Matrix Product States and Tensor Network States

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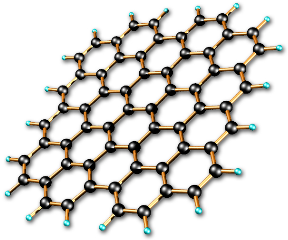
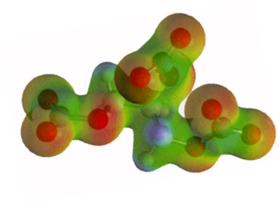
Overview

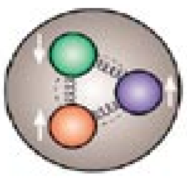
- **Quantum many-body systems** are all around!



condensed matter



quantum chemistry



high-energy physics

- Can exhibit **complex quantum correlations** (=multipartite entanglement)
  $\rightarrow$ rich and unconventional physics, but difficult to understand!

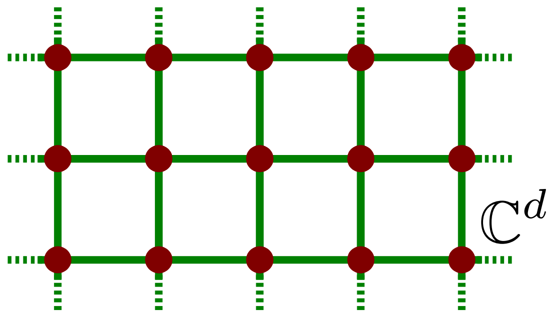
- **Quantum information** and **Entanglement Theory**:
  Toolbox to characterize and utilize **entanglement**

Aim: Study strongly correlated quantum many-body systems from the perspective of quantum information + entanglement theory.

Entanglement structure of quantum many-body states

Quantum many-body systems

- Wide range of quantum many-body (QMB) systems exists
- Our focus: **spin models** (=qudits) on lattices:

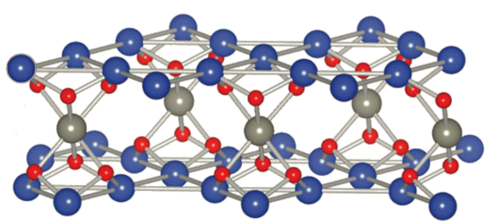


local interactions

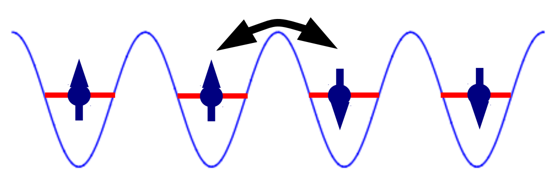
$$H = \sum_{\langle ij \rangle} h_{ij}$$

... typically **transl. invariant**

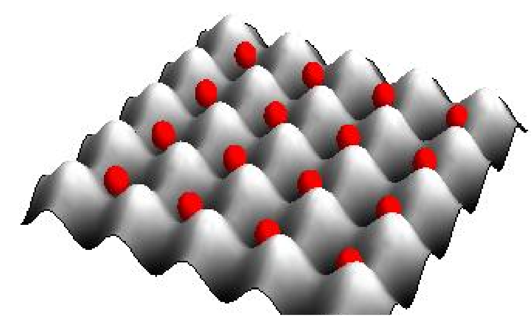
- Realized in many systems:



localized d/f electrons



half-filled band



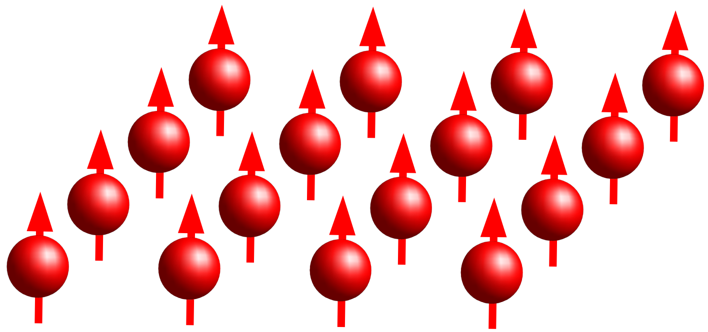
quantum simulators, e.g. optical lattices

- Expecially interested in the **ground state** $|\Psi_0\rangle$, i.e., the lowest eigenvector $H\,|\Psi_0\rangle = E_0\,|\Psi_0\rangle$

(It is the “most quantum” state, and it also carries relevant information about excitations.)

Mean-field theory

- In many cases, **entanglement** in QMB systems is **negligible**
- System can be studied with **product state ansatz**



$$H = \sum_{\langle ij \rangle} h_{ij}$$

"mean field theory"

$$|\Phi\rangle = |\phi\rangle \otimes |\phi\rangle \otimes \cdots$$

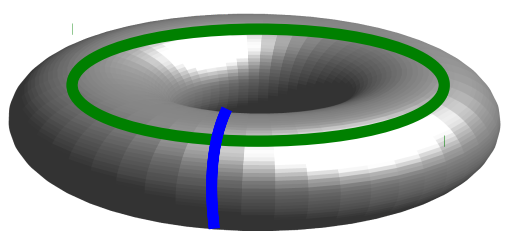
- Consequence of **"monogamy of entanglement"** ($\rightarrow$ de Finetti theorem)

- Behavior fully characterized by **a single spin** $|\phi\rangle$ –
  – a **local property** (order parameter) $\rightarrow$ **Landau theory** of phases

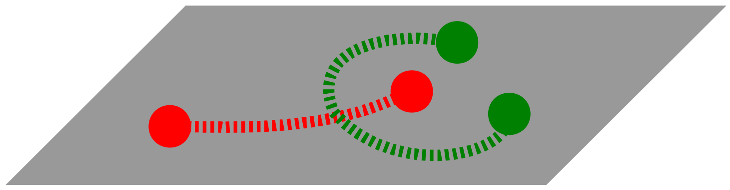
- Behavior insensitive to boundary conditions, topology, ...

Exotic phases and topological order

• Systems exist which **cannot be described by mean field theory**



degeneracy depends on
global properties



system supports
exotic excitations (“anyons”)

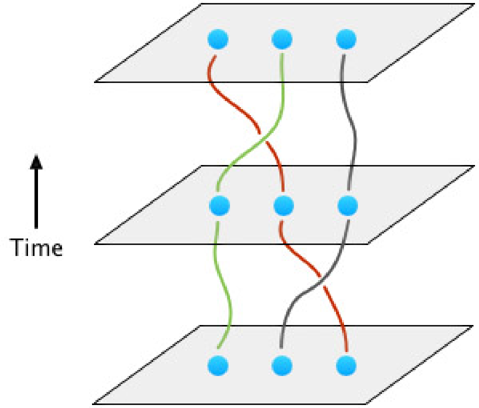
... e.g. Kitaev's “Toric Code”.

→ impossible within mean-field ansatz

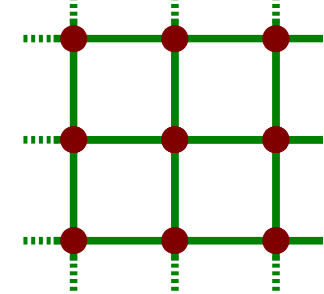
→ **ordering in entanglement**

→ To understand these systems: need to **capture their entanglement**!

• Useful as **quantum memories**
and for **topological quantum computing**



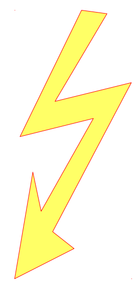
The physical corner of Hilbert space

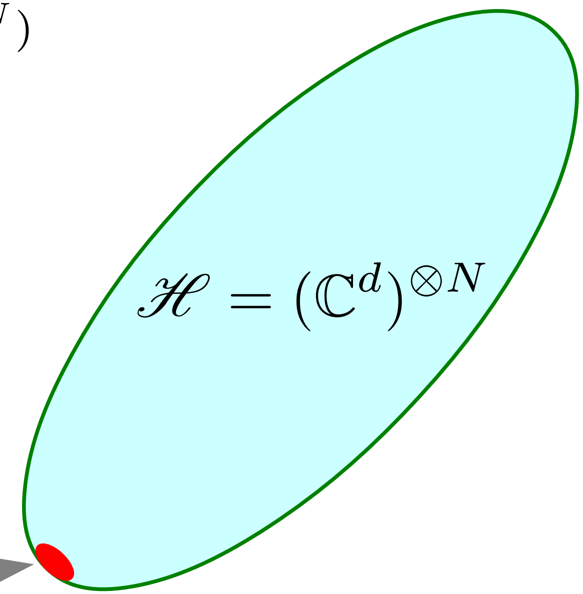


- How can we describe **entangled QMB states**?
- general state of N spins:

$$|\Psi_0\rangle = \sum_{i_1,\ldots,i_N} c_{i_1\cdots i_N} |i_1,\ldots,i_N\rangle \;\in\; (\mathbb{C}^d)^{\otimes N} = \mathbb{C}^{(d^N)}$$

exponentially large Hilbert space!





- but then again ...

$H = \sum_{\langle ij\rangle} h_{ij}$ has only $O(N)$ parameters

$\rightarrow$ ground state $|\Psi_0\rangle$ must live in a small **"physical corner" of Hilbert space**!

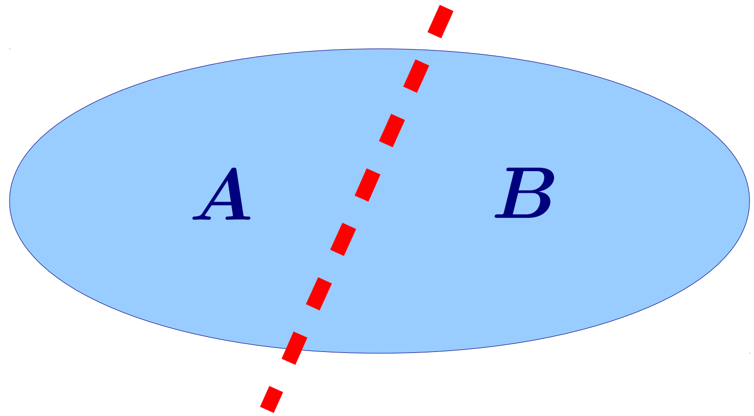
- Is there a "nice" way to **describe** states in the **physical corner**?

$\rightarrow$ use **entanglement structure**!

Entanglement

- Consider **bipartition** of QMB system into A and B

$$|\Phi_{AB}\rangle =$$



Schmidt decomposition $|\Phi_{AB}\rangle = \sum_k \sqrt{p_k}|\alpha_k\rangle_A|\beta_k\rangle_B$ ($|\alpha_k\rangle$, $|\beta_k\rangle$ ONB)

- **Schmidt coefficients** p_k characterize bipartite **entanglement**

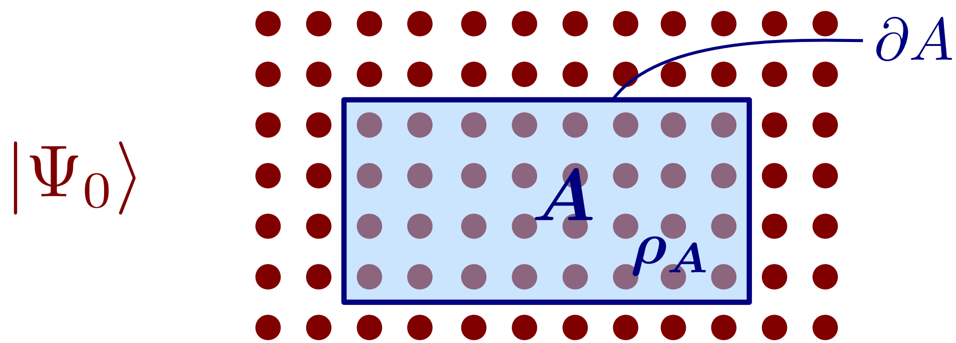
  more **disorder** $\rightarrow$ more **entanglement**

- Measure of entanglement:

Entanglement entropy $E(\Phi_{AB}) = S(\rho_A) = -\sum p_k \log p_k$

Entanglement structure: The area law

- How much is a region of a QMB system entangled with the rest?

