}[\mathbf{Y}_n > M \log M] \leq c'(m, \mathcal{D})p(m)^2/M$. Setting $c(m, \mathcal{D}) = c'(m, \mathcal{D})p(m)^2$ as the new constant, this would give us part (i) of the proposition.

It remains to show that each of the summands in $\mathbf{Y}_n$ satisfies $\mathbb{P}[|X| > M \log M] < c(m, \mathcal{D})/M$ with the quantity $c(m, \mathcal{D})$ not depending on n . The understanding is that c will depend on the particular summand, and we will take a maximum over the $p(m)$ summands in order to obtain $c'(m, \mathcal{D})$ as in the previous paragraph. We consider the cases of $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{N}$ separately.

Estimating $\mathbf{L}_{ij}^{(n)}$, $i \vee j \geq 2$:

We have

$$\mathbb{E}\left[|\mathbf{L}_{ij}^{(n)}|^2\right] = \sum_{l=0}^{n-i \wedge n-j} \frac{1}{(l+i)_i(l+j)_j} \leq \sum_{l=0}^{\infty} \frac{1}{(l+i)_i(l+j)_j} = c(i, j) \leq c < \infty$$

(since either i or $j \geq 2$). An application of Chebyshev's inequality proves the desired tail bound on $\mathbf{L}_{ij}^{(n)}$.

Estimating $\mathbf{M}_{ij}^{(n)}$, $j \geq 2$:

By Proposition 10.6 we can assume (after discarding an event A of probability $\leq e^{-c(\log M)^2}$) that $\max_{l \geq 1} \eta_l \leq \log M / l^{1/8}$. Applying the triangle inequality to the definition of $\mathbf{M}_{ij}^{(n)}$ and using the upper bound on η_{l+j} we have

$$\mathbb{E} \left[|\mathbf{M}_{ij}^{(n)}| \mathbb{1}_{A^c} \right] \leq \log M \mathbb{E} \left[\sum_{l=0}^{n-i \wedge n-j} \frac{|\xi_{l+i}|}{(l+j)^{1/8} \sqrt{(l+i)_i (l+j)_j}} \right] \leq m(i, j) \log M \leq c \log M < \infty,$$

where $m(i, j) = \sum_{l=0}^{\infty} \frac{\mathbb{E}[|\xi|]}{(l+j)^{1/8} \sqrt{(l+i)_i (l+j)_j}} \leq c < \infty$, and ξ is a standard complex Gaussian. In the last step, we have used the fact that $j \geq 2$ in order to upper bound $m(i, j)$ uniformly in i and j . Applying Markov inequality gives us the desired tail estimate $\mathbb{P}(|\mathbf{M}_{ij}^{(n)}| \mathbb{1}_{A^c} > M \log M) \leq c/M$. This, along with $\mathbb{P}(A) < e^{-c(\log M)^2}$ completes the proof.

Estimating $\mathbf{N}_{ij}^{(n)}$, $i \wedge j \geq 2$:

We consider the event A as in the case of $\mathbf{M}_{ij}^{(n)}$. On A^c , we use $\eta_{l+i} \eta_{l+j} \leq (\log M)^2 < M \log M$ (for large M), and the rest is bounded above by

$$\sum_{l=0}^{n-i \wedge n-j} \frac{1}{l^{1/4} \sqrt{(l+i)_i (l+j)_j}} \leq \sum_{l=0}^{\infty} \frac{1}{(l+i)^{1/8} (l+j)^{1/8} \sqrt{(l+i)_i (l+j)_j}} = c(i, j) \leq c < \infty.$$

This, along with $\mathbb{P}[A] < e^{-c(\log M)^2}$ establishes the desired tail bound on $\mathbf{N}_{ij}^{(n)}$.

All these arguments complete the proof of part (i) of Proposition 10.7.

Part(ii):

We simply observe that $|\xi|^2$ -s are i.i.d. exponentials and by the strong law of large numbers, $\mathbf{E}_n/n \rightarrow 1$ a.s. From this, the statement of part (ii) follows. ■

Define

$$\begin{aligned} \mathbf{L}_{ij} &= \sum_{l=0}^{\infty} \frac{(-1)^{i+j} \overline{\xi_{l+i}} \xi_{l+j}}{\sqrt{(l+i)_i (l+j)_j}}, & \tau(\mathbf{L}_{ij}^{(n)}) &= \mathbf{L}_{ij} - \mathbf{L}_{ij}^{(n)}, \quad i \vee j \geq 2 \\ \mathbf{M}_{ij} &= \sum_{l=0}^{\infty} \frac{|\xi_{l+i}| \eta_{l+j}}{\sqrt{(l+i)_i (l+j)_j}}, & \tau(\mathbf{M}_{ij}^{(n)}) &= \mathbf{M}_{ij} - \mathbf{M}_{ij}^{(n)}, \quad j \geq 2 \\ \mathbf{N}_{ij} &= \sum_{l=0}^{\infty} \frac{\eta_{l+i} \eta_{l+j}}{\sqrt{(l+i)_i (l+j)_j}}, & \tau(\mathbf{N}_{ij}^{(n)}) &= \mathbf{N}_{ij} - \mathbf{N}_{ij}^{(n)}, \quad i \wedge j \geq 2. \end{aligned}$$

For $i = 0$ the product $(l+i)_i$ in the denominator is replaced by 1. The above random variables have finite first moments, as can be seen by arguing on similar lines to the estimates in Proposition 10.7. Similar arguments with first moments of $\tau(\mathbf{L}_{ij}^{(n)})$, $\tau(\mathbf{M}_{ij}^{(n)})$, $\tau(\mathbf{N}_{ij}^{(n)})$ also show that:

Proposition 10.8. *As $n \rightarrow \infty$, each of the random variables $\tau(\mathbf{L}_{ij}^{(n)})$, $\tau(\mathbf{M}_{ij}^{(n)})$, $\tau(\mathbf{N}_{ij}^{(n)})$ converges to 0 in L^1 , and hence, in probability.*

Now we are ready to prove Proposition 10.3.

Proof of Proposition 10.3. We refer back to equation (26):

$$1 - K(m, \mathcal{D}) \frac{\mathbf{Y}_n}{\mathbf{E}_n} \leq \frac{D(\underline{\zeta}', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})} \leq 1 + K(m, \mathcal{D}) \frac{\mathbf{Y}_n}{\mathbf{E}_n} .$$

By virtue of Proposition 10.7 we have, dropping an event of probability $\leq c/M$, the inequality $|\mathbf{Y}_n/\mathbf{E}_n| \leq 2M \log M/n$ (for $n \geq n_0$, where n_0 is large enough such that Proposition 10.7 part(ii) holds). Raising this to the $(n+1)$ th power we obtain the desired result. All constants are absorbed in $K(m, \mathcal{D})$. $\blacksquare$

Corollary 10.9. *Let $(\underline{\zeta}, \underline{\omega})$ be picked randomly from the distribution $\mathbb{P}[\mathcal{Z}_n]$ and $s = \sum_{i=1}^m \zeta_i$. Let $\Sigma_s = \{\underline{\zeta}' : \sum_{i=1}^m \zeta'_i = s\}$. Given $M > 0$ large enough, there exists n_0 such that for all $n \geq n_0$, we have, on $\Omega_n^{m, \delta}$ (except on an event of probability $\leq C/M$)*

$$e^{-4K(m, \mathcal{D})M \log M} \leq \left(D(\underline{\zeta}'', \underline{\omega}) / D(\underline{\zeta}', \underline{\omega}) \right)^{n+1} \leq e^{4K(m, \mathcal{D})M \log M}$$

for all $(\underline{\zeta}'', \underline{\zeta}') \in \Sigma_s \times \Sigma_s$.

Proof. We consider the identity

$$\frac{D(\underline{\zeta}'', \underline{\omega})}{D(\underline{\zeta}', \underline{\omega})} = \frac{D(\underline{\zeta}'', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})} \bigg/ \frac{D(\underline{\zeta}', \underline{\omega})}{D(\underline{\zeta}, \underline{\omega})}$$

for $(\underline{\zeta}', \underline{\zeta}'') \in \Sigma_s \times \Sigma_s$ where $s = \sum_{i=1}^m \zeta_i$, and applying the bounds in Proposition 10.3. $\blacksquare$

10.1.3 Estimate for Ratio of Conditional Densities

Proposition 10.10. *Let $(\underline{\zeta}, \underline{\omega})$ be picked randomly from the distribution $\mathbb{P}[\mathcal{Z}_n]$ and $s = \sum_{i=1}^m \zeta_i$. Let $\Sigma_s = \{\underline{\zeta}' : \sum_{i=1}^m \zeta'_i = s\}$. There exists a constant $K(m, \mathcal{D}, \delta)$ such that given $M > 0$ large enough, for $n \geq n_0(m, M, \mathcal{D})$ the following inequalities hold on $\Omega_n^{m, \delta}$, except on an event of probability $\leq c(m, \mathcal{D})/M$:*

$$\exp \left(-K(m, \mathcal{D}, \delta) M \log M \right) \frac{|\Delta(\underline{\zeta}'')|^2}{|\Delta(\underline{\zeta}')|^2} \leq \frac{\rho_{\underline{\omega}, s}^n(\underline{\zeta}'')}{\rho_{\underline{\omega}, s}^n(\underline{\zeta}')} \leq \exp \left(K(m, \mathcal{D}, \delta) M \log M \right) \frac{|\Delta(\underline{\zeta}'')|^2}{|\Delta(\underline{\zeta}')|^2}$$

for all $(\underline{\zeta}', \underline{\zeta}'') \in \Sigma_s \times \Sigma_s$.

Proof. We simply put together the estimates for the ratios of the Vandermondes as well as the symmetric functions and subsume all relevant constants in $c(m, \mathcal{D})$ and $K(m, \mathcal{D}, \delta)$. $\blacksquare$

10.2 Estimates for Inverse Powers of Zeroes

In this section we prove estimates on the (smoothed) sum of inverse powers of GAF zeroes.

Proposition 10.11. *Let Φ be a C_c^∞ radial function supported on the annulus between r_0 and $3r_0$. Let $\Phi_R = \Phi(z/R)$. We have*

$$\begin{aligned} \text{(i)} \quad & \mathbb{E} \left[\left| \int \frac{\Phi_R(z)}{z^l} d[\mathcal{Z}_n](z) \right| \right] \leq C_1(r_0, \Phi, l)/R^l \\ \text{(ii)} \quad & \mathbb{E} \left[\int \frac{\Phi_R(z)}{|z|^l} d[\mathcal{Z}_n](z) \right] \leq C_2(r_0, \Phi, l)/R^{l-2}. \end{aligned}$$

The same is true for $\mathcal{Z}$ in place of $\mathcal{Z}_n$. The quantities $C_1(r_0, \Phi, l)$ and $C_2(r_0, \Phi, l)$ do not depend on n .

Proof. Part (i): We begin with

$$\int \frac{1}{z^l} \Phi_R(z) d[\mathcal{Z}_n](z) = \int \frac{1}{z^l} \Phi_R(z) \Delta \log(|f_n(z)|) d\mathcal{L}(z).$$

Now, $\log(\sqrt{\mathbb{E}[|f_n(z)|^2]})$ is a radial function, and Laplacian of a radial function is also radial. Hence,

$$\int \frac{1}{z^l} \Phi_R(z) \Delta \log(\sqrt{\mathbb{E}[|f_n(z)|^2]}) d\mathcal{L}(z) = 0$$

because for $l \geq 1$, $1/z^l$ when integrated against a radial function always gives 0 due to rotational symmetry. Let $\hat{f}_n(z) = f_n(z)/\sqrt{\mathbb{E}[|f_n(z)|^2]}$. Then the above argument implies

$$\int \frac{1}{z^l} \Phi_R(z) d[\mathcal{Z}_n](z) = \int \frac{1}{z^l} \Phi_R(z) \Delta \log(|\hat{f}_n(z)|) d\mathcal{L}(z). \quad (27)$$

Integrating by parts the right hand side of (27) we have

$$\mathbb{E} \left[\left| \int \frac{1}{z^l} \Phi_R(z) d[\mathcal{Z}_n](z) \right| \right] \leq \int \left| \Delta \left(\frac{1}{z^l} \Phi_R(z) \right) \right| \mathbb{E} \left[\left| \log(|\hat{f}_n(z)|) \right| \right] d\mathcal{L}(z). \quad (28)$$

Now, the integrand is non-zero only for $Rr_0 \leq |z| \leq 3Rr_0$. Hence it is easy to see that $\left| \Delta \left(\frac{1}{z^l} \Phi_R(z) \right) \right| \leq C(r_0, \Phi, l)/R^{l+2}$. Further, $\mathbb{E} \left[\left| \log(|\hat{f}_n(z)|) \right| \right]$ is a constant because $\hat{f}_n$ is $N_{\mathbb{C}}(0, 1)$.

Finally, $\int_{3Rr_0 \cdot D \setminus Rr_0 \cdot D} d\mathcal{L}(z) = 8\pi r_0^2 R^2$, where D is the unit disk.

Putting all these together, we have

$$\int \left| \Delta \left(\frac{1}{z^l} \Phi_R(z) \right) \right| \mathbb{E} \left[\left| \log(|\hat{f}_n(z)|) \right| \right] d\mathcal{L}(z) \leq C_1(r_0, \Phi, l)/R^l$$

as desired (all constants are subsumed in $C_1(r_0, \Phi, l)$).

Part (ii): Since Φ is a radial function on $\mathbb{C}$, there exists a function $\tilde{\Phi}$ on $\mathbb{R}_{\geq 0}$ such that $\Phi(z) = \tilde{\Phi}(|z|)$. We have, with $r = |z|$,

$$\mathbb{E} \left[\int \frac{1}{|z|^l} \Phi_R(z) d[\mathcal{Z}_n](z) \right] = c \int \frac{1}{r^l} \tilde{\Phi}_R(r) \Delta \log(\sqrt{K_n(z, z)}) r dr$$

where the integral on the right hand side is over the non-negative reals. Integrating by parts,

$$\left| \int \frac{1}{r^l} \tilde{\Phi}_R(r) \Delta \log(\sqrt{K_n(z, z)}) r dr \right| \leq \int \left| \Delta \left(\frac{1}{r^l} \tilde{\Phi}_R(r) \right) \right| \log(\sqrt{K_n(z, z)}) r dr.$$

But $\log(\sqrt{K_n(z, z)}) \leq \log(\sqrt{K(z, z)}) = \frac{1}{2}r^2 \leq \frac{9}{2}r_0^2 R^2$ and $\left| \Delta \left(\frac{1}{r^l} \tilde{\Phi}_R(r) \right) \right| \leq C(r_0, \Phi, l)/R^{l+2}$, hence

$$\int \left| \Delta \left(\frac{1}{r^l} \tilde{\Phi}_R(r) \right) \right| \log(\sqrt{K_n(z, z)}) r dr \leq C_2(r_0, \Phi, l)/R^{l-2}$$

for another constant C_2 (which clearly does not depend on n). ■

Remark 10.2. The quantity $C_1(r_0, \Phi, l)$ above is of the form $p(l) \left(\frac{1}{r_0} \right)^l C_1(\Phi)$ where p is a fixed polynomial (of degree 2), and $C_1(\Phi)$ is a quantity that depends only on Φ . Both p and $C_1(\Phi)$ do not depend on n . Similarly, $C_2(r_0, \Phi, l)$ is of the form $p_1(l) \left(\frac{1}{r_0} \right)^{l-2} C_2(\Phi)$; where p_1 is another degree 2 polynomial and $C_2(\Phi)$ does not depend on n .

Let r_0 be the radius of $\mathcal{D}$. Let φ be a non-negative radial C_c^∞ function supported on $[r_0, 3r_0]$ such that $\varphi = 1$ on $[\frac{3r_0}{2}, 2r_0]$ and $\varphi(r_0 + r) = 1 - \varphi(2r_0 + 2r)$, for $0 \leq r \leq \frac{1}{2}r_0$. In other words, φ is a test function supported on the annulus between r_0 and $3r_0$ and its “ascent” to 1 is twice as fast as its “descent”.

Let $\tilde{\varphi}$ be another radial test function with the same support as φ , and $\tilde{\varphi}(r) = 1$ for $r_0 \leq r \leq \frac{3r_0}{2}$ and $\tilde{\varphi} = \varphi$ otherwise.

Proposition 10.12.

$$\mathbb{E} \left[\left| \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{Z}_n](z) \right| \right] \leq \mathbb{E} \left[\left| \int \frac{\tilde{\varphi}(z)}{|z|^l} d[\mathcal{Z}_n](z) \right| \right] \leq c(r_0, \tilde{\varphi}, l)$$

where $c(r_0, \tilde{\varphi}, l)$ does not depend on n , and the same result remains true when $\mathcal{Z}_n$ is replaced by $\mathcal{Z}$.

Proof. For fixed l , let $a(n) = \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{Z}_n](z)$ and $b(n) = \int \frac{\tilde{\varphi}(z)}{|z|^l} d[\mathcal{Z}_n](z)$. We have, $\mathbb{E}[|a(n)|] \leq \mathbb{E}[|b(n)|] = C \int \frac{\tilde{\varphi}(z)}{|z|^l} \Delta \log(K_n(z, z)) d\mathcal{L}(z)$ for some constant C . Using the uniform convergence of the continuous functions $\Delta \log(K_n(z, z)) \rightarrow \Delta \log(K(z, z)) < \infty$ on the (compact) support of $\tilde{\varphi}$, we deduce that $\mathbb{E}[b(n)]$ -s are bounded by constants that do not depend on n but depend on $r_0, \tilde{\varphi}$ and l . It is obvious from the above argument that the same holds true for $\mathcal{Z}$ instead of $\mathcal{Z}_n$. ■

Proposition 10.13. Let r_0 be the radius of $\mathcal{D}$. Let φ and $\tilde{\varphi}$ be defined as above.

(i) The random variables

$$S_l(n) := \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{Z}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} d[\mathcal{Z}_n](z) = \sum_{\omega \in \mathcal{Z}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \quad (\text{for } l \geq 1)$$

and

$$\tilde{S}_l(n) := \int \frac{\tilde{\varphi}(z)}{|z|^l} d[\mathcal{Z}_n](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l} d[\mathcal{Z}_n](z) = \sum_{\omega \in \mathcal{Z}_n \cap \mathcal{D}^c} \frac{1}{|\omega|^l} \quad (\text{for } l \geq 3)$$

have a finite first moment which, for every fixed l , is bounded above uniformly in n .

(ii) There exists $k_0 = k_0(\varphi) \geq 1$, not depending on n and l , such that for $k \geq k_0$ the “tails” of $S_l(n)$ and $\tilde{S}_l(n)$ beyond the disk $2^k \cdot \mathcal{D}$, given by

$$\tau_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} d[\mathcal{Z}_n](z) \quad (\text{for } l \geq 1) \quad \text{and} \quad \tilde{\tau}_l^n(2^k) := \sum_{j=k}^{\infty} \int \frac{\varphi_{2^j}(z)}{|z|^l} d[\mathcal{Z}_n](z) \quad (\text{for } l \geq 3)$$

satisfy the estimates

$$\mathbb{E} \left[\left| \tau_l^n(2^k) \right| \right] \leq C_1(\varphi, l)/2^{kl/2} \quad \text{and} \quad \mathbb{E} \left[\left| \tilde{\tau}_l^n(2^k) \right| \right] \leq C_2(\varphi, l)/2^{k(l-2)/2}.$$

All of the above remain true when $\mathcal{Z}_n$ is replaced by $\mathcal{Z}$, for which we use the notations S_l and $\tilde{S}_l$ to denote the quantities corresponding to $S_l(n)$ and $\tilde{S}_l(n)$.

Remark 10.3. For $\mathcal{Z}$, by the sum $\left(\sum_{\omega \in \mathcal{Z} \cap \mathcal{D}^c} \frac{1}{\omega^l} \right)$ we denote the quantity

$$S_l = \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{Z}](z) + \sum_{j=1}^{\infty} \int \frac{\varphi_{2^j}(z)}{z^l} d[\mathcal{Z}](z)$$

due to the obvious analogy with $\mathcal{Z}_n$, where the corresponding sum $S_l(n)$ is indeed equal to $\left(\sum_{\omega \in \mathcal{Z}_n \cap \mathcal{D}^c} \frac{1}{\omega^l} \right)$ with its usual meaning.

Proof of Proposition 10.13. The functions $\tilde{\varphi}$ and $\varphi_{2^j}, j \geq 1$ form a partition of unity on $\mathcal{D}^c$, hence the identities appearing in part (i). That the left hand side in part (i) has finite expectation can be seen from Proposition 10.11 and Proposition 10.12; it is uniformly bounded in n because so are $C(r_0, \varphi, l)$ in Proposition 10.11 and $c(r_0, \varphi_0)$ in Proposition 10.12.