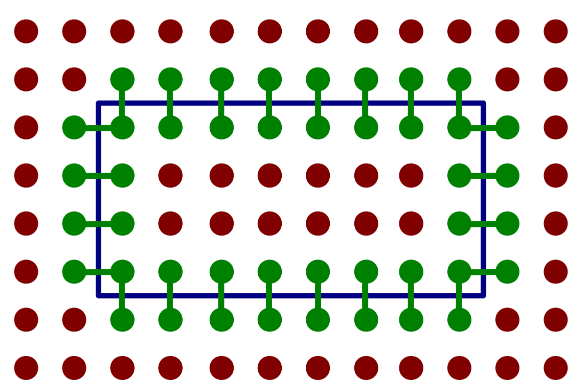


- **entanglement entropy** $S(\rho_A)$ of a region **scales as boundary** (vs. volume)

"area law" for entanglement

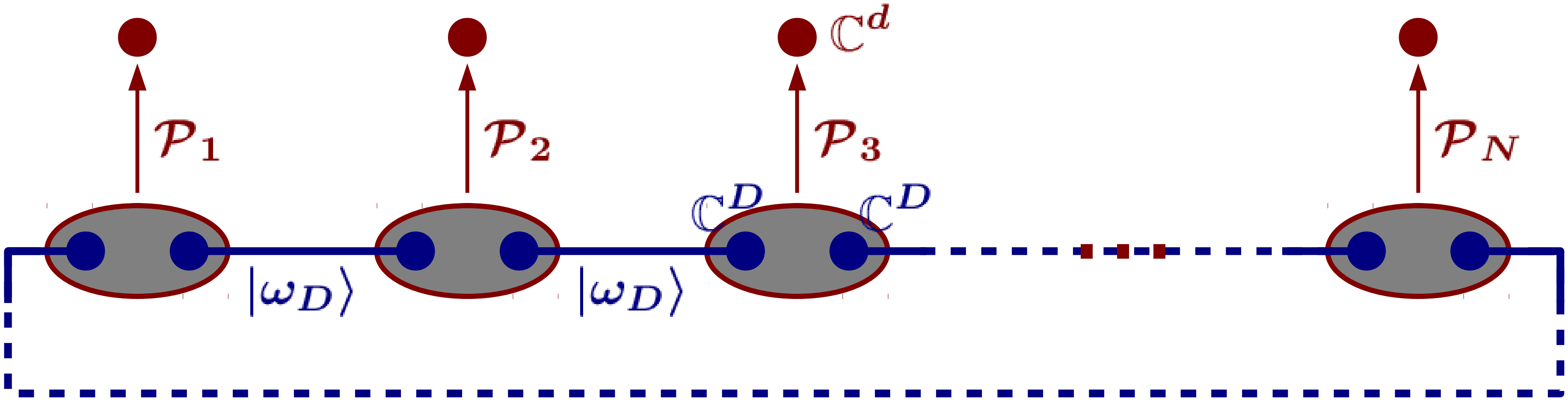
(for Hamiltonians with a **spectral gap**; but approx. true even without gap)

- Interpretation: entanglement is **distributed locally**



One dimension: Matrix Product States

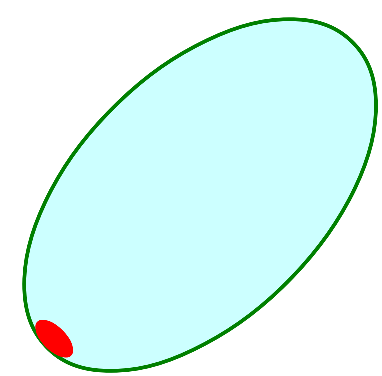
An ansatz for states with an area law



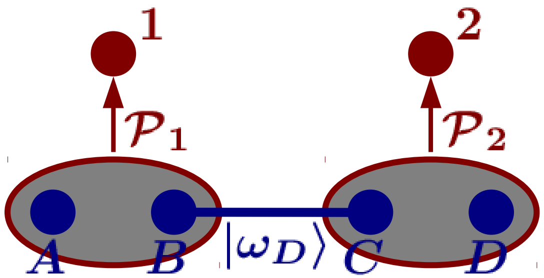
- each site composed of two **auxiliary particles** (“virtual particles”) forming max. entangled **bonds** $|\omega_D\rangle := \sum_{i=1}^{D} |i,i\rangle$ (D: “bond dimension”)
- apply **linear map** (“projector”) $\mathcal{P}_k : \mathbb{C}^D \times \mathbb{C}^D \to \mathbb{C}^d$

$$\Rightarrow \quad \boxed{|\psi\rangle = (\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_N)|\omega_D\rangle^{\otimes N}}$$

- satisfies **area law** by construction
- state characterized by $\mathcal{P}_1, \ldots, \mathcal{P}_N \;\; \rightarrow \; NdD^2$ parameters
- family of states: enlarged by increasing D



Formulation in terms of Matrix Products



$$\mathcal{P}_s = \sum_{i,\alpha,\beta} A^{[s],i}_{\alpha\beta} |i\rangle\langle\alpha,\beta|$$

$A^{[s],i}$: $D \times D$ matrices

$$(\mathcal{P}_1 \otimes \mathcal{P}_2)|\omega_D\rangle = \Big[\sum_{i,\alpha,\beta} A^{[1],i}_{\alpha\beta}|i\rangle_1\langle\alpha,\beta|_{AB}\Big]\Big[\sum_{j,\gamma,\delta} A^{[2],j}_{\gamma\delta}|j\rangle_2\langle\gamma,\delta|_{CD}\Big]\Big[\sum_k |k,k\rangle_{BC}\Big]$$

$$= \sum_{i,j,\alpha,\delta}\Big[\sum_\beta A^{[1],i}_{\alpha\beta}A^{[2],j}_{\beta\delta}\Big]|i,j\rangle_{12}\langle\alpha,\delta|_{AD} \qquad \boldsymbol{\beta=\gamma}$$

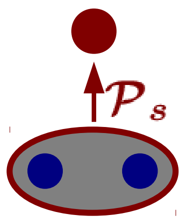
$$= \sum_{i,j,\alpha,\delta} \boldsymbol{(A^{[1],i}A^{[2],j})_{\alpha\delta}}|i,j\rangle_{12}\langle\alpha,\delta|_{AD}$$

- iterate this for the whole state $|\psi\rangle = (\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_N)|\omega_D\rangle^{\otimes N}$:

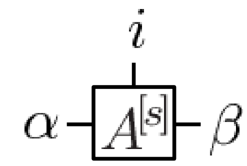
$$|\psi\rangle = \sum_{i_1,\dots,i_N} [A^{[1],i_1}A^{[2],i_2}\cdots A^{[N],i_N}]|i_1,\dots,i_N\rangle$$ **"Matrix Product State" (MPS)**

(or $|\psi\rangle = \sum_{i_1,\dots,i_N} \langle l|A^{[1],i_1}A^{[2],i_2}\cdots A^{[N],i_N}|r\rangle|i_1,\dots,i_N\rangle$ for open boundaries)

Formulation in terms of Tensor Networks

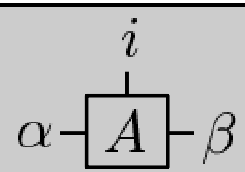


$$\mathcal{P}_s = \sum_{i,\alpha,\beta} A^{[s],i}_{\alpha,\beta} |i\rangle\langle\alpha,\beta| \qquad A^{[s],i}_{\alpha\beta} \equiv$$

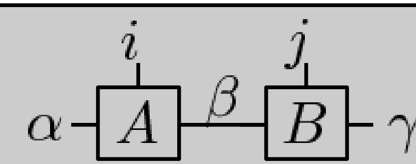


- **Tensor Network** notation:

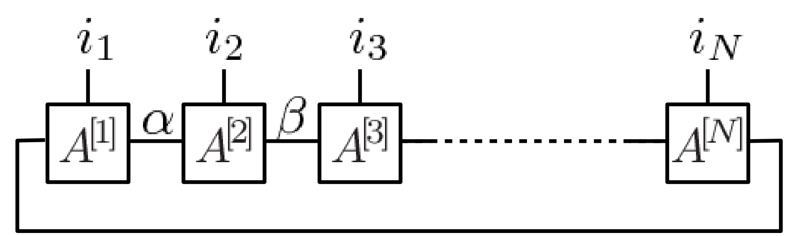
$$A^i_{\alpha\beta} \equiv$$



$$\sum_\beta A^i_{\alpha\beta} B^j_{\beta\gamma} \equiv$$

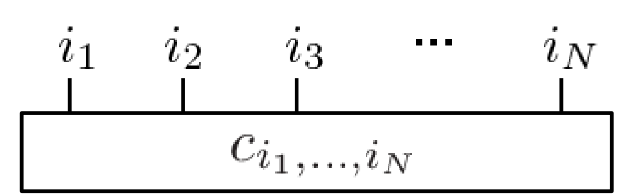


$$\mathbf{tr}[A^{[1],i_1} A^{[2],i_2} \cdots A^{[N],i_N}] =$$



- Matrix Product States can be written as

$$|\Psi_0\rangle = \sum_{i_1,\ldots,i_N} c_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle \quad \text{with}$$

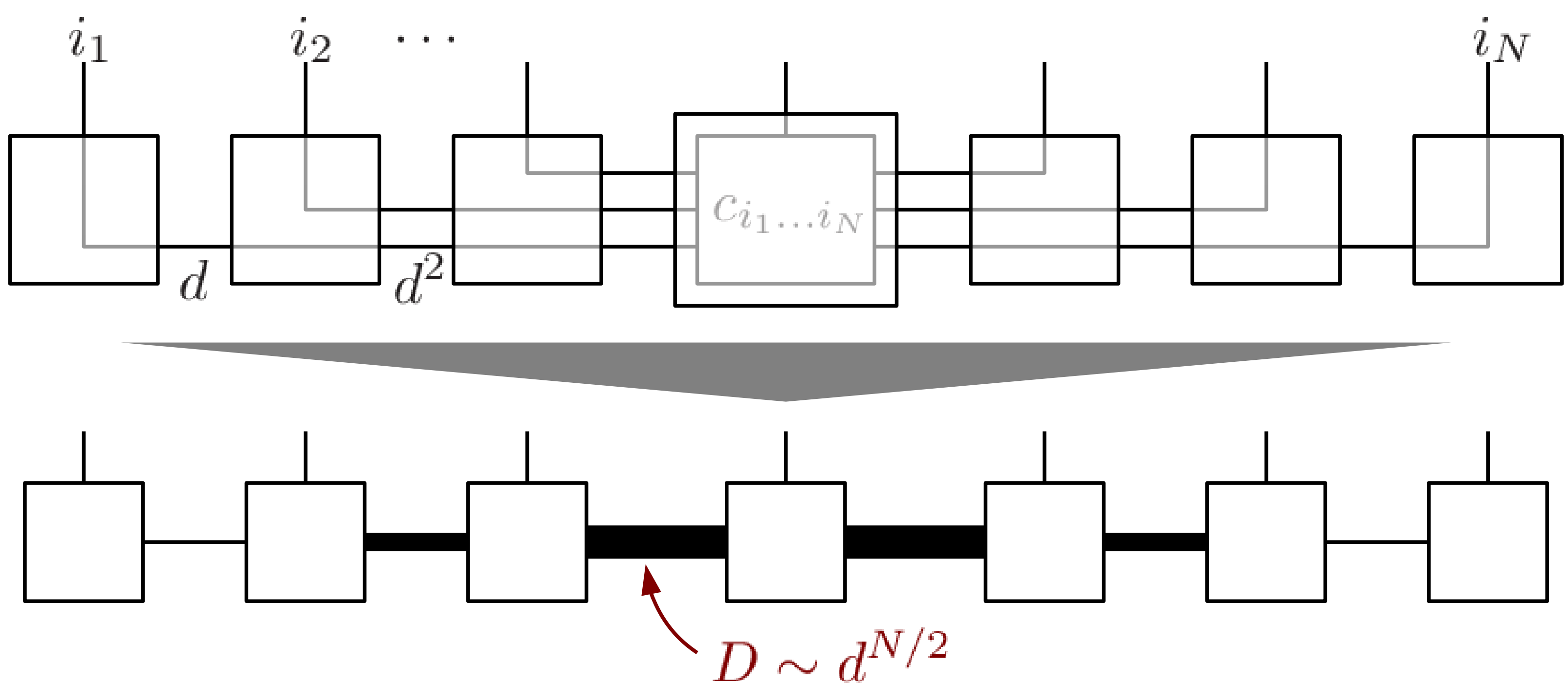


"Tensor Network States"