Fix $n, l \geq 1$. Set $\psi_k = \int \frac{\varphi_{2^k}(z)}{z^l} d[\mathcal{Z}_n](z)$ for $k \geq 1$, and $\psi_0 = \int \frac{\tilde{\varphi}(z)}{z^l} d[\mathcal{Z}_n](z)$. When $l \geq 3$ we also define $\gamma_k = \int \frac{\varphi_{2^k}(z)}{|z|^l} d[\mathcal{Z}_n](z)$ for $k \geq 1$, and $\gamma_0 = \int \frac{\tilde{\varphi}(z)}{|z|^l} d[\mathcal{Z}_n](z)$. Let Ψ_k and Γ_k denote the analogous quantities defined with respect to $\mathcal{Z}$ instead of $\mathcal{Z}_n$.

For $k \geq 1$, we apply Proposition 10.11 part (i) to the function $\Phi = \varphi$ and $R = 2^k$ to obtain

$$\mathbb{E}[|\psi_k|] \leq C(r_0, \varphi, l)/(2^k)^l. \quad (29)$$

We also observe that $\mathbb{E}[|\psi_0|] \leq \mathbb{E}[|\gamma_0|]$ which, for fixed l , is uniformly bounded above in n , using Proposition 10.12.

The above arguments imply that for $l \geq 1$

$$\mathbb{E}[|S_l(n)|] \leq \sum_{k=0}^{\infty} \mathbb{E}[|\psi_k|] < \infty.$$

The results for $\tilde{S}_l(n)$ are similar, utilizing part (ii) of Proposition 10.11.

To obtain k_0 as in part (ii), we recall Remark 10.2 and find k_0 such that $p(l) \left(\frac{1}{r_0}\right)^l 2^{-kl/2} C(\varphi) \leq 1/2$ for all $k \geq k_0$. Such a k_0 can be obtained as follows: there exists $C_1(\varphi) > 0$ such that for all $l \geq 1$ we have $(2 \cdot C(\varphi) \cdot p(l))^{1/l} \leq C_1(\varphi)$. We now choose k_0 such that $\frac{2^{-k_0/2}}{r_0} \leq \frac{1}{C_1(\varphi)}$. This k_0 will clearly be independent of n , because so is $C(\varphi)$, and by choice it is independent of l . To estimate $\mathbb{E} [|\tau_l^n(2^k)|]$, we now simply sum (29) for $k \geq k_0$.

The result for $\tilde{\tau}_l^n(2^k)$ follows from a similar argument.

The same arguments yield the corresponding results when F_n is replaced by F . ■

Corollary 10.14. *For $R = 2^k, k \geq k_0$ as in Proposition 10.13. We have $\mathbb{P}[|\tau_l^n(R)| > R^{-l/4}] \leq R^{-l/4}$ and $\mathbb{P}[|\tilde{\tau}_l^n(R)| > R^{-(l-2)/4}] \leq R^{-(l-2)/4}$, and these estimates remain true when f_n is replaced with f .*

Proof. We use the estimates on the expectation on $|\tau_l^n(R)|$ and $|\tilde{\tau}_l^n(R)|$ from Proposition 10.13 and apply Markov's inequality. ■

With notations as above, we have

Proposition 10.15. *For each $l \geq 1$ we have $S_l(n) \rightarrow S_l$ in probability, and for each $l \geq 3$ we have $\tilde{S}_l(n) \rightarrow \tilde{S}_l$ in probability, and hence we have such convergence a.s. along some subsequence, simultaneously for all l .*

Proof. We argue on similar lines to the proof of Proposition 8.8. ■

Define $S_k(\mathcal{D}, n) = \sum_{z \in \mathcal{Z}_n} 1/z^k$ and $S_k(\mathcal{D}) = \sum_{z \in \mathcal{Z}} 1/z^k$. Set $\alpha_k(n) = S_k(\mathcal{D}, n) + S_k(n)$ and $\alpha_k = S_k(\mathcal{D}) + S_k$. Observe that $\alpha_k(n) = \sum_{z \in \mathcal{Z}_n} 1/z^k$. Then we have:

Proposition 10.16. *For each k , $\alpha_k(n) \rightarrow \alpha_k$ in probability as $n \rightarrow \infty$. Hence, there is a subsequence such that $\alpha_k(n) \rightarrow \alpha_k$ a.s. when $n \rightarrow \infty$ along this subsequence, simultaneously for all k .*

Proof. Since the finite point configurations given by $\mathcal{Z}_n|_{\mathcal{D}} \rightarrow \mathcal{Z}|_{\mathcal{D}}$ a.s. and a.s. there is no point at the origin, therefore $S_k(\mathcal{D}, n) \rightarrow S_k(\mathcal{D})$ a.s. This, combined with Proposition 10.15, gives us the desired result. ■

11 Limiting procedure for GAF zeroes

In this section, we will work in the framework of Section 7. More specifically, we will show that the conditions for Theorem 7.2 are satisfied, which will give us the desired conclusion. We will introduce the definitions and check all the conditions here except the fact $\Omega(j)$ exhausts Ω^m . This last criterion will be verified in the subsequent Proposition 11.1.

In terms of the notation used in Section 7, we have $X^n = \mathcal{Z}_n$ and $X = \mathcal{Z}$.

Proof of Theorem 1.4 for a disk. We will invoke Theorem 7.2. We will define the relevant quantities (as in the statement of Theorem 7.2) and verify that they satisfy the required conditions for that theorem to apply.

Following the notation in Theorem 7.2, we begin with Ω^m , $m \geq 0$. If $m = 0$, there is nothing to be proved, and if $m = 1$ then from Theorem 1.3 the position of the zero inside $\mathcal{D}$ is known, and therefore the result is trivial. So we focus on the case $m \geq 2$.

Our candidate for $\nu(X_{\text{out}}(\xi), \cdot)$ (refer to Theorem 7.2) is the probability measure $\Theta^{-1}|\Delta(\underline{\zeta})|^2 d\mathcal{L}_\Sigma(\underline{\zeta})$ on $\Sigma_{S(X_{\text{out}}(\xi))}$, where $\Delta(\underline{\zeta})$ is the Vandermonde determinant formed by the coordinates of $\underline{\zeta}$, $\mathcal{L}_\Sigma$ is the Lebesgue measure on $\Sigma_{S(X_{\text{out}}(\xi))}$ and Θ is the normalizing factor. Note that this form of the candidate measure, heuristically, is in tune with the fact the conditional measure is supported on $\Sigma_{S(X_{\text{out}}(\xi))}$ and with the estimate in Proposition 10.10 on conditional densities. It is also a mathematical expression of the notion that ω determines only the number and the sum of the points in $\mathcal{D}$, and “nothing more”. Here we recall the definition of $S(X_{\text{out}}(\xi))$ from Theorem 1.3 and the definition of $\Sigma_{S(X_{\text{out}}(\xi))}$ from Section 10.1.2. Note that as soon as we define $\nu(X_{\text{out}}, \cdot)$ which maps ξ to $\mathcal{M}(\mathcal{D}^m)$, this automatically induces a map from $\mathcal{S}_{\text{out}}$ to $\mathcal{M}(\mathcal{D}^m)$ which satisfies the required measurability properties.

To find the sequence n_k (which will be the same for every j in our case), we proceed as follows. Let $N_g(K)$ denote the number of zeroes of the function g in a set K , Γ denote the closed annulus of width 1 around $\mathcal{D}$, and in the next statement let Z be any of the variables L, M or N as in Proposition 10.7 (and the immediately preceding discussion) with $p = 0$ or $2 \leq p \leq m$ and $2 \leq q \leq m$. We have the probabilities of each of the following events converging to 0 as $n \rightarrow \infty$: $\{|S_j(n) - S_j| > 1\}_{j=1,2}$, $\{|\tilde{S}_3(n) - \tilde{S}_3| > 1\}$, $\{|N_f(\overline{\mathcal{D}}) \neq N_{f_n}(\overline{\mathcal{D}})|\}$, $\{|N_f(\Gamma) \neq N_{f_n}(\Gamma)|\}$, $\{|\tau(Z_{pq}^{(n)})| > 1\}$, $\{\frac{1}{n} \left(\sum_{j=0}^n |\xi_j|^2 \right) \leq 1/2\}$ and $\{||\Delta(\underline{\zeta}(n))|^2 - |\Delta(\underline{\zeta})|^2| > \frac{1}{M_j}\}$ (where $\underline{\zeta}(n)$ and $\underline{\zeta}$ correspond to the zeroes inside $\mathcal{D}$ of f_n and f respectively). Call the union of these events $\mathfrak{B}_n$. For a given $k \geq 1$, let n'_k be such that $\mathbb{P}(\mathfrak{B}_n) < 2^{-k}$ for all $n \geq n'_k$. From Proposition 10.15 we have a subsequence such that $S_j(n) \rightarrow S_j$ ($j = 1, 2$) and $\tilde{S}_3(n) \rightarrow \tilde{S}_3$ a.s. as $n \rightarrow \infty$ along that subsequence. For a given $k \geq 1$, we define n_k to be the least integer in that subsequence which is $\geq n'_k$.

Fix a sequence of positive real numbers $M_j \uparrow \infty$, such that $M_1 = \min_j M_j > \sqrt{2}$.

On the event $\Omega_{n_k}^m$ (which entails that f_{n_k} has m zeroes inside $\mathcal{D}$), let $\underline{\zeta}(n_k)$ and $\underline{\omega}(n_k)$ denote the vector of inside and outside zeroes of f_{n_k} (taken in uniform random order) respectively. Let $s(n_k)$ denote the sum of the inside zeroes: $s(n_k) := \sum_{j=1}^m \underline{\zeta}(n_k)_j$, where $\underline{\zeta}(n_k)_j$ are the coordinates of the vector $\underline{\zeta}(n_k)$. By $\Sigma_{s(n_k)}$ we will denote the (random) set $\{\underline{\zeta}' \in \mathcal{D}^m : \sum_{j=1}^m \underline{\zeta}'_j = s(n_k)\}$. Also recall the notation $\rho_{\underline{\omega}, s}^{n_k}(\underline{\zeta})$ to be the conditional density (with respect to the Lebesgue measure on Σ_s) of the inside zeroes (at $\underline{\zeta} \in \mathcal{D}^m$) given the vector of outside zeroes to be $\underline{\omega}$ and the sum of the inside zeroes to be s .

We define our event $\Omega_{n_k}(j)$ by the following conditions on the zero set $(\underline{\zeta}(n_k), \underline{\omega}(n_k))$ of f_{n_k} :

1. There are exactly m zeroes of f_{n_k} in $\mathcal{D}$
2. There are no zeroes of f_{n_k} in the closed annulus of width $1/M_j$ around $\mathcal{D}$.
3. $|S_1(n_k)| \leq M_j, |S_2(n_k)| \leq M_j, |\tilde{S}_3(n_k)| \leq M_j$.
4. There exists $\underline{\zeta}' \in \Sigma_{s(n_k)}$ such that $(\underline{\zeta}', \underline{\omega}(n_k))$ satisfies ($\underline{\omega}$ here is an abbreviation for $\underline{\omega}(n_k)$):

$$\begin{aligned}
\text{(a)} \quad & \left| \sum_{r=0}^{n_k} \frac{\overline{\sigma_r(\underline{\zeta}', \underline{\omega})}}{\sqrt{\binom{n}{r}} r!} \frac{\sigma_{r-i}(\underline{\omega})}{\sqrt{\binom{n}{r}} r!} \right| / \left(\sum_{r=0}^{n_k} \left| \frac{\sigma_r(\underline{\zeta}', \underline{\omega})}{\sqrt{\binom{n}{r}} r!} \right|^2 \right) \leq M_j/n_k \text{ for each } 2 \leq i \leq m. \\
\text{(b)} \quad & \left| \sum_{r=0}^{n_k} \frac{\overline{\sigma_{r-i}(\underline{\omega})}}{\sqrt{\binom{n}{r}} r!} \frac{\sigma_{r-j}(\underline{\omega})}{\sqrt{\binom{n}{r}} r!} \right| / \left(\sum_{r=0}^{n_k} \left| \frac{\sigma_r(\underline{\zeta}', \underline{\omega})}{\sqrt{\binom{n}{r}} r!} \right|^2 \right) \leq M_j/n_k \text{ for all } 2 \leq i, j \leq m. \\
\text{(c)} \quad & \frac{1}{M_j} \leq |\Delta(\underline{\zeta}')|^2 \leq M_j
\end{aligned}$$

Clearly, $\Omega_{n_k}(j)$ depends only on the quantities $\underline{\omega}(n_k)$ and $s(n_k)$ (and on no other information from $\underline{\zeta}(n_k)$). In particular, for a vector $\underline{\omega}$ of outside zeroes of f_{n_k} , if there exists $\underline{\zeta} \in \Sigma_s$ such that $(\underline{\zeta}, \underline{\omega})$ satisfies the conditions required in the definition of $\Omega_{n_k}(j)$, then all $(\underline{\zeta}', \underline{\omega})$ (with $\underline{\zeta}' \in \Sigma_s$) satisfies these conditions. From (10), Proposition 10.1, (14), (15) and the discussion therein, it follows that on the event $\Omega_{n_k}(j)$ we have

$$b(j)|\Delta(\underline{\zeta})|^2 \leq \rho_{\underline{\omega}, s}^{n_k}(\underline{\zeta}) \leq B(j)|\Delta(\underline{\zeta})|^2 \quad (30)$$

for positive quantities $B(j)$ and $b(j)$ (that depend on M_j). More explicitly, if $(\underline{\zeta}, \underline{\omega}) \in \Omega_{n_k}(j)$ such that $(\underline{\zeta}', \underline{\omega})$ is as in condition 4 (in the definition of $\Omega_{n_k}(j)$), then from (10), we have

$$\rho_{\underline{\omega}, s}^{n_k}(\underline{\zeta}) = \rho_{\underline{\omega}, s}^{n_k}(\underline{\zeta}') \frac{|\Delta(\underline{\zeta}, \underline{\omega})|^2}{|\Delta(\underline{\zeta}', \underline{\omega})|^2} \left(\frac{D(\underline{\zeta}, \underline{\omega})}{D(\underline{\zeta}', \underline{\omega})} \right)^{n+1}.$$

But from Proposition 10.1 (with $\delta = 1/M_j$), we have

$$\left(-2mK(\mathcal{D}, \delta) \mathbf{X}_n(\underline{\omega}) \right) \frac{|\Delta(\underline{\zeta})|^2}{|\Delta(\underline{\zeta}')|^2} \leq \frac{|\Delta(\underline{\zeta}, \underline{\omega})|^2}{|\Delta(\underline{\zeta}', \underline{\omega})|^2} \leq \left(2mK(\mathcal{D}, \delta) \mathbf{X}_n(\underline{\omega}) \right) \frac{|\Delta(\underline{\zeta})|^2}{|\Delta(\underline{\zeta}')|^2}.$$

where $X_n(\underline{\omega}) = |S_1(n_k)| + |S_2(n_k)| + |\tilde{S}_3(n_k)|$. Using conditions 3 and 4(c) in the definition of $\Omega_{n_k}(j)$ (and recalling that m, D is fixed for our purposes) we deduce that

$$C_1(j)|\Delta(\underline{\zeta})|^2 \leq \frac{|\Delta(\underline{\zeta}, \underline{\omega})|^2}{|\Delta(\underline{\zeta}', \underline{\omega})|^2} \leq C_2(j)|\Delta(\underline{\zeta})|^2$$

for positive numbers $C_1(j)$ and $C_2(j)$. In order to bound $\left(\frac{D(\underline{\zeta}, \underline{\omega})}{D(\underline{\zeta}', \underline{\omega})} \right)^{n+1}$, we refer to (13), (14), (15) and the discussion therein. From the assumptions 4(a) and 4(b) in the definition of $\Omega_{n_k}(j)$, it follows that

$$C_3(j) \leq \left(\frac{D(\underline{\zeta}, \underline{\omega})}{D(\underline{\zeta}', \underline{\omega})} \right)^{n+1} \leq C_4(j)$$

for positive quantities $C_3(j)$ and $C_4(j)$. Combining the inequalities in the two displays above, we get (30).

To obtain condition (C3), we introduce some further notations. On the event $\Omega_{n_k}^m$, let $\gamma_{n_k}(\underline{\omega}; s)$ denote the conditional probability measure on the sum s of inside zeroes given the vector of outside zeroes of f_{n_k} to be $\underline{\omega}$. Further, let $\mathcal{L}_s$ denote the Lebesgue measure on the set Σ_s .

For any $A \in \mathfrak{A}^m$ and $B \in \mathcal{B}$, we can write

$$\mathbb{P}[(X_{\text{in}} \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] = \int_{B \cap \Omega_{n_k}(j)} \left(\int_{A \cap \Sigma_s} \rho_{\underline{\omega}, s}^{n_k}(\underline{\zeta}) d\mathcal{L}_s(\underline{\zeta}) \right) d\gamma_{n_k}(\underline{\omega}; s) d\mathbb{P}_{X_{\text{out}}^{n_k}}(\underline{\omega}) \quad (31)$$

Setting $h(s, A) := \left(\int_{A \cap \Sigma_s} |\Delta(\underline{\zeta})|^2 d\mathcal{L}_s(\underline{\zeta}) \right) / \left(\int_{\Sigma_s} |\Delta(\underline{\zeta})|^2 d\mathcal{L}_s(\underline{\zeta}) \right)$, we get from (30) that the right hand side of (31) is

$$\int_{B \cap \Omega_{n_k}(j)} \left(\int_{A \cap \Sigma_s} \rho_{\underline{\omega}, s}^{n_k}(\underline{\zeta}) d\mathcal{L}_s(\underline{\zeta}) \right) d\gamma_{n_k}(\underline{\omega}; s) d\mathbb{P}_{X_{\text{out}}^{n_k}}(\underline{\omega}) \asymp_j \int_{B \cap \Omega_{n_k}(j)} h(s, A) d\gamma_{n_k}(\underline{\omega}; s) d\mathbb{P}_{X_{\text{out}}^{n_k}}(\underline{\omega}). \quad (32)$$

The integral on the right hand side can be written as $\mathbb{E} \left[h(s(n_k), A) \mathbb{1}[X_{\text{out}}^{n_k} \in B] \mathbb{1}[\Omega_{n_k}(j)] \right]$, where $\mathbb{1}[E]$ is the indicator function of the event E and the expectation is with respect to the random variable $(s(n_k), X_{\text{out}}^{n_k})$. Note that for fixed A , the function $h(s, A)$ is continuous in s , because $A \in \mathfrak{A}^m$, and that $0 \leq h(s, A) \leq 1$. Since, as $k \rightarrow \infty$, we have $s(n_k) \rightarrow S(X_{\text{out}})$ a.s. Using the Dominated Convergence Theorem we get

$$\mathbb{E} [h(s(n_k), A) \mathbb{1}[X_{\text{out}}^{n_k} \in B] \mathbb{1}[\Omega_{n_k}(j)]] = \mathbb{E} [h(S(X_{\text{out}}), A) \mathbb{1}[X_{\text{out}}^{n_k} \in B] \mathbb{1}[\Omega_{n_k}(j)]] + o_k(1), \quad (33)$$

where $o_k(1)$ is a quantity which $\rightarrow 0$ as $k \rightarrow \infty$, for fixed A and B (actually, in this particular case, it can be easily seen that the convergence is uniform in $B \in \mathcal{B}$). Also, as $k \rightarrow \infty$, we have $\mathbb{1}[X_{\text{out}}^{n_k} \in B] \rightarrow \mathbb{1}[X_{\text{out}} \in B]$ since B is a compact set. Since $0 \leq h(s, A) \leq 1$ a.s., this implies that

$$\mathbb{E} [h(S(X_{\text{out}}), A) \mathbb{1}[X_{\text{out}}^{n_k} \in B] \mathbb{1}[\Omega_{n_k}(j)]] = \mathbb{E} [h(S(X_{\text{out}}), A) \mathbb{1}[X_{\text{out}} \in B] \mathbb{1}[\Omega_{n_k}(j)]] + o_k(1). \quad (34)$$

Observe that for $\xi \in \Xi$, we have $h(S(X_{\text{out}}(\xi)), A) = \nu(X_{\text{out}}(\xi), A)$. Therefore, putting (31)-(34) together, we obtain (C3). The only condition in Theorem 7.2 that remains to be verified is the fact that $\Omega(j)$ -s exhaust Ω^m , which will be taken up in Proposition 11.1.