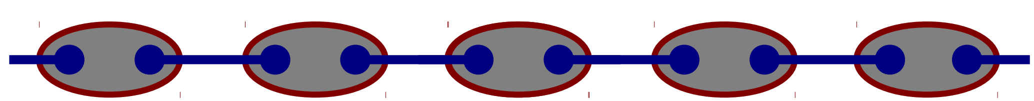
Completeness of MPS

- MPS form a **complete family** – every state can be written as an MPS:

$$|\psi\rangle = \sum_{i_1,\dots,i_N} c_{i_1\dots i_N} |i_1,\dots,i_N\rangle$$

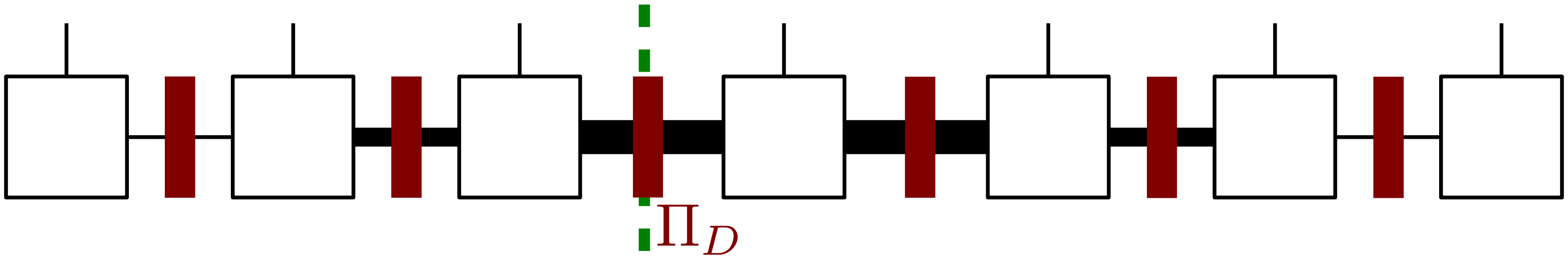


- Can be understood in terms of **teleporting** $|\psi\rangle$ using the entangled bonds



Approximation by MPS

- **General MPS** with possibly very **large bond dimension**



- **Schmidt decomposition** across some cut:

$$|\Phi_{AB}\rangle = \sum_k \sqrt{p_k}\, |\alpha_k\rangle\, |\beta_k\rangle$$

- Project onto D **largest Schmidt values** $p_1, \ldots, p_D$:

$\rightarrow$ error $\epsilon(D) = \sum_{k>D} p_k$

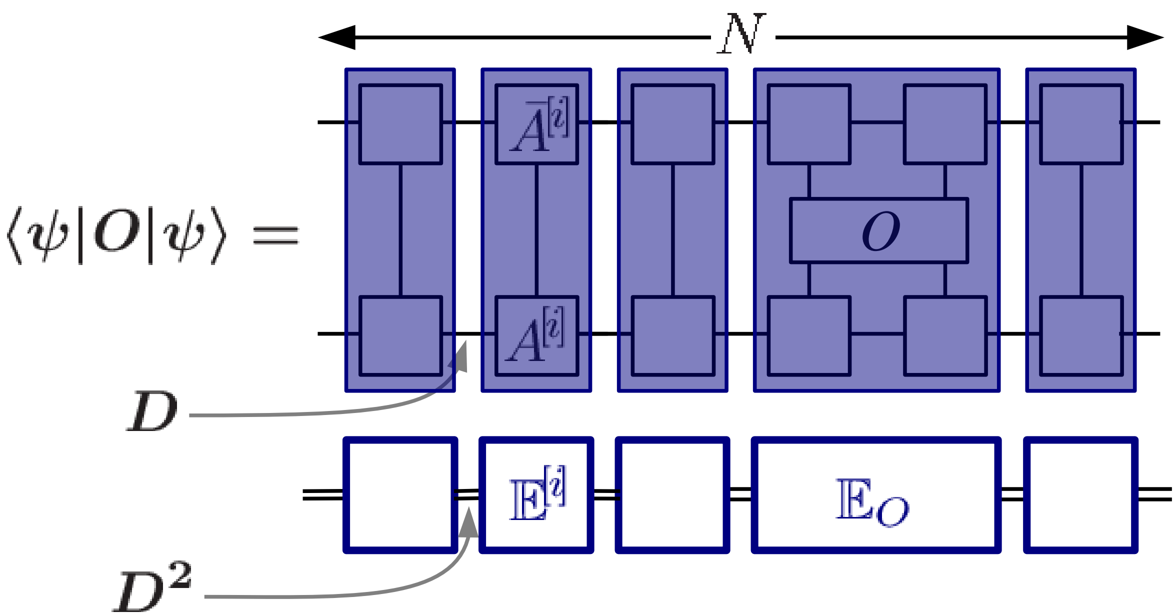
- **Rapidly decaying p_k** ($\leftrightarrow$ bounded entropy): total error $\sim \mathrm{poly}(N, 1/D)$

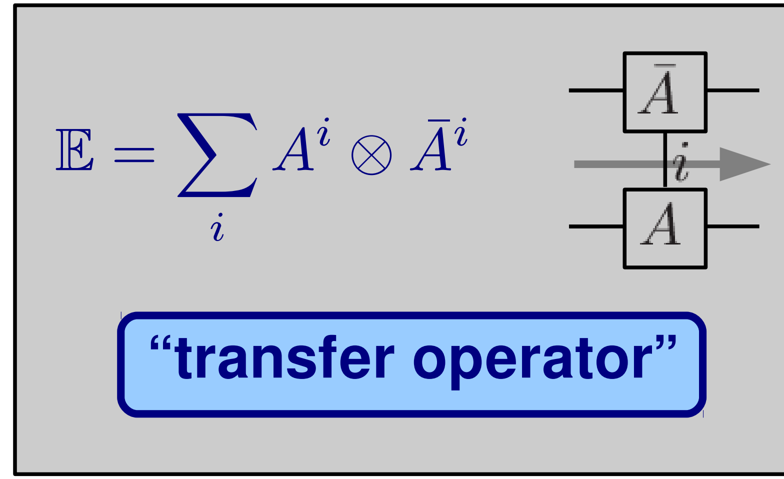
- **Efficient approximation** of states with **area law** (and thus ground states)

Matrix Product States can efficiently approximate states with an area law, and ground states of (gapped) one-dimensional Hamiltonians.

Computing properties of MPS

- Given an MPS $|\psi\rangle$, can we compute exp. values $\langle\psi|O|\psi\rangle$ for local O ?





$$\langle\psi|O|\psi\rangle = [\mathbb{E}^{[1]}\mathbb{E}^{[2]}\ldots\mathbb{E}^{[k-1]}\ \mathbb{E}_O\ \mathbb{E}^{[k+2[}\ldots\mathbb{E}^{[N]}]$$

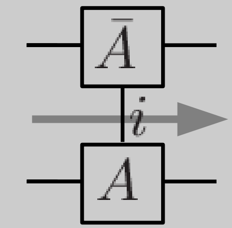
- computing $\langle\psi|O|\psi\rangle$ = multiplication of $D^2{\times}D^2$ matrices

  $\rightarrow$ computation time $\propto \boldsymbol{N\cdot D^6} = \mathrm{poly}(N)$

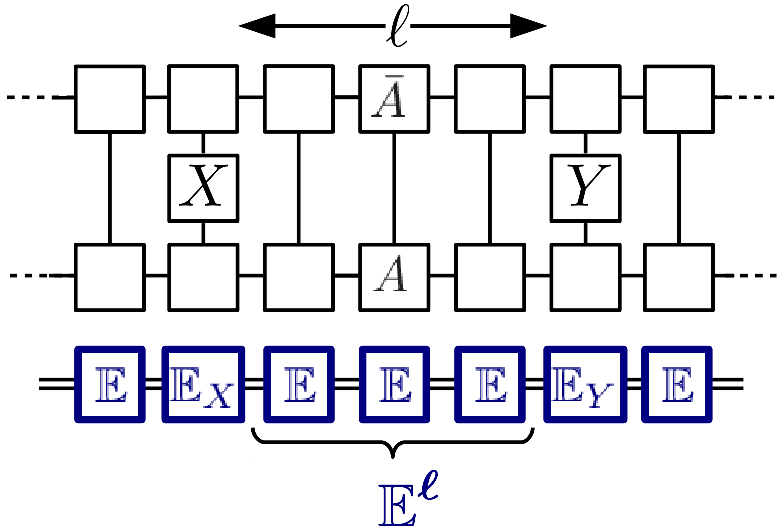
- OBC scaling: D^4 [and if done properly, even D^5 (PBC) and D^3 (OBC)]

The transfer operator

$$\mathbb{E} = \sum_i A^i \otimes \bar{A}^i$$



- consider **translational invariant** system:



$$\mathbb{E} = \sum_k \lambda_k |r_k\rangle\langle l_k|$$

$$\mathbb{E}^\ell = \sum_k \lambda_k^\ell |r_k\rangle\langle l_k|$$

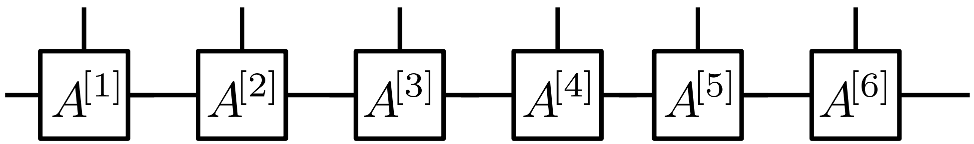
- **spectrum of transfer operator** governs **scaling of correlations**
  - (a) largest eigenvalue unique: **exponential decay** of correlations
  - (b) largest eigenvalue degenerate: long-range correlations

- uniqueness of purification: $\mathbb{E}$ contains **all non-local information** about state

- $\mathbb{E} = \sum A^i \otimes \bar{A}^i$ is Choi matrix of **quantum channel** $\mathcal{E} : \rho \mapsto \sum A^i \rho (A^i)^\dagger$

Numerical optimization of MPS

- MPS approximate ground states efficiently
- expectation values can be computed efficiently
- can we efficiently **find the** $|\psi\rangle$ which **minimizes** $\langle\psi|\boldsymbol{H}|\psi\rangle$?