By Theorem 7.2, this proves that a.s. we have $\rho(X_{\text{out}}, \cdot) \equiv \nu(X_{\text{out}}, \cdot)$. Since a.s. $\nu(X_{\text{out}}, \cdot) \equiv \mathcal{L}_\Sigma$, we have $\rho(X_{\text{out}}, \cdot) \equiv \mathcal{L}_\Sigma$, as desired. $\blacksquare$

We end this section with a proof that $\Omega(j)$ -s exhaust Ω^m :

Proposition 11.1. *With definitions as above, $\Omega(j) := \varliminf_{k \rightarrow \infty} \Omega_{n_k}(j)$ exhausts Ω^m as $j \rightarrow \infty$.*

Proof. We begin by showing that $\Omega(j) \subset \Omega^m$ for each j . Due to the convergence of $X^{n_k} \rightarrow X$ on compact sets, we have $\varliminf_{k \rightarrow \infty} \Omega_{n_k}^m = \Omega^m$. Since $\Omega_{n_k}(j) \subset \Omega_{n_k}^m$ for each k , therefore $\Omega(j) \subset \Omega^m$ for each j . Since $M_j < M_{j+1}$, it is also clear that $\Omega_{n_k}(j) \subset \Omega_{n_k}(j+1)$ for each k . Hence $\Omega(j) \subset \Omega(j+1)$.

To show that $\mathbb{P}(\Omega^m \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$, we will define for each j three events $\Omega^1(j)$, $\Omega^2(j)$ and $\Omega^3(j)$; the last event being the one which allows us to obtain the desired conclusion. The main reason for introduction of several events is as follows. The events $\Omega(j)$ are defined in terms of the polynomials, whereas the right setting in which the exhaustion property can be understood easily is that of the GAF. Thus, the proof of the exhaustion property involves a transition from events defined in terms of the polynomials to events defined in terms of the GAF. The sequence of events $\Omega^1(j)$, $\Omega^2(j)$ and $\Omega^3(j)$ accomplishes this in steps, such that the correct relationships between the events (in terms of containment or equality except on a set of small probability) is transparent from the successive definitions.

For each j we first **define the event** $\Omega_{n_k}^1(j)$ by demanding that the zeroes $(\underline{\zeta}, \underline{\omega})$ of f_{n_k} satisfy the following conditions :

1. There are exactly m zeroes of f_{n_k} in $\mathcal{D}$
2. There are no zeroes of f_{n_k} in the closed annulus of width $1/M_j$ around $\mathcal{D}$.
3. $|S_1(n_k)| \leq M_j, |S_2(n_k)| \leq M_j, |\tilde{S}_3(n_k)| \leq M_j$.
4. $\left| \sum_{t=0}^{n_k} \frac{\overline{\sigma_t(\underline{\zeta}, \underline{\omega})}}{\sqrt{\binom{n_k}{t}} t!} \frac{\sigma_{t-i}(\underline{\omega})}{\sqrt{\binom{n_k}{t}} t!} \right| / \left(\sum_{t=0}^{n_k} \left| \frac{\sigma_t(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n_k}{t}} t!} \right|^2 \right) \leq M_j/n_k$ for each $2 \leq i \leq m$.
5. $\left| \sum_{t=0}^{n_k} \frac{\overline{\sigma_{t-i}(\underline{\omega})}}{\sqrt{\binom{n_k}{t}} t!} \frac{\sigma_{t-j}(\underline{\omega})}{\sqrt{\binom{n_k}{t}} t!} \right| / \left(\sum_{k=0}^{n_k} \left| \frac{\sigma_t(\underline{\zeta}, \underline{\omega})}{\sqrt{\binom{n_k}{t}} t!} \right|^2 \right) \leq M_j/n_k$ for all $2 \leq i, j \leq m$.
6. $\frac{1}{M_j} \leq |\Delta(\underline{\zeta})|^2 \leq M_j$.

Clearly, $\Omega_{n_k}^1(j) \subset \Omega_{n_k}(j)$ and hence $\lim_{k \rightarrow \infty} \Omega_{n_k}^1(j) = \Omega^1(j) \subset \Omega(j)$.

Next, we **define the event** $\Omega_{n_k}^2(j)$ by the following conditions (refer to Proposition 10.7 and the discussion immediately preceding it to recall the notations):

1. There are exactly m zeroes of f_{n_k} in $\mathcal{D}$
2. There are no zeroes of f_{n_k} in the closed annulus of width $1/M_j$ around $\mathcal{D}$.
3. $|S_1(n_k)| \leq M_j, |S_2(n_k)| \leq M_j, |\tilde{S}_3(n_k)| \leq M_j$.
4. Each of $|L_{0q}^{(n_k)}|, |M_{0q}^{(n_k)}|, |L_{pq}^{(n_k)}|, |M_{pq}^{(n_k)}|, |N_{pq}^{(n_k)}|$ is $\leq M_j/8$ for all $2 \leq p, q \leq m$.
5. $E_{n_k}/n_k \geq 1/2$.
6. $\frac{1}{M_j} \leq |\Delta(\underline{\zeta}(n_k))|^2 \leq M_j$, where $\underline{\zeta}(n_k)$ corresponds to the zeroes of f_{n_k} inside $\mathcal{D}$.

It is clear from (25) that $\Omega_{n_k}^2(j) \subset \Omega_{n_k}^1(j)$.

Finally, we define an event $\Omega^3(j)$ by the conditions:

1. There are exactly m zeroes of f in $\mathcal{D}$
2. There are no zeroes of f in the closed annulus of width $1/M_j$ around $\mathcal{D}$.
3. $|S_1| \leq M_j, |S_2| \leq M_j - 1, |\tilde{S}_3| \leq M_j - 1$.
4. Each of $|L_{0q}|, |M_{0q}|, |L_{pq}|, |M_{pq}|, |N_{pq}|$ is $\leq \frac{M_j}{8} - 1$ for all $2 \leq p, q \leq m$.
5. $\frac{2}{M_j} \leq |\Delta(\underline{\zeta})|^2 \leq \frac{M_j}{2}$, where $\underline{\zeta}$ corresponds to the zeroes of the GAF f inside $\mathcal{D}$.

Notice (refer to Proposition 10.8) that for each of the random variables $Z_{pq}^{(n_k)}$ in condition 4 defining $\Omega_{n_k}^2(j)$ and Z_{pq} in condition 4 defining $\Omega^3(j)$ (where $Z = L, M, N$) we have

$$|Z_{pq}^{(n_k)}| \leq |Z_{pq}| + |\tau(Z_{pq}^{(n_k)})|. \quad (35)$$

Now consider the event F_{n_k} where any one of the following conditions hold:

1. There are m zeroes of f in $\mathcal{D}$, but the same assertion does not hold for f_{n_k} .
2. There are no zeroes of f in the closed annulus of width $1/M_j$ around $\mathcal{D}$, but this is not true for f_{n_k} .
3. $|S_1 - S_1(n_k)| > 1$ or $|S_2 - S_2(n_k)| > 1$ or $|\tilde{S}_3 - \tilde{S}_3(n_k)| > 1$.
4. $|\tau(Z_{pq}^{(n_k)})| > 1$ for any of the random variables appearing in condition 4 of $\Omega_{n_k}^2(j)$.
5. $E_{n_k}/n_k < 1/2$.
6. $||\Delta(\underline{\zeta}(n_k))|^2 - |\Delta(\underline{\zeta})|^2| > \frac{1}{M_j}$, where $\underline{\zeta}(n_k)$ corresponds to the zeroes of f_{n_k} inside $\mathcal{D}$ and $\underline{\zeta}$ corresponds to the zeroes of f inside $\mathcal{D}$.

It is easy to see (using (35)) that $\Omega^3(j) \setminus F_{n_k} \subset \Omega_{n_k}^2(j) \subset \Omega_{n_k}(j)$ (recall that by choice of M_j -s, for each j we have $M_j > \sqrt{2}$, implying $\frac{M_j}{2} + \frac{1}{M_j} \leq M_j$). However, by our choice of the sequence n_k , we have $\mathbb{P}(F_{n_k}) < 2^{-k}$, hence by the Borel Cantelli lemma, $\varliminf_{k \rightarrow \infty} (\Omega^3(j) \setminus F_{n_k}) = \Omega^3(j)$. Hence, $\Omega^3(j) \subset \Omega(j) = \varliminf_{k \rightarrow \infty} \Omega_{n_k}(j)$.

However, each random variable that appears in the definition of $\Omega^3(j)$ does not put any mass at 0 or ∞ , and the width of the annulus around $\mathcal{D}$ in condition 2 in its definition goes to 0 as $j \rightarrow \infty$. Hence, the probability that each of the conditions 2-5 defining $\Omega^3(j)$ holds goes to 1 as $j \rightarrow \infty$. Finally, condition 1 is just the definition of Ω^m . Hence, as $j \rightarrow \infty$, we have $\mathbb{P}(\Omega^m \setminus \Omega^3(j)) \rightarrow 0$. But $\Omega^3(j) \subset \Omega(j)$, hence $\mathbb{P}(\Omega^m \setminus \Omega(j)) \rightarrow 0$ as $j \rightarrow \infty$, as desired. ■

12 Reconstruction of GAF from Zeroes and an Analog of Vieta's formula

In this section we prove the reconstruction Theorem 1.5 for the planar GAF using the estimates we obtained in Section 10.2. En route, we establish an analogue of Vieta's formula for the planar GAF.

12.1 Vieta's formula for the planar GAF

It is an elementary fact that for a polynomial

$$p(z) = \sum_{j=0}^N b_j z^j$$

whose roots are $\{z_j\}_{j=1}^N$, we have for any $1 \leq k \leq N$,

$$(-1)^k b_{N-k}/b_N = \sum_{i_1 < i_2 < \dots < i_k} z_{i_1} \cdots z_{i_k}. \quad (36)$$

When $b_0 \neq 0$, we equivalently have

$$(-1)^k b_k / b_0 = \sum_{i_1 < i_2 < \dots < i_k} \frac{1}{z_{i_1} \dots z_{i_k}}. \quad (37)$$

This kind of result is referred to as Vieta's formula. For entire functions, it is not clear in general how to relate the coefficients to the zeroes of the function; for example it may not be possible to provide any reasonable interpretation to the function of the zeroes appearing on the right hand side of (37).