- various methods:
  - DMRG: **optimize sequentially** $A^{[1]}, A^{[2]}, \ldots$ & iterate
  - gradient methods: optimize all $A^{[s]}$ **simultaneously**
  - hybrid methods

... $\langle\psi|H|\psi\rangle$ is **quadratic in each** $\boldsymbol{A}^{[s]}$ $\rightarrow$ each step can be done efficiently

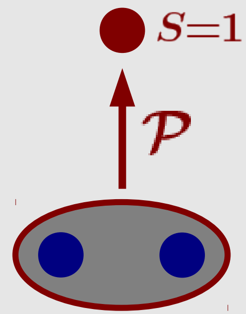
- **hard instances** exist (NP-hard), but methods practically **converge very well**
- **provably working** poly-time method exists

MPS form the basis for powerful variational methods for the simulation of one-dimensional spin chains

Example: The AKLT state – a rotationally invariant model

singlet $|\omega\rangle = |01\rangle - |10\rangle$

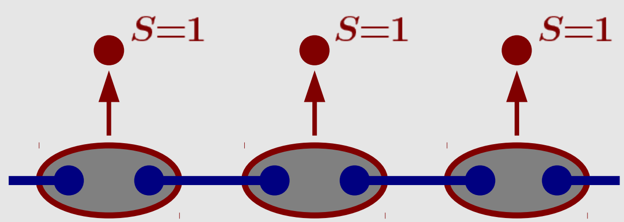
$$\boldsymbol{u \otimes u\, |\omega\rangle = |\omega\rangle}$$



$\mathcal{P}$: projector onto the spin-1 representation of $u \otimes u = 1 \oplus V_u$ ("$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$")

$$\Rightarrow \boldsymbol{\mathcal{P}(u \otimes u) = V_u \mathcal{P}}$$

$$|\Psi\rangle = \mathcal{P}^{\otimes N} |\omega\rangle^{\otimes N}$$



"AKLT state"
[Affleck, Kennedy, Lieb & Tasaki, '87]

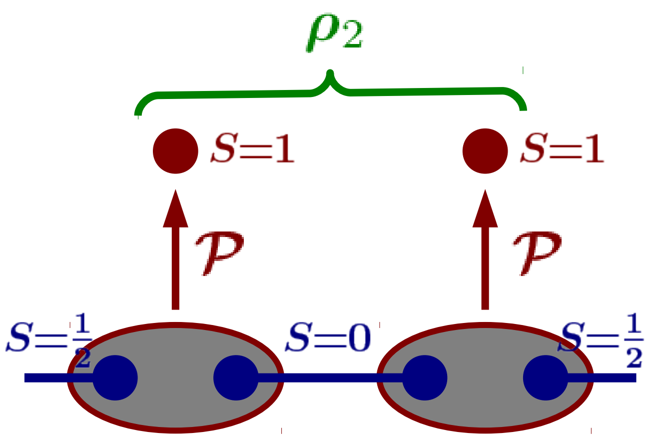
- Resulting state is **invariant under $\mathrm{SU}(2)$** (=spin rotation) by construction:

$$V_u^{\otimes N} |\Psi\rangle = (V_u \mathcal{P})^{\otimes N} |\omega\rangle^{\otimes N} = (\mathcal{P}\,(u \otimes u))^{\otimes N} |\omega\rangle^{\otimes N} = \mathcal{P}^{\otimes N} |\omega\rangle^{\otimes N} = |\Psi\rangle$$

- Can construct states w/ symmetries by **encoding symmetries locally**

The AKLT Hamiltonian

- consider **2 sites of AKLT model**



2 sites have spin $1 \otimes 1 = 0 \oplus 1 \oplus \cancel{2}$

impossible!

- $\boldsymbol{h := \Pi_{S=2}}$: $h \geq 0$, and $h|\Psi_{\mathrm{AKLT}}\rangle = 0$

$\Rightarrow |\boldsymbol{\Psi_{\mathrm{AKLT}}}\rangle$ is a (frustration free) **ground state** of $\boldsymbol{H = \sum h_i}$

(frustration free = it minimized each h_i individually)

"parent Hamiltonian"

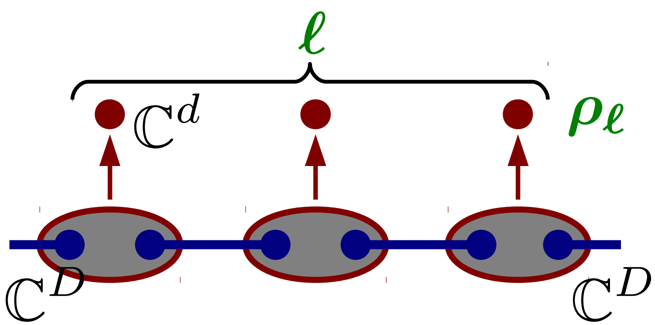
- H **inherits spin-rotation symmetry** of state by construction

(specifically, $h_i = \frac{1}{2}\left[\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2\right] + \frac{1}{3}$)

- One can prove:
  - $|\Psi_{\mathrm{AKLT}}\rangle$ is the **unique ground state** of H
  - H has a **spectral gap** above the ground state

Parent Hamiltonians

- A **parent Hamiltonian** can be constructed for any MPS:



ρ_ℓ lives in d^ℓ–dimensional space

D^2 possible boundary conditions

choose ℓ s.th. $d^\ell > D^2$ $\rightarrow \rho_\ell$ doesn't have full rank

- Construct **parent Hamiltonian** $h = \mathbb{1} - \Pi_{\mathrm{ker}(\rho_\ell)}$, $H = \sum h$

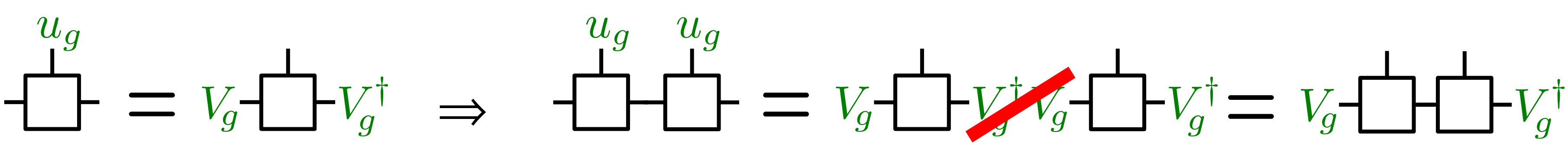
- Can prove:
  - has **unique ground state**
  - has a **spectral gap** above the ground state

- This + ability of MPS to approximate ground states of general Hamiltonians

  $\rightarrow$ MPS form right framework to **study physics of 1D QMB systems**

MPS and symmetries

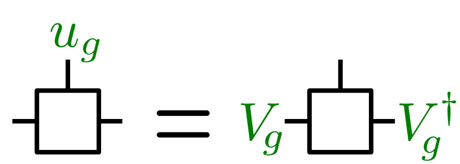
- **Symmetries in MPS** can always be **encoded locally**



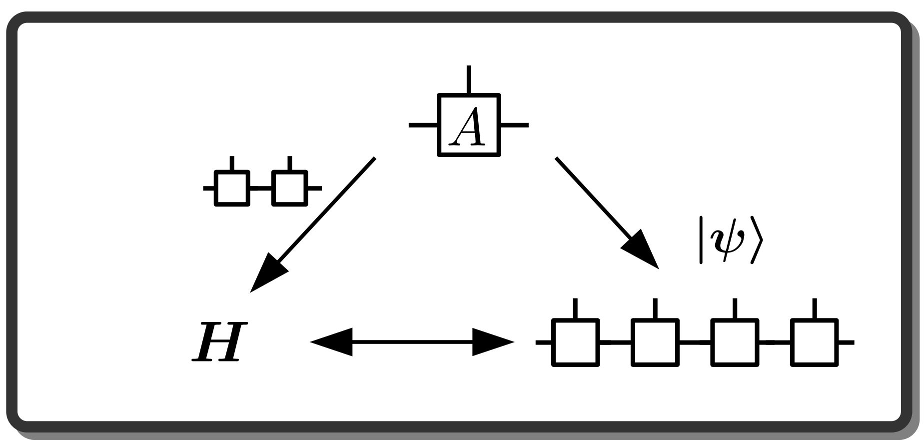
$$\Rightarrow \quad \ldots \quad \Rightarrow \quad |\psi\rangle = u_g^{\otimes N}\,|\psi\rangle$$

and conversely

$$|\psi\rangle = u_g^{\otimes N}\,|\psi\rangle \quad \Rightarrow$$

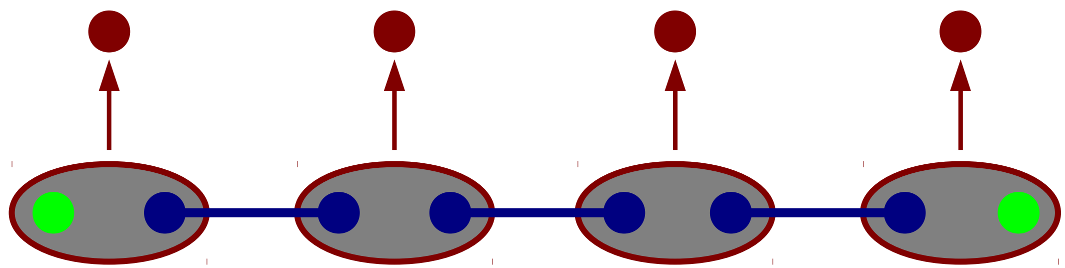


- Symmetries are **inherited by the parent Hamiltonian**!

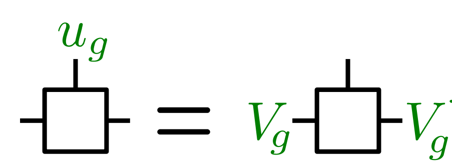


Fractionalization

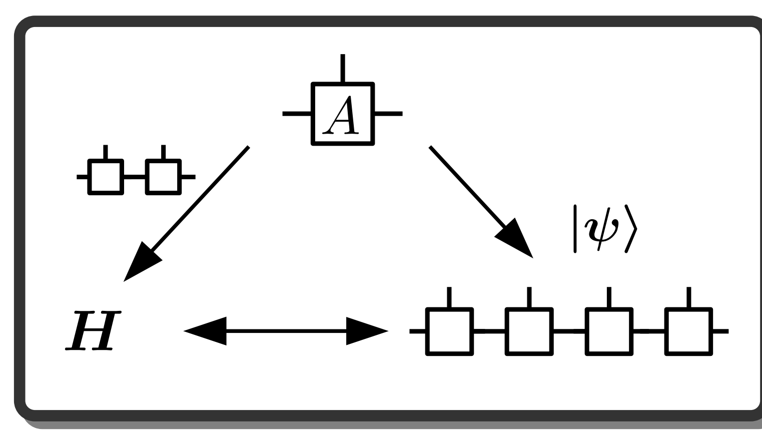
- Consider AKLT model on chain with **open boundaries**



- all choices of boundaries ● are **ground states** of parent Hamiltonian
  - $\rightarrow$ zero energy **"edge excitations"** with spin $S = \frac{1}{2}$
- **"fractionalization"** of physical **spin** $S = 1$ into $S = \frac{1}{2}$ at the **boundary**
  - $\rightarrow$ impossible in mean-field theory
  - $\rightarrow$ **non-trivial "topological" phase** ("Haldane phase")



- can prove: cannot smoothly connect MPS with integer and half-integer spin at edge
  - $\rightarrow$ **inequivalent phases**!

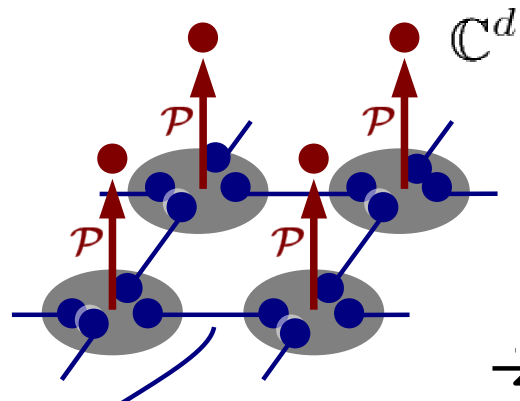


MPS encode physical symmetries locally, and can be used to model physical systems and study their different non-trivial phases.

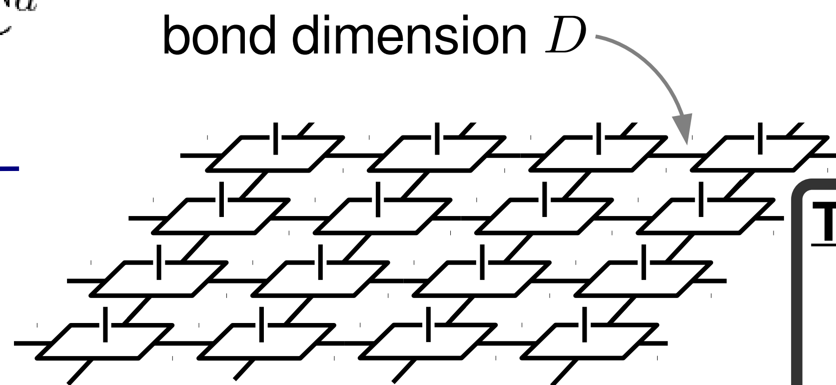
Two dimensions: Projected Entangled Pair States

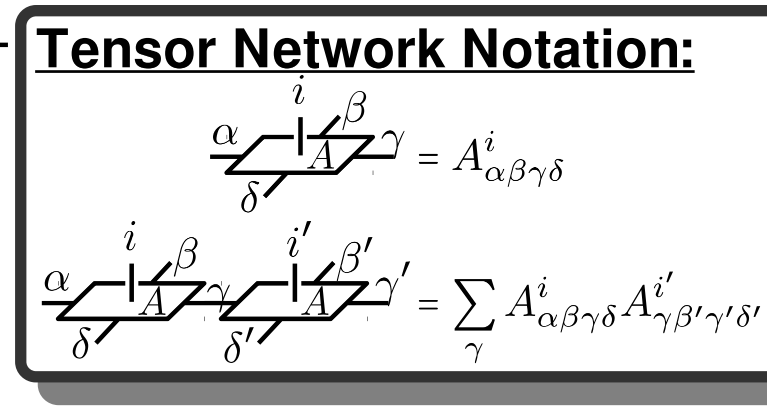
Two dimensions: Projected Entangled Pair States

- Natural generalization of MPS to two dimensions:



$$|\omega_D\rangle = \sum_{i=1}^{D} |i\rangle|i\rangle$$



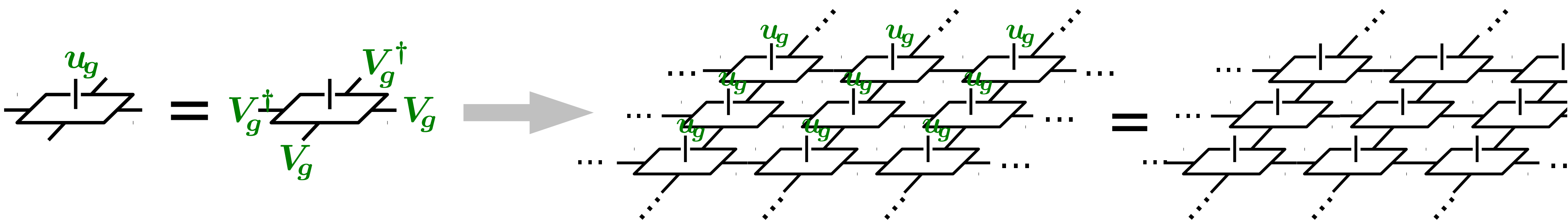


Projected Entangled Pair States (PEPS)

- **approximate ground states** of local Hamiltonians well
- PEPS form a **complete family** with accuracy parameter D.
- PEPS can also be defined on **other lattices**, in **three and more dimensions**, even on **any graph**

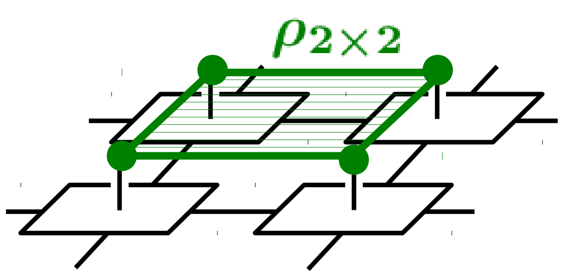
2D: Symmetries and parent Hamiltonians

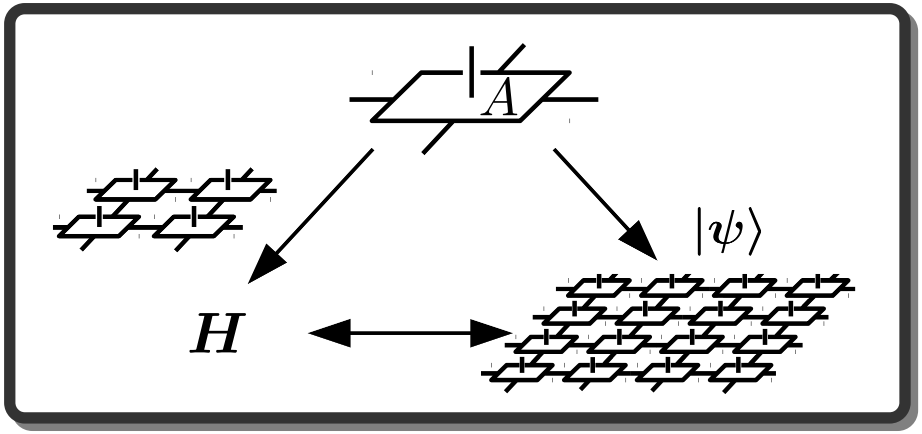
- **symmetries** can be encoded locally in **entanglement** degrees of freedom:



- however, a **general characterization** of inverse direction is **still missing** ...
  (but there are partial results)

- we can also define **parent Hamiltonians**

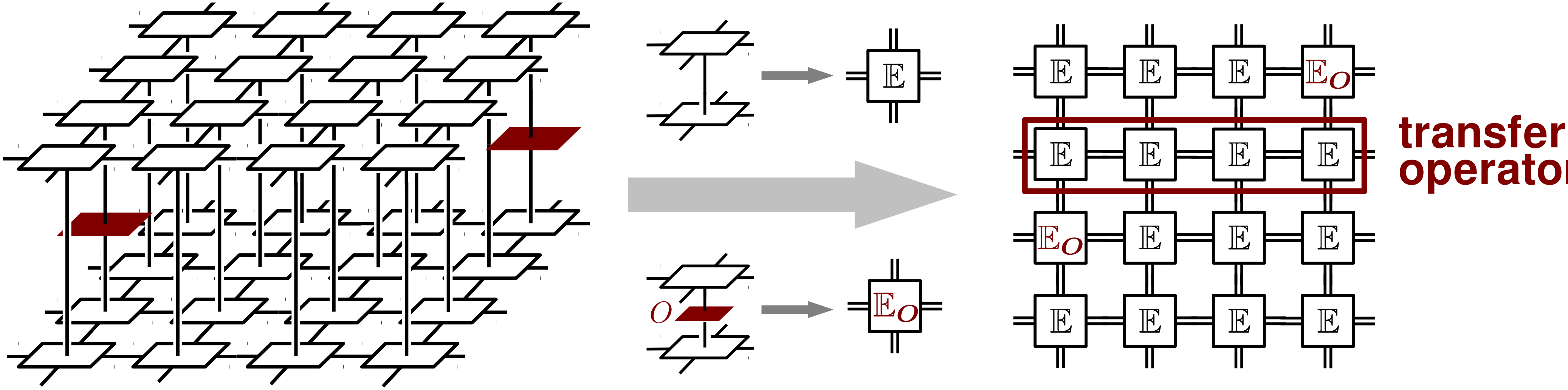




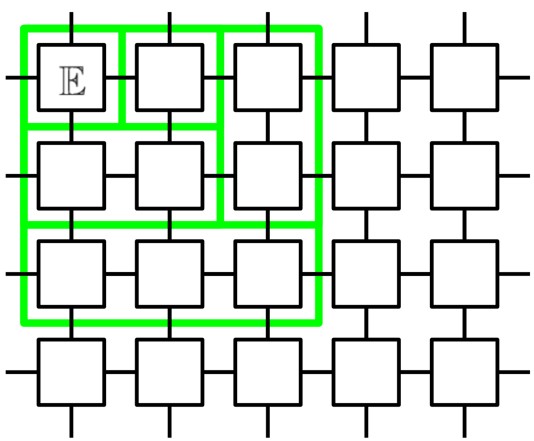
- again, a full characterization of **ground space and spectral gap** is missing ...
  (and again, there are partial results)

Computational complexity of PEPS

- **expectation values** in PEPS (e.g. correlation functions):



- resembles 1D situation, but ...



... **exact contraction** is a **hard problem**

(more precisely, #P-hard)

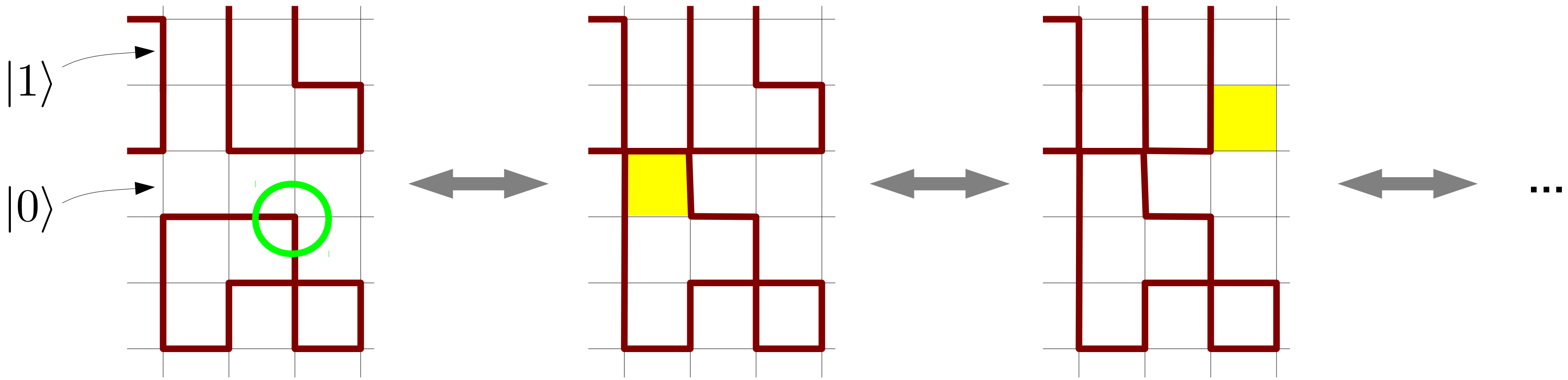
- **approximation methods** necessary – e.g. by again **using MPS**