The quantity on the right hand side of (36) is the elementary symmetric function of order k in the variables $z_1, \dots, z_N$, denoted by $e_k(z_1, \dots, z_N)$. If we introduce the power sum $\beta_k = \sum_{j=1}^N z_j^k$, then it is known from Newton's identities that for each k we have

$$e_k(z_1, \dots, z_N) = P_k(\beta_1, \dots, \beta_k)$$

where P_k is a homogeneous symmetric polynomial of degree k in the variables $(\beta_1, \dots, \beta_k)$. The polynomial P_k is called the Newton polynomial of degree k , and it is known that the coefficients of P_k depend only on k and do not depend on n (refer [Sta99], chapter 7).

Recall from Section 1 the notation $\alpha_k(n) = (\sum_{z \in \mathcal{Z}_n} 1/z^k)$ and Proposition 10.16 shows that for each k the random variables $\alpha_k(n)$ converge in probability as $n \rightarrow \infty$. The limit α_k is an analogue of the sum $(\sum_{z \in \mathcal{Z}} 1/z^k)$; in fact for $k \geq 3$ this inverse power sum converges absolutely, and coincides with α_k .

Proposition 12.1. *For the planar GAF zero process, we have, a.s. $\frac{1}{\sqrt{k!}} \frac{\xi_k}{\xi_0} = (-1)^k P_k(\alpha_1, \alpha_2, \dots, \alpha_k)$ for each $k \geq 1$.*

Proof. We begin by observing that under the natural coupling of the planar GAF f and its approximating polynomials f_n , we have by the Vieta's formula for each f_n :

$$\frac{1}{\sqrt{k!}} \frac{\xi_k}{\xi_0} = (-1)^k P_k(\alpha_1(n), \dots, \alpha_k(n)) \quad (38)$$

where P_k is, as before, the Newton polynomial of degree k . Clearly, P_k is continuous in the input variables. We also know, from Proposition 10.15 that, as $n \rightarrow \infty$ (possibly along some appropriately chosen subsequence), $\alpha_k(n) \rightarrow \alpha_k$ a.s., simultaneously for all $k \geq 1$. Taking this limit in (38) and using the continuity of P_k , we get a.s.

$$\frac{1}{\sqrt{k!}} \frac{\xi_k}{\xi_0} = (-1)^k P_k(\alpha_1, \dots, \alpha_k). \quad (39)$$

■

12.2 Proof of Theorem 1.5

Since ξ_0 is a complex Gaussian, $|\xi_0| \neq 0$ a.s. For $|\xi_0| \neq 0$, we can write

$$f(z) = \frac{\xi_0}{|\xi_0|} \cdot |\xi_0| \left(1 + \frac{\xi_1}{\xi_0} \frac{z}{\sqrt{1!}} + \frac{\xi_2}{\xi_0} \frac{z^2}{\sqrt{2!}} + \dots + \frac{\xi_k}{\xi_0} \frac{z^k}{\sqrt{k!}} + \dots \infty \right). \quad (40)$$

Recall that, for two random variables U and V defined on the same probability space, we say that U is measurable with respect to V if U is measurable with respect to the sigma-algebra generated by V . From Proposition 12.1 we have that for each k the random variable $\frac{\xi_k}{\xi_0}$ is measurable with respect to $\mathcal{Z}$. From the strong law of large numbers, a.s. we have

$$\frac{|\xi_0|^2 + \dots + |\xi_{k-1}|^2}{k} \rightarrow 1.$$

Therefore,

$$|\xi_0| = \lim_{k \rightarrow \infty} k^{1/2} \left(\sum_{j=0}^{k-1} \frac{|\xi_j|^2}{|\xi_0|^2} \right)^{-1/2} = \chi,$$

where we recall the definition of χ from Section 1. Since $k^{1/2} \left(\sum_{j=0}^{k-1} \frac{|\xi_j|^2}{|\xi_0|^2} \right)^{-1/2}$ is measurable with respect to $\mathcal{Z}$ for each k , therefore $\chi = |\xi_0|$ is also measurable with respect to $\mathcal{Z}$.

Let us define the random variable $\zeta = \xi_0/|\xi_0|$ (it is set to be equal to 0 when $|\xi_0| = 0$). The random function g as in the statement of the theorem clearly satisfies a.s.

$$g(z) = |\xi_0| \left(1 + \frac{\xi_1}{\xi_0} \frac{z}{\sqrt{1!}} + \frac{\xi_2}{\xi_0} \frac{z^2}{\sqrt{2!}} + \dots + \frac{\xi_k}{\xi_0} \frac{z^k}{\sqrt{k!}} + \dots \right).$$

Then (40) can be re-written as

$$f(z) = \zeta g(z) \tag{41}$$

almost surely, and g is measurable with respect to $\mathcal{Z}$ and ζ is distributed uniformly on $\mathbb{S}^1$.

Therefore, all that remains to complete the proof is to show that ζ and $\mathcal{Z}$ are independent. For this, note that for $\theta \in \mathbb{S}^1$ the random function

$$|\xi_0| + \sum_{j \geq 1} \frac{\theta \xi_j}{\sqrt{j!}} z^j$$

has the same distribution (irrespective of θ). This is because since the ξ_i -s are independent complex Gaussians with mean 0, the vectors $(|\xi_0|, \theta \xi_j)_{j \geq 1}$ and $(|\xi_0|, \xi_j)_{j \geq 1}$ have the same distribution for each fixed $\theta \in \mathbb{S}^1$. Now note that

$$g(z) = |\xi_0| + \sum_{j \geq 1} \frac{\bar{\zeta} \xi_j}{\sqrt{j!}} z^j.$$

Therefore, the distribution of $g(z)$ given ζ does not depend on the value of ζ . Hence, the random function g and ζ are independent. But, g and f have the same zero set a.s. and consequently ζ and $\mathcal{Z}$ are independent random variables. This completes the proof.

13 Proof of Theorem 7.2

In this section, we will complete the proof of Theorem 7.2. Before the main proof, we will state two useful propositions, whose proofs will be deferred to the end of this section. First, we look at the heuristics which motivate some of our main steps. In the statement of Theorem 7.2, think of the

point processes X^n to be approximations of the limiting point process X , e.g., matrix eigenvalues approximating the infinite Ginibre ensemble, or the polynomial zeroes approximating the zeroes of the GAF. Often, the X^n -s have finitely many points and have joint probability densities for their points, which makes it possible to estimate probabilities of events defined with respect to such processes, like in the case of the Ginibre or the Gaussian zero ensembles. On the other hand, our goal is to draw conclusions about conditional probabilities defined with respect to the presumably infinite point process X . The task at hand is to make a transition from the former to the latter. With this in mind, if we look at the setting in which Theorem 7.2 is stated, then the hypotheses involve probabilities of events defined with respect to the approximations X^n ; whereas the conclusion of the theorem involves conditional probabilities for the limiting process X . Thus, broadly speaking, we need to make two kinds of transitions: one from probabilities without conditioning to conditional probabilities, and the other from X^n to X . The main goal of these somewhat technical propositions is to make a connection between statements about probabilities of events (defined with respect to X^n), as in (C3), with conditional probabilities for X , as in (C4).

Proposition 13.1 makes a transition between the inequalities connecting probabilities of the kind that appear on the right hand side of (C3) to domination relations between conditional expectations. Thus, Proposition 13.1 connects quantities as on the right hand side of (C3) with conditional measures, which appear in the conclusion (C4) of Theorem 7.2.

For stating Propositions 13.1 and 13.2, we assume the hypotheses of Theorem 7.2, namely, conditions (a) and (b). Recall the notation that the Borel sigma-algebra on $\mathcal{D}^m$ is denoted by $\mathfrak{B}(\mathcal{D}^m)$.

Proposition 13.1. *Let $\mu_j, j = 1, 2$ be two mappings with $\mu_j : \Xi \times \mathfrak{B}(\mathcal{D}^m) \rightarrow [0, 1]$, such that for each $\xi \in \Xi$ we have $\mu_j(\xi, \cdot) \in \mathcal{M}(\mathcal{D}^m)$, and for each Borel set $A \subset \mathcal{D}^m$, the function $\xi \rightarrow \mu_j(\xi, A)$ is measurable. Suppose μ_1 and μ_2 satisfy, for each positive integer j_0 , $A \in \mathfrak{A}^m$ and Borel set $B \subset \mathcal{S}_{\text{out}}$*

$$\int_{X_{\text{out}}^{-1}(B) \cap \Omega(j_0)} \mu_1(\xi, A) d\mathbb{P}(\xi) \leq c(j_0) \cdot \int_{X_{\text{out}}^{-1}(B) \cap \Omega^m} \mu_2(\xi, A) d\mathbb{P}(\xi) \quad (42)$$

for some positive number $c(j_0)$. Then a.s. on the event Ω^m we have

$$\mathbb{E} [\mu_1(\xi, \cdot) | X_{\text{out}}] \ll \mathbb{E} [\mu_2(\xi, \cdot) | X_{\text{out}}]. \quad (43)$$

Remark 13.1. *Here the conditional expectation is with respect to the sigma-algebra generated by the random variable X_{out} .*

Proposition 13.2 connects probabilistic expressions of the type appearing on the left hand side of (C3), involving X^n , with probabilities of similar events involving X . Thus, Proposition 13.2 broadly effects a transition from the X^n -s to X , in the setting of Theorem 7.2.

Proposition 13.2. *Suppose we have measurable functions h_1 and h_2 mapping $\Xi \rightarrow [0, 1]$, and measurable sets $U_1, U_2 \in \mathfrak{A}^m$. Define $\mathbb{P}_{h_i}$ to be the finite non-negative measure on Ξ given by $d\mathbb{P}_{h_i}(\xi) = h_i(\xi) d\mathbb{P}(\xi)$, $i = 1, 2$. Suppose there is a positive number $c(j_0)$ such that the following inequality holds for all $\tilde{V} \in \mathcal{B}$:*

$$\mathbb{P}_{h_1} \left[(X_{\text{in}}^{n_k} \in U_1) \cap (X_{\text{out}}^{n_k} \in \tilde{V}) \cap \Omega_{n_k}(j_0) \right] \leq c(j_0) \cdot \mathbb{P}_{h_2} \left[(X_{\text{in}}^{n_k} \in U_2) \cap (X_{\text{out}}^{n_k} \in \tilde{V}) \cap \Omega_{n_k}(j_0) \right] + \vartheta(k) \quad (44)$$

where $\vartheta(k) = \vartheta(k; j_0, U_1, U_2, \tilde{V}) \rightarrow 0$ as $k \rightarrow \infty$ for fixed $j_0, U_1, U_2, \tilde{V}$. Then for all Borel sets V in $\mathcal{S}_{\text{out}}$, we have

$$\mathbb{P}_{h_1} \left[(X_{\text{in}} \in U_1) \cap (X_{\text{out}} \in V) \cap \Omega(j_0) \right] \leq c(j_0) \cdot \mathbb{P}_{h_2} \left[(X_{\text{in}} \in U_2) \cap (X_{\text{out}} \in V) \right]. \quad (45)$$

We are now ready to complete the proof of Theorem 7.2.