**Projected Entangled Pair States (PEPS)
approximate two-dimensional systems faithfully,
can be used for numerical simulations,
and allow to locally encode the physics of 2D systems.**

Tensor Networks and Topological Order

The Toric Code model

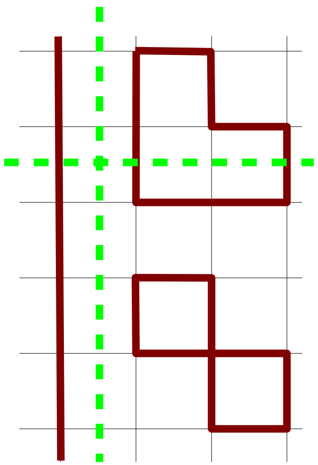
• **Toric Code**: ground state = superposition of all **loop patterns**



• Hamiltonian: (i) vertex term → enforce **closed loops**

(ii) plaquette term → **fix phase** when flipping plaquette

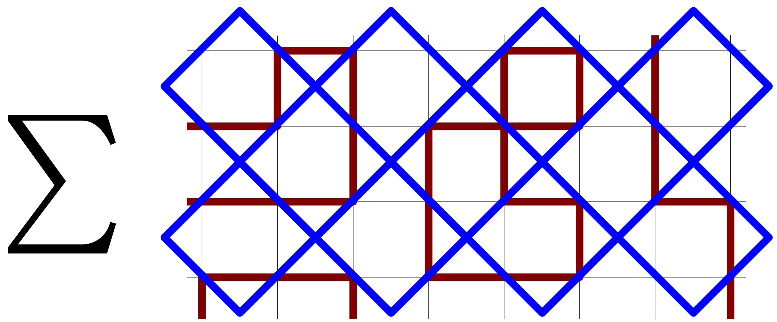
• degenerate **ground states**:
labeled by **parity of loops around torus**

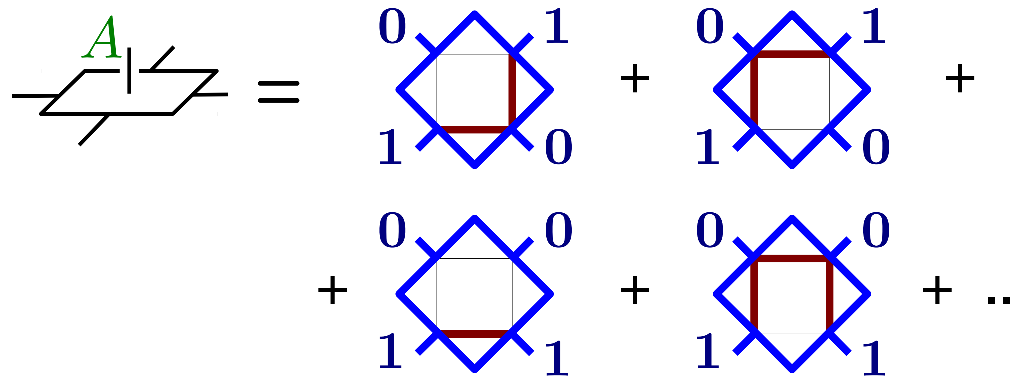


• non-trivial **excitations**:
(i) broken strings (come in pairs)
(ii) wrong relative phase (also in pairs)

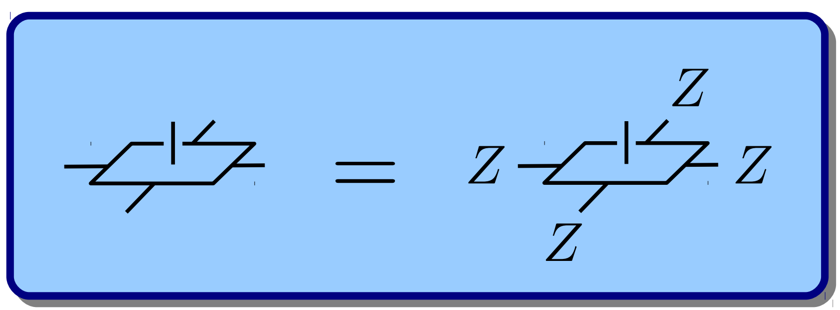
Tensor networks for topological states

- Tensor network for Toric Code:





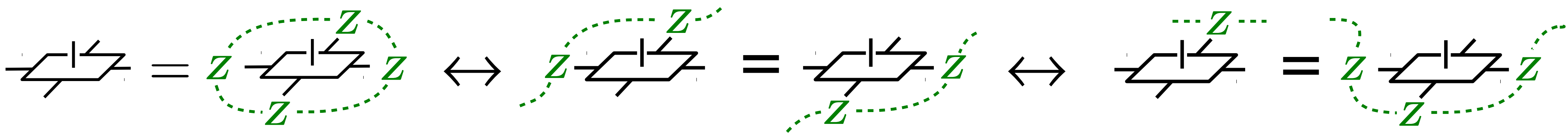
- Toric Code tensor has $\mathbb{Z}_2$ **symmetry** (=even parity):



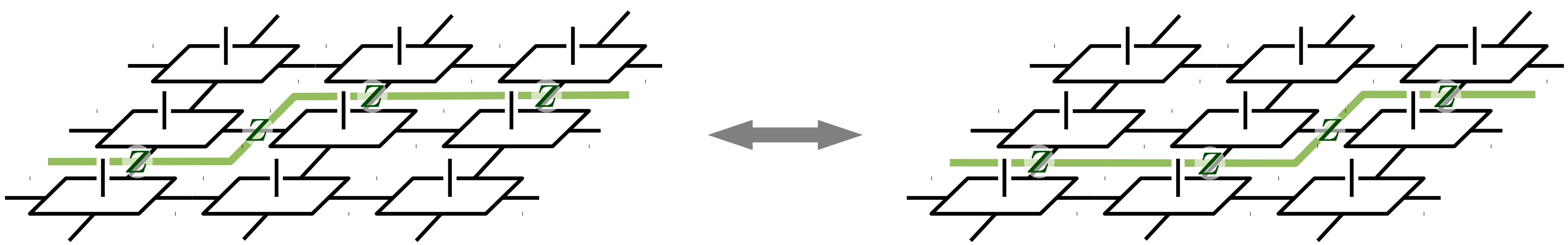
- What are consequences of such an **entanglement symmetry** in a PEPS?

Entanglement symmetry and pulling through

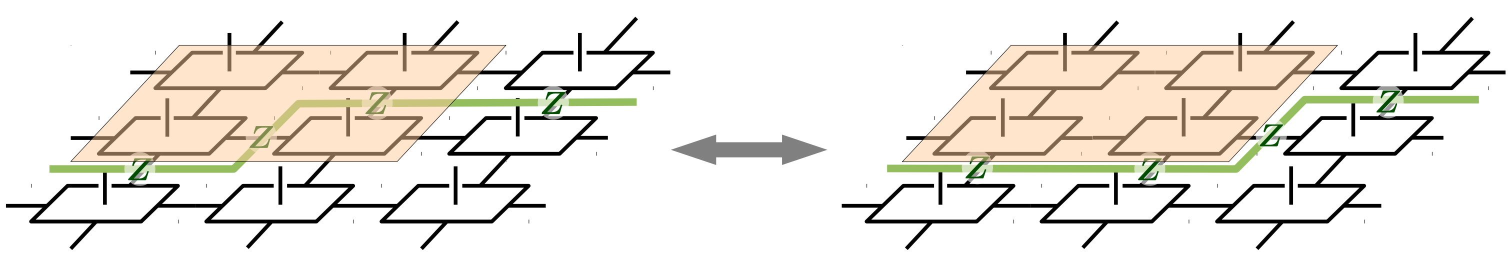
- Symmetry can be rephrased as **"pulling-through condition"**:



- pulling-through condition ⇒ **Strings can be freely moved!**



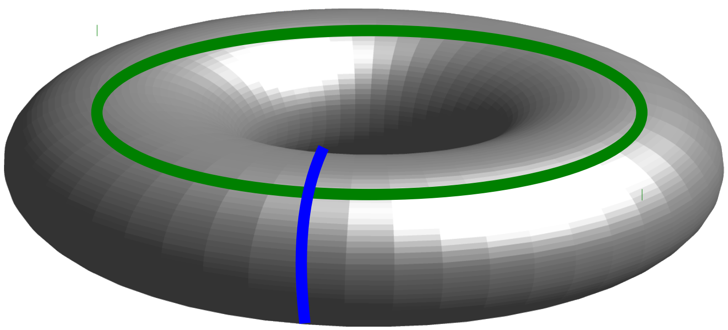
- Strings are **invisible locally** (e.g. to Hamiltonian)

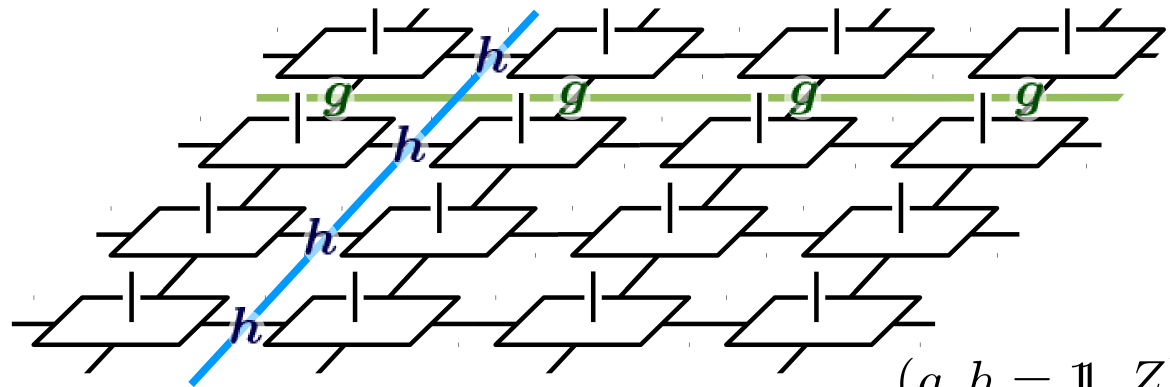


- Note: Generalization of **"pulling-through condition"** allows to characterize **all known (non-chiral) topological phases**

Topological ground space manifold

• Torus: **closed strings** yield **different ground states**





$(g, h = \mathbb{1}, Z)$

• **degeneracy** depends on **topology** (genus): **Topological order!**

$\rightarrow$ **local characterization** of topological order

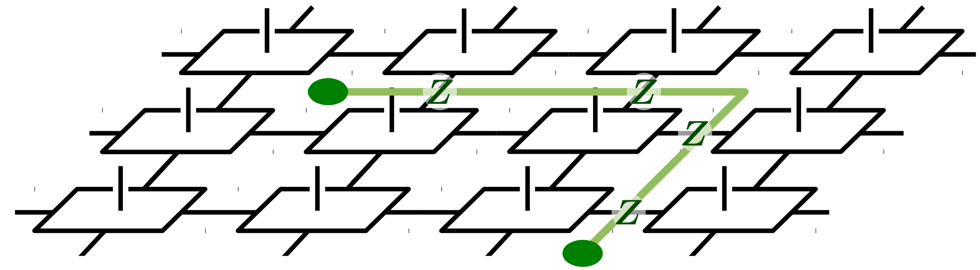
$\rightarrow$ parametrization of **ground space manifold** based on **symmetry** of **single tensor**

$\rightarrow$ gives us the tools to **explicitly construct & study ground states**

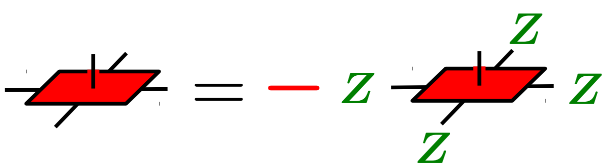
$\rightarrow$ works for systems with **finite correlation length**

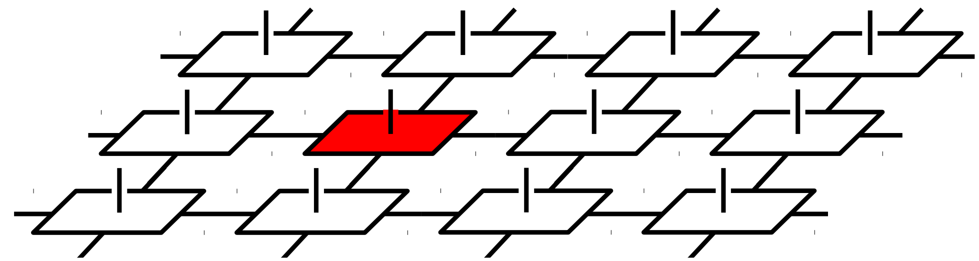
Symmetries and excitations

- **Strings w/ open ends:**
  $\rightarrow$ endpoints = **excitations**
  $\rightarrow$ excitations come in **pairs**



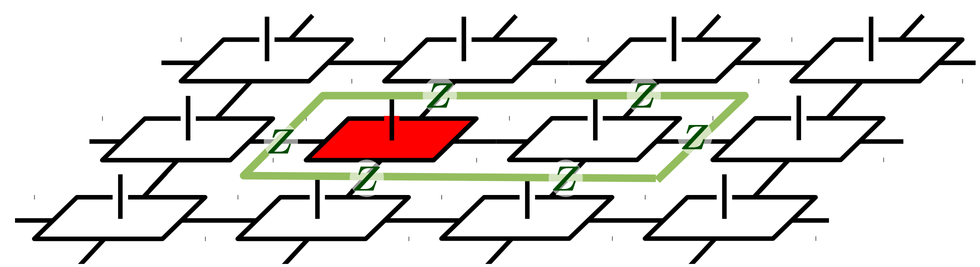
- tensors with odd parity:





  $\rightarrow$ cannot be created locally
  $\rightarrow$ must also come in **pairs**

- these two types of excitations have **non-trivial mutual statistics**!



- **modeling of anyonic excitations** from local symmetries of tensor
- fully **local description** also at finite correlation length

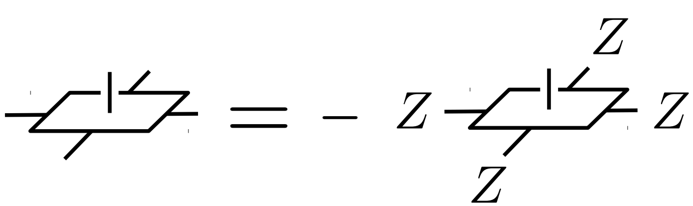
Topological order in PEPS can be comprehensively modeled based on a local entanglement symmetry.

Interplay of physical and entanglement symmetries

- spin-$\frac{1}{2}$ model: how can we **encode $\mathrm{SU}(2)$ symmetry**?

$\Rightarrow V_g$ must **combine integer & half-integer representations**!

- constraint: number of half-integer representations must be odd



$S=\frac{1}{2}$

$$Z = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$ counts half-int. spins

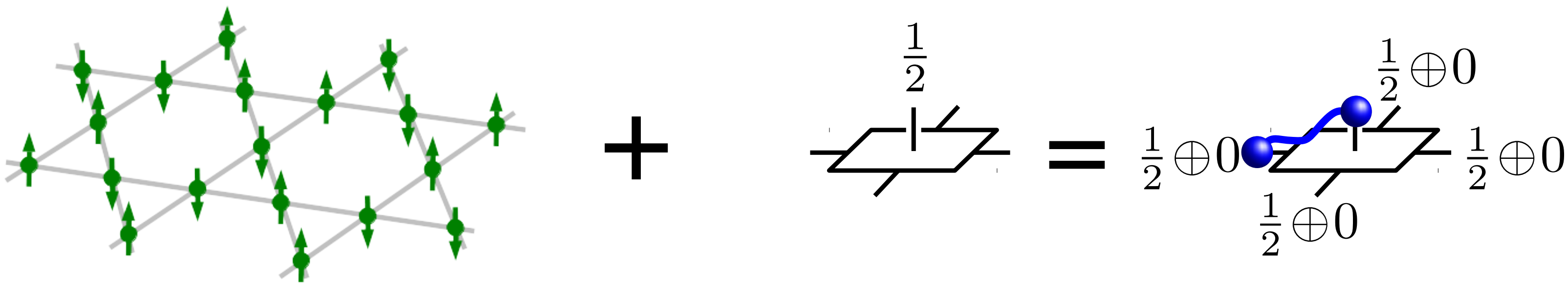
$S=0$

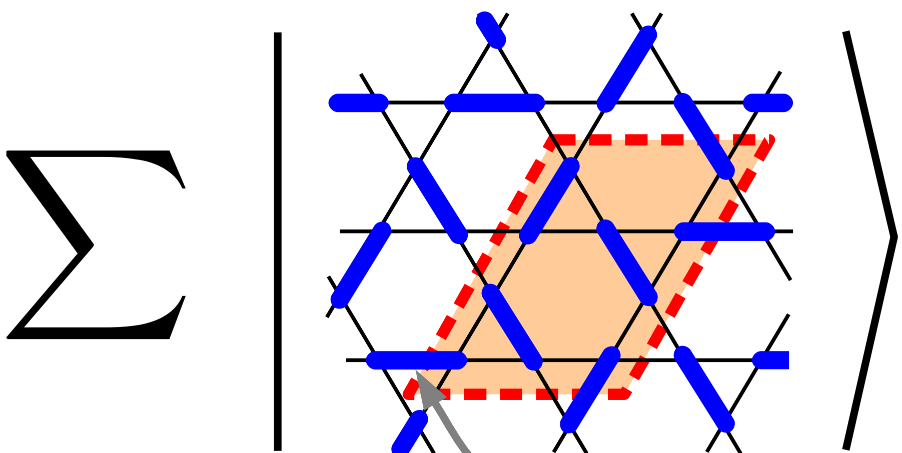
- **Entanglement symmetry** can **emerge** from physical symmetries

- Open: Full understanding of **interplay** between **physical and entanglement symmetries**!

Example: Study of Resonating Valence Bond states

• $\mathrm{SU}(2)$ **invariant PEPS** on the kagome lattice:





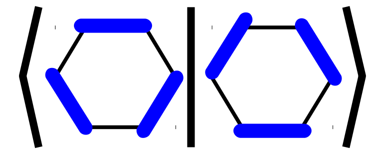
"Resonating Valence Bond State" (RVB)

singlets $\frac{1}{\sqrt{2}}|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle$

• Natural interpretation of $\mathbb{Z}_2$ **constraint**: fixed parity of singlets along cut

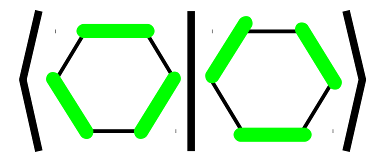
RVB and dimer models

- RVB difficult to study:
  - configurations **not orthogonal**, negative signs
  - Topological? Magnetically ordered?



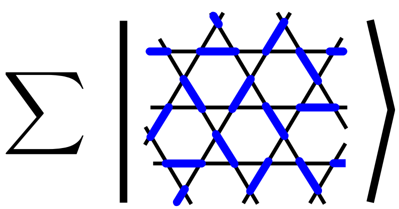
$= \frac{2}{2^{\ell/2}}$

- resort to **dimer models** with orthogonal dimers
  - can be exactly solved
  - topologically ordered

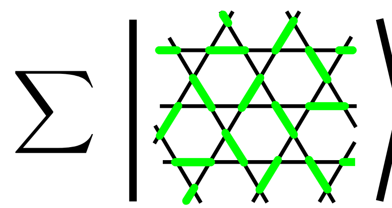


$= 0$

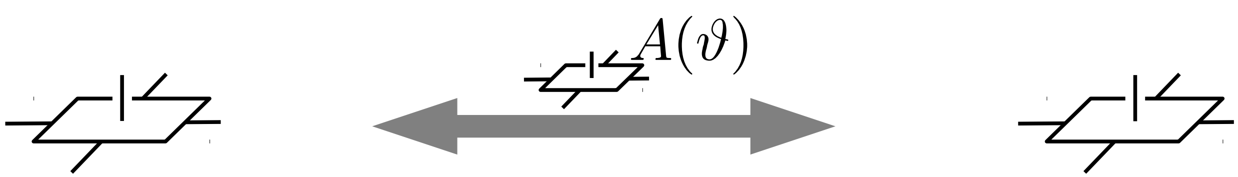
- **Interpolation** in PEPS (w/ smooth Hamiltonian!):



RVB



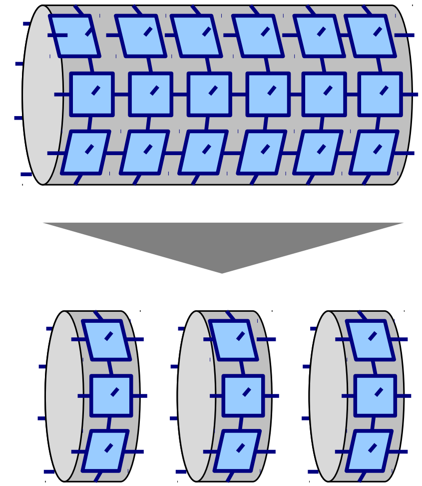
dimer model



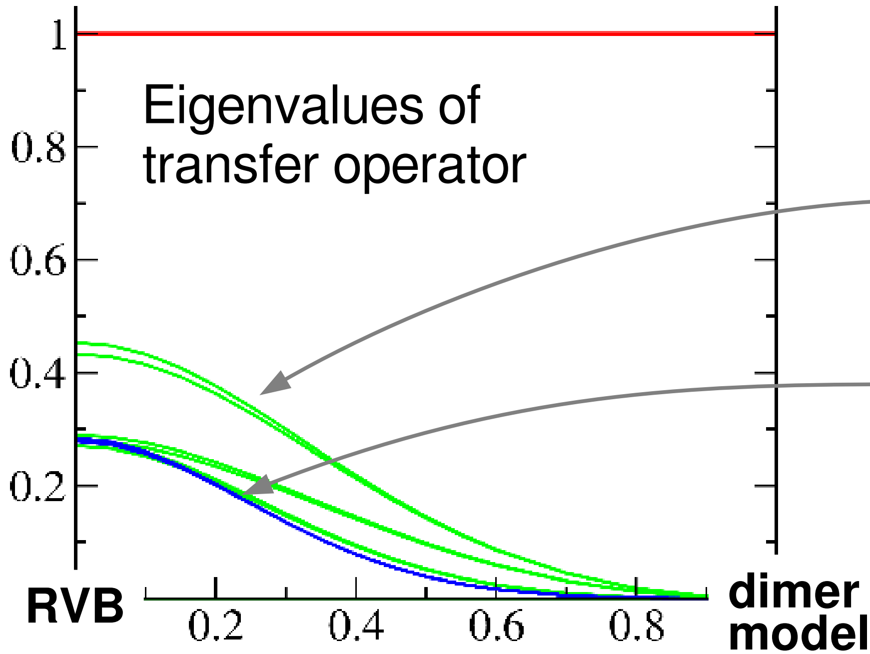
$\mathbb{Z}_2$ (topological)
$\mathrm{SU}(2)$ (physical)

$\mathbb{Z}_2$ (topological)

Numerical study of the RVB state



- numerical study of interpolation RVB $\leftrightarrow$ dimer model
- "transfer operator": **- governs all correlation functions**
  - topological sector labeled by symmetry



no overlap of topological sectors
$\Rightarrow$ **topologically ordered**

Finite correlation length
$\Rightarrow$ **no long range order**
$\Rightarrow$ spin liquid

$\Rightarrow$ RVB state on kagome lattice is a $\mathbb{Z}_2$ **topological spin liquid**

- can be proven: **RVB** is (topo. degenerate) ground state of **parent Hamiltonian**

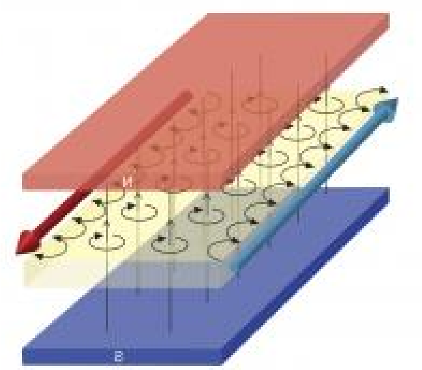
PEPS allow to study the interplay of physical and entanglement symmetries and to separately analyze their effect.