Proof of Theorem 7.2. We claim that it suffices to show that for every j_0 , there are positive real numbers $M(j_0)$ and $m(j_0)$ such that for any $A \in \mathfrak{A}^m$ and any Borel set B in $\mathcal{S}_{\text{out}}$

$$\mathbb{P}((X_{\text{in}} \in A) \cap (X_{\text{out}} \in B) \cap \Omega(j_0)) \leq M(j_0) \int_{X_{\text{out}}^{-1}(B) \cap \Omega^m} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi) \quad (46)$$

and

$$m(j_0) \int_{X_{\text{out}}^{-1}(B) \cap \Omega(j_0)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi) \leq \mathbb{P}((X_{\text{in}} \in A) \cap (X_{\text{out}} \in B)). \quad (47)$$

Once we have (46), we can invoke Proposition 13.1. Setting $\mu_1(\xi, \cdot) = \rho(X_{\text{out}}(\xi), \cdot)$, $\mu_2(\xi, \cdot) = \nu(X_{\text{out}}(\xi), \cdot)$ and $c(j_0) = M(j_0)$ in (42) we get that a.s. on the event Ω^m we have $\rho(X_{\text{out}}, \cdot) \ll \nu(X_{\text{out}}, \cdot)$.

If we have (47), we can again appeal to Proposition 13.1. Setting $\mu_1(\xi, \cdot) = \nu(X_{\text{out}}(\xi), \cdot)$, $\mu_2(\xi, \cdot) = \rho(X_{\text{out}}(\xi), \cdot)$ and $c(j_0) = 1/m(j_0)$ in (42) we get that a.s. on the event Ω^m we have $\nu(X_{\text{out}}, \cdot) \ll \rho(X_{\text{out}}, \cdot)$.

The last two paragraphs together imply that $\rho(X_{\text{out}}(\xi), \cdot) \equiv \nu(X_{\text{out}}(\xi), \cdot)$ a.s. on the event Ω^m , as desired.

To establish (46) and (47), we begin with a fixed j_0 , a set $A \in \mathfrak{A}^m$ and a Borel set B in $\mathcal{S}_{\text{out}}$. We will invoke Proposition 13.2 in the following manner. Recall (C3): for all $A \in \mathfrak{A}^m$ and $B \in \mathcal{B}$ we have

$$\mathbb{P}[(X_{\text{in}}^{n_k} \in A) \cap (X_{\text{out}}^{n_k} \in B) \cap \Omega_{n_k}(j)] \asymp_j \int_{(X_{\text{out}}^{n_k})^{-1}(B) \cap \Omega_{n_k}(j)} \nu(X_{\text{out}}(\xi), A) d\mathbb{P}(\xi) + \vartheta(k; j, A, B).$$

where $\lim_{k \rightarrow \infty} \vartheta(k; j, A, B) = 0$ for each fixed j, A and B .

Let the multiplicative constants appearing in the $\asymp_j$ relation in (C3) with $j = j_0$ be $m(j_0)$ and $M(j_0)$ respectively, with $m(j_0) \leq M(j_0)$ (to be more explicit, in the notations of Remark 7.2, $M(j_0)$ is playing the role of C_j and $m(j_0)$ is playing the role of c_j). In Proposition 13.2 we set $h_1(\xi) = 1, h_2(\xi) = \nu(X_{\text{out}}(\xi), A), U_1 = A, U_2 = \mathcal{D}^m, V = B, c(j_0) = M(j_0)$ to obtain (46). On the other hand, setting $h_1(\xi) = \nu(X_{\text{out}}(\xi), A), h_2(\xi) = 1, U_1 = \mathcal{D}^m, U_2 = A, V = B, c(j_0) = 1/m(j_0)$ in Proposition 13.2 we obtain (47).

This completes the proof of theorem 7.2. ■

In the above proof, we have used the fact that the random measures $\rho(X_{\text{out}}, \cdot)$ and $\nu(X_{\text{out}}, \cdot)$ are X_{out} -measurable, which is true because of the measurability properties of ν as laid down in the assumptions of Theorem 7.2 and the measurability properties of regular conditional distributions on Polish spaces (pertaining to ρ).

We end this section with the proofs of Propositions 13.1 and 13.2 which are used in the proof of Theorem 7.2 above.

Proof of Proposition 13.1. For each $A \in \mathfrak{A}^m$ a.s. in X_{out} it follows from (42) that

$$\mathbb{E} [\mathbb{1}_{\Omega(j_0)} \mu_1(\xi, A) | X_{\text{out}}] \leq c(j_0) \mathbb{E} [\mathbb{1}_{\Omega^m} \mu_2(\xi, A) | X_{\text{out}}]. \quad (48)$$

On the event Ω^m (which is measurable with respect to X_{out}) the last inequality becomes

$$\mathbb{E} [\mathbb{1}_{\Omega(j_0)} \mu_1(\xi, A) | X_{\text{out}}] \leq c(j_0) \mathbb{E} [\mu_2(\xi, A) | X_{\text{out}}]. \quad (49)$$

To see how (48) follows from (42), we note that (42) implies

$$\int_{X_{\text{out}}^{-1}(B)} \mathbb{1}_{\Omega(j_0)} \mu_1(\xi, A) d\mathbb{P}(\xi) \leq \int_{X_{\text{out}}^{-1}(B)} c(j_0) \mathbb{1}_{\Omega^m} \mu_2(\xi, A) d\mathbb{P}(\xi).$$

Since this is true of every Borel set $B \subset \mathcal{S}_{\text{out}}$, therefore by the definition of conditional expectation, we get

$$\mathbb{E} [\mathbb{1}_{\Omega(j_0)} \mu_1(\xi, A) | X_{\text{out}}] \leq \mathbb{E} [c(j_0) \mathbb{1}_{\Omega^m} \mu_2(\xi, A) | X_{\text{out}}].$$

For almost every configuration $\omega \in \mathcal{S}_{\text{out}}$ such that $N(\omega) = m$ (with respect to the measure $\mathbb{P}_{X_{\text{out}}}$, and N being the function as in Definition 3), or equivalently, on the event Ω^m which is measurable with respect to X_{out} , we have (49) for all $A \in \mathfrak{A}^m$. Therefore by the regularity of the Borel measure on the right hand side, (49) extends to all Borel sets $A \subset \mathcal{D}^m$ (see Proposition 7.1). Now, for $\omega \in \mathcal{S}_{\text{out}}$ such that $N(\omega) = m$, suppose that $A \subset \mathcal{D}^m$ is a Borel set such that $\mathbb{E} [\mu_2(\xi, A) | X_{\text{out}} = \omega] = 0$. Then (49) implies that

$$\mathbb{E} [\mathbb{1}_{\Omega(j_0)} \mu_1(\xi, A) | X_{\text{out}} = \omega] = 0, \quad (50)$$

for each j_0 . But $\Omega(j_0)$ exhausts Ω^m . Hence, for ω such that $X_{\text{out}} = \omega$ implies Ω^m occurs (equivalently, $N(\omega) = m$), using the Dominated Convergence Theorem we have

$$\mathbb{E} [\mathbb{1}_{\Omega(j_0)} \mu_1(\xi, A) | X_{\text{out}} = \omega] \rightarrow \mathbb{E} [\mathbb{1}_{\Omega^m} \mu_1(\xi, A) | X_{\text{out}} = \omega] = \mathbb{E} [\mu_1(\xi, A) | X_{\text{out}} = \omega]$$

as $j_0 \uparrow \infty$. By letting $j_0 \uparrow \infty$ in (50), we obtain the fact that a.s. on Ω^m we have $\mathbb{E} [\mu_2(\xi, A) | X_{\text{out}}] = 0$ implies $\mathbb{E} [\mu_1(\xi, A) | X_{\text{out}}] = 0$. In other words, a.s. on Ω^m , we have (43). $\blacksquare$

Proof of Proposition 13.2. In what follows, we will denote by $\mathbb{P}_h$ the non-negative finite measure on Ξ obtained by setting $d\mathbb{P}_h(\xi) = h(\xi)d\mathbb{P}(\xi)$ where $h : \Xi \rightarrow [0, 1]$ is a measurable function. We will proceed in stages as follows: we will first perform certain computations that hold true for a general $\mathbb{P}_h$. Subsequently, we use the result of these computations successively for $h = h_1$ and $h = h_2$, and then use (44) to connect the resulting quantities via an inequality.

We note that for any event E , we have $0 \leq \mathbb{P}_h(E) \leq \mathbb{P}(E)$. Fix a $U \in \mathfrak{A}^m$.

For any Borel set V in $\mathcal{S}_{\text{out}}$ and $\varepsilon > 0$ we can find a $V_\varepsilon \in \mathcal{B}$ such that

$$\mathbb{P} (X_{\text{out}}^{-1}(V) \Delta X_{\text{out}}^{-1}(V_\varepsilon)) < \varepsilon.$$

This can be seen by considering the fact that the push forward probability measure $(X_{\text{out}})_* \mathbb{P}$ is a regular Borel measure on $\mathcal{S}_{\text{out}}$, and $\mathcal{B}$ consists of finite unions of sets that form a topological basis for $\mathcal{S}_{\text{out}}$. The purpose of this reduction is to exploit the fact that as $k \rightarrow \infty$, we have $\mathbb{1}_{V_\varepsilon}(X_{\text{out}}^{n_k}) \rightarrow \mathbb{1}_{V_\varepsilon}(X_{\text{out}})$ a.s.

Let us consider the quantity $\mathbb{P}_h[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in V_\varepsilon) \cap \Omega_{n_k}(j_0)]$. This is equal to

$$\mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V_\varepsilon) \cap \Omega_{n_k}(j_0) \right] + o_k(1; U, V_\varepsilon),$$

where $o_k(1; U, V_\varepsilon)$ stands for a quantity that tends to 0 as $k \rightarrow \infty$ for fixed sets U and V_ε . This step uses the fact that $\mathbb{1}[X_{\text{out}}^{n_k} \in V_\varepsilon] \rightarrow \mathbb{1}[X_{\text{out}} \in V_\varepsilon]$ and $\mathbb{1}[X_{\text{in}}^{n_k} \in U] \rightarrow \mathbb{1}[X_{\text{in}} \in U]$ a.s. The expression in the last display above equals

$$\mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \right] + o_\varepsilon(1) + o_k(1; U, V_\varepsilon), \quad (51)$$

where $o_\varepsilon(1)$ denotes a quantity that tends to 0 uniformly in k as $\varepsilon \rightarrow 0$.

We can bound the probability in (51) simply by $\mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \right]$. Putting all these together, we have

$$\mathbb{P}_h[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in V_\varepsilon) \cap \Omega_{n_k}(j_0)] \leq \mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \right] + o_\varepsilon(1) + o_k(1; U, V_\varepsilon). \quad (52)$$

To obtain a comparable lower bound requires some more work. We begin with

$$\mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \right] \geq \mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \cap \Omega(j_0) \right].$$

Observe that $\Omega(j_0) = \varliminf_{k \rightarrow \infty} \Omega_{n_k}(j_0)$ and $\mathbb{P} \left(\left(\varliminf_{l \rightarrow \infty} \Omega_{n_l}(j_0) \right) \Delta \left(\bigcap_{l \geq k} \Omega_{n_l}(j_0) \right) \right) = \mathfrak{o}_k(1)$ where $\mathfrak{o}_k(1)$ denotes a quantity which, for a fixed j_0 , converges to 0 as $k \rightarrow \infty$, uniformly in all the other quantities.

Hence

$$\begin{aligned} & \mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \cap \Omega(j_0) \right] \\ &= \mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega_{n_k}(j_0) \cap \left(\bigcap_{l \geq k} \Omega_{n_l}(j_0) \right) \right] + \mathfrak{o}_k(1). \end{aligned}$$

But $\Omega_{n_k}(j_0) \cap \left(\bigcap_{l \geq k} \Omega_{n_l}(j_0) \right) = \left(\bigcap_{l \geq k} \Omega_{n_l}(j_0) \right)$ and hence the probability on the right hand side of the last display equals

$$\mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega(j_0) \right] + \mathfrak{o}_k(1).$$

The arguments above result in

$$\begin{aligned} \mathbb{P}_h \left[(X_{\text{in}}^{n_k} \in U) \cap (X_{\text{out}}^{n_k} \in V_\varepsilon) \cap \Omega_{n_k}(j_0) \right] &\geq \mathbb{P}_h \left[(X_{\text{in}} \in U) \cap (X_{\text{out}} \in V) \cap \Omega(j_0) \right] \\ &+ o_k(1; U, V_\varepsilon) + o_\varepsilon(1) + \mathfrak{o}_k(1). \end{aligned} \quad (53)$$

Now, we wish to prove (45). We appeal to (52) with $h = h_2, U = U_2$ and to (53) with $h = h_1, U = U_1$ to obtain, using (44) (applied with $\tilde{V} = V_\varepsilon$),

$$\begin{aligned} \mathbb{P}_{h_1} \left[(X_{\text{in}} \in U_1) \cap (X_{\text{out}} \in V) \cap \Omega(j_0) \right] &\leq c(j_0) \mathbb{P}_{h_2} \left[(X_{\text{in}} \in U_2) \cap (X_{\text{out}} \in V) \right] \\ &+ o_\varepsilon(1) + o_k(1; U, V_\varepsilon) + \mathfrak{o}_k(1) + \vartheta(k). \end{aligned} \quad (54)$$

We first keep all the other quantities fixed and let $k \rightarrow \infty$, after that we let $\varepsilon \rightarrow 0$ to obtain (45), as desired. ■

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References

- [AGZ09] G. Anderson, A. Guionnet, O. Zeitouni, *An Introduction to Random Matrices*, Cambridge University Press, 2009.
- [Bh05] J. Ben Hough, *Large deviations for the zero set of an analytic function with diffusing coefficients*, <http://arxiv.org/abs/math/0510237>.
- [Bo11] A. Borodin, *Determinantal Point Processes*, The Oxford Handbook of Random Matrix Theory, Oxford University Press, 2011.
- [BK89] R.M. Burton, M. Keane, *Density and Uniqueness in Percolation*, Comm. Math. Phys., Volume 121, Number 3, 501-505, 1989.
- [Du10] R. Durrett, *Probability: Theory and Examples, Fourth Edition*, Cambridge University Press, 2010.
- [DG09] P. Deift, D. Gioev, *Random Matrix Theory: Invariant Ensembles and Universality*, Courant Lecture Notes in Mathematics 18, American Mathematical Society, 2009.
- [DaV97] D.J. Daley, D. Vere Jones, *An Introduction to the Theory of Point Processes (Vols. I & II)*, Springer, 1997.
- [FH99] P.J. Forrester, G. Honner, *Exact statistical properties of the zeros of complex random polynomials*, J. Phys. A: Math. Gen., Vol. 32, No. 16, 1999.
- [G12-I] S. Ghosh, *Determinantal processes and completeness of random exponentials: the critical case*, <http://arxiv.org/abs/1211.2435>.
- [G12-II] S. Ghosh, *Rigidity and Tolerance in Gaussian zeroes and Ginibre eigenvalues: quantitative estimates*, <http://arxiv.org/abs/1211.3506>.
- [Gin65] J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices*, Journal of Mathematical Physics, 1965.
- [Go10] A. Goldman, *The Palm measure and the Voronoi tessellation for the Ginibre process*, Ann. Appl. Probab., Volume 20, Number 1 (2010), 90-128.
- [GK] S. Ghosh, M. Krishnapur, *Rigidity hierarchy in random point fields: random polynomials and determinantal processes*, in preparation.
- [GKP12] S. Ghosh, M. Krishnapur, Y. Peres, *Continuum Percolation for Gaussian zeroes and Ginibre eigenvalues*, <http://arxiv.org/abs/1211.2514v1>.

- [HKPV10] J.B. Hough, M.Krishnapur, Y.Peres, B.Virag, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, A.M.S., 2010.
- [HLY03] D. Hecklen, R. Lyons, *Change Intolerance in Spanning Forests*, Journal of Theoretical Probability, Volume 16, Number 1 (2003), 47-58.
- [HP05] A.E. Holroyd, Y. Peres, *Extra heads and invariant allocations*, Ann. Probab., 33(1):31-52, 2005.
- [HS10] A.E. Holroyd, T.Soo, *Insertion and deletion tolerance of point processes*, arXiv:1007.3538v2, 2010.
- [Kr06] M. Krishnapur, *Zeros of Random Analytic Functions*, PhD Thesis, University of California, Berkeley [arXiv.math.PR/0607504], 2006.
- [Ly03] R. Lyons, *Determinantal Probability Measures*, Publ. Math. Inst. Hautes Etudes Sci., 98, 167-212, 2003.
- [NS10] F. Nazarov, M.Sodin, *Random Complex Zeroes and Random Nodal Lines*, Proceedings of the International Congress of Mathematicians, Vol. III, 1450-1484, 2010.
- [NS11] F. Nazarov, M. Sodin, *Fluctuations in Random Complex Zeroes: Asymptotic Normality Revisited*, Int. Math. Res. Not., Vol. 2011, No. 24, 5720-5759, 2011.
- [NSV07] F. Nazarov, M. Sodin, A. Volberg, *Transportation to random zeroes by the gradient flow. Geometric and Functional Analysis*, Vol 17-3, 887-935, 2007.
- [Pa00] K.R. Parthasarathy, *Probability Measures on Metric Spaces*, Amer. Math. Soc., 2000.
- [PS14] Y. Peres, A. Sly, *Rigidity and tolerance for perturbed lattices*, arXiv:1409.4490.
- [PV05] Y. Peres, B. Virag, *Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process*, Acta Mathematica, Volume 194, Issue 1, pp 1-35, 2005.
- [Rud87] W Rudin, *Real and Complex Analysis*, McGraw-Hill, 3rd edition, 1987.
- [RV07] B. Rider, B. Virag, *The Noise in the Circular Law and the Gaussian Free Field*, Int. Math. Res. Not., Vol. 2007, 2007.
- [Sos00] A. Soshnikov, *Russian Math. Surveys* 55:5 923975 (*Uspekhi Mat. Nauk* 55:5 107160), 2000.
- [Sta99] R. Stanley, *Enumerative Combinatorics* Volume 2, Cambridge University Press, 1999.
- [S00] M. Sodin, *Zeros of Gaussian analytic functions*, Mathematical Research Letters, Volume 7, Number 4, 371381, 2000.
- [STs1-04] M. Sodin, B. Tsirelson, *Random complex zeroes, I. Asymptotic normality*, Israel Journal of Mathematics, Volume 144, Number 1, 125-149, 2004.
- [STs2-06] M. Sodin, B. Tsirelson, *Random complex zeroes, II. Perturbed lattice*, Israel Journal of Mathematics, Volume 152, Number 1, 105-124, 2006.
- [STs3-05] M. Sodin, B. Tsirelson, *Random complex zeroes, III. Decay of the hole probability*, Israel Journal of Mathematics, Volume 147, Number 1, 371-379, 2005.

- [ShZe08] B. Shiffman, S. Zelditch, *Number Variance of Random Zeros on Complex Manifolds*, Geom. Funct. Anal., Volume 18 Number 4, 1422-1475, 2008.
- [ShZeZ-08] B. Shiffman, S. Zelditch, S. Zrebiec, *Overcrowding and hole probabilities for random zeros on complex manifolds*, Indiana Univ. Math. J. 57 , 1977-1997, 2008.
- [ShZe10] B. Shiffman, S. Zelditch, *Number variance of random zeros on complex manifolds, II: smooth statistics*, Special issue in honor of Joseph J. Kohn, Pure Appl. Math. Q. 6 , 1145-1167, 2010.
- [ShZeZh11] B. Shiffman, S. Zelditch, Q. Zhong, *Random zeros on complex manifolds: conditional expectations*, J. Math. Inst. Jussieu 10 , 753-783, 2011.
- [Str65] V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Statist., 36, 423-439, 1965.
- [Ze10] O. Zeitouni, S. Zelditch, *Large deviations of empirical zero point measures on Riemann surfaces, I: $g=0$* , I.M.R.N. Vol. 2010, No. 20, 3939-3992, 2010.
- [Ze13] S. Zelditch, *Large deviations of empirical measures of zeros on Riemann surfaces*, I.M.R.N., Vol. 2013, 592-664, 2013.

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