Tensor networks: boundary and entanglement

Edge physics of topological models

- Fractional Quantum Hall effect (FQHE):

edge exhibits **precisely quantized currents**
which are **robust to any perturbation**



- Such a behavior **cannot occur** in a truly **one-dimensional system**:

Physics at the **edge** has an **anomaly**!

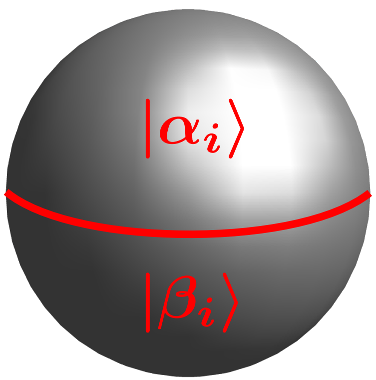
- Origin of anomalous edge physics: presence of **topologically entangled bulk**!

- **Nature of anomaly** characterizes **topological order** in the bulk

Entanglement spectra

• Entanglement spectra: [Li & Haldane, PRL '08]

FQHE state $|\psi\rangle$



$$|\psi\rangle = \sum e^{-E_i} |\alpha_i\rangle \otimes |\beta_i\rangle$$

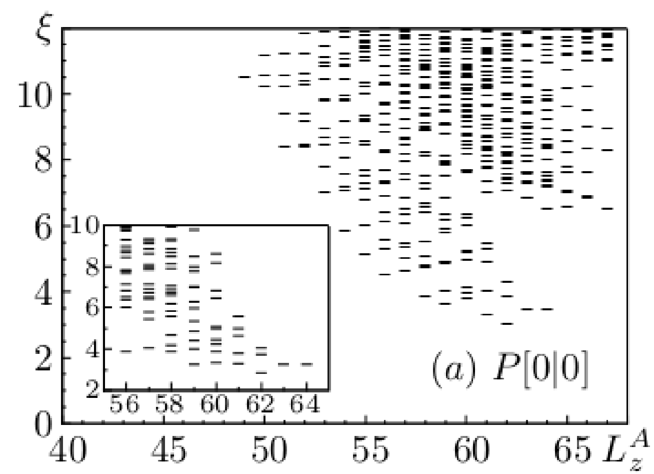
"Entanglement spectrum (ES)" $E_i \equiv E_i(k)$

momentum k associated to 1D boundary

$\rightarrow$ spectrum of 1D "entanglement Hamiltonian"?

• FQHE: **Entanglement spectrum** resembles spectrum of anomalous edge theory (a conformal field theory)

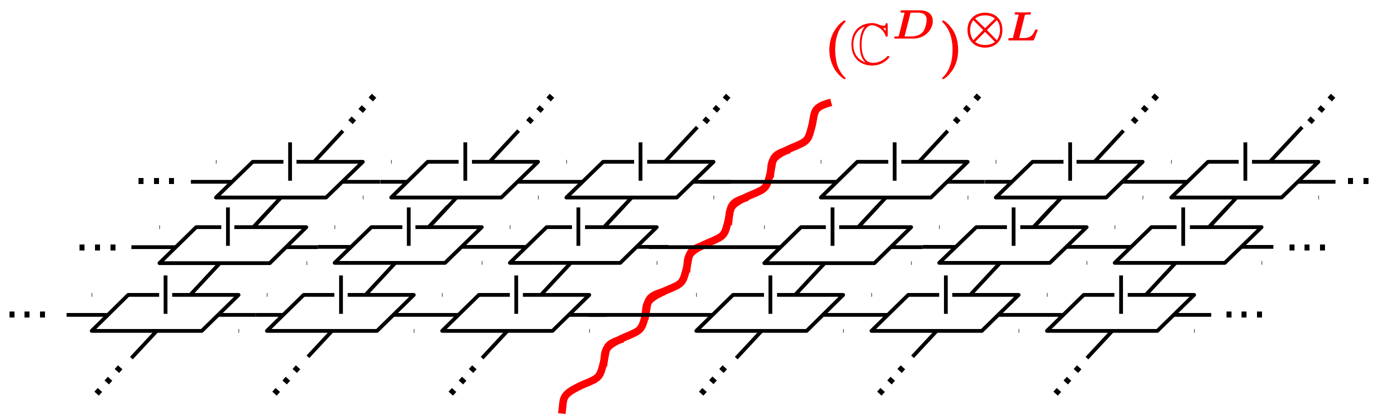
$\rightarrow$ Entanglement spectrum can help to **characterize topological phases**



• Can we understand the relation between **entanglement spectrum**, **edge physics**, and **topological order** in the bulk?

• Can we understand why the **entanglement spectrum** relates to a **1D system**?

Bulk-edge correspondence in PEPS



- Bipartition $|\Phi_{AB}\rangle = \sum_i \sqrt{p_i}|\alpha_i\rangle|\beta_i\rangle$ $\rightarrow$ **entanglement** carried by **degrees of freedom $i = (i_1, \ldots, i_L)$ at boundary**

- Allows for direct derivation of **entanglement Hamiltonian**

$$\boxed{e^{-H_{\text{ent}}} = \sigma}$$

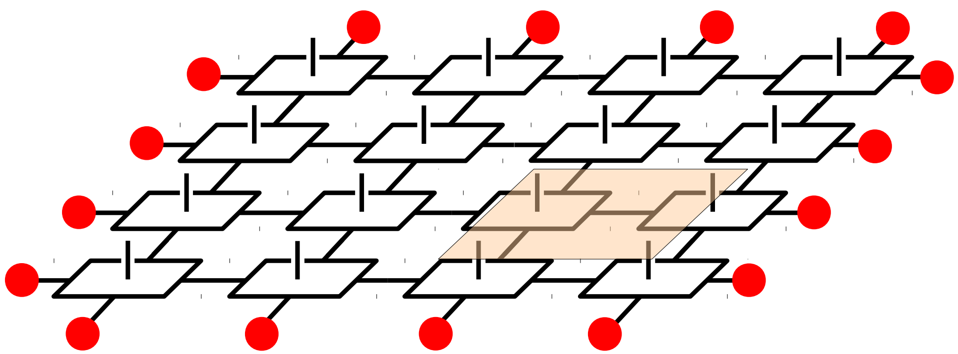
lives on **entanglement degrees of freedom**

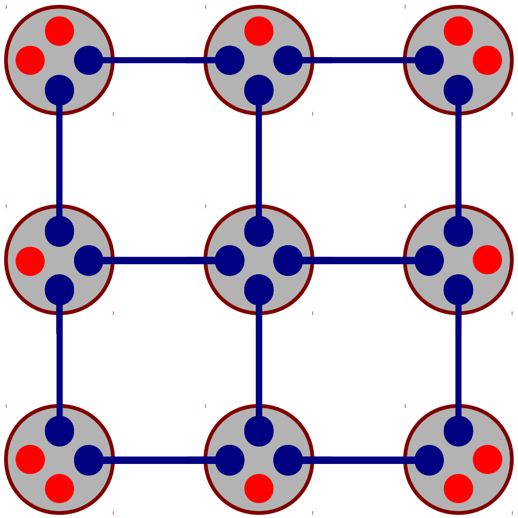
$\rightarrow$ H_{ent} has **natural 1D structure**!

- **H_{ent} inherits all symmetries** from tensor

Edge physics

- How to describe **low-energy edge physics** for parent Hamiltonian?



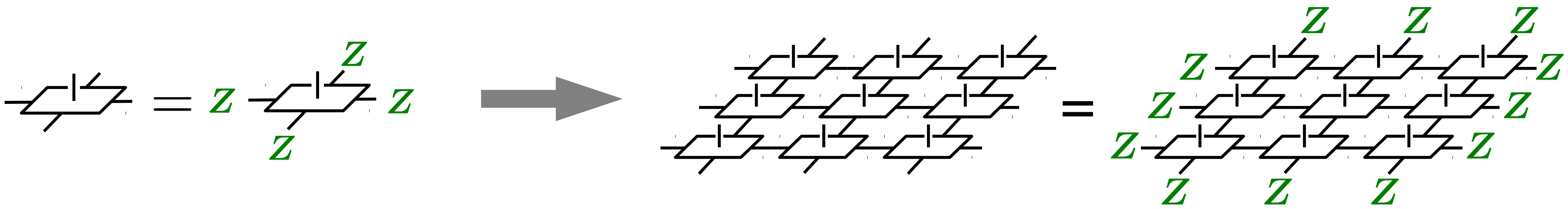


- Parametrized by choosing **all possible boundary conditions** ● !

- **Edge physics** lives on the **entanglement degrees of freedom**

Topological symmetries at the edge

- **Entanglement symmetry** inherited by the **edge**:



- **global constraint** (here, parity) on entanglement degrees of freedom:
  Only **states in even parity sector** can appear at boundary!

  $\longrightarrow$ **topological correction** to entanglement entropy

  $\longrightarrow$ entanglement Hamiltonian has an anomalous term:

$$\rho = \Pi_{\mathrm{even}} e^{-H} \Pi_{\mathrm{even}} = e^{-H+\beta_{\mathrm{topo}}\cdot H_{\mathrm{topo}}}$$

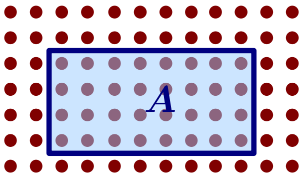
  $\longrightarrow$ **edge physics constrained** to even parity sector: anomalous!

- **entanglement spectrum** and **edge physics** exhibit the same anomaly,
  which originates in the **topological order in the bulk**

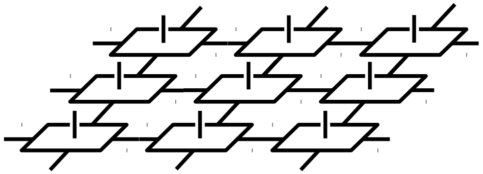
PEPS provide a natural one-dimensional Hilbert space which describes the edge physics and entanglement spectrum, and yield an explicit connection between edge physics, entanglement spectrum, and bulk topological order.

Summary

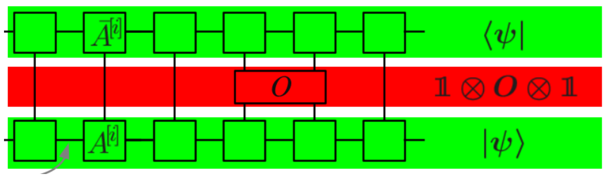
- **Entanglement** of quantum many-body systems: **Area law**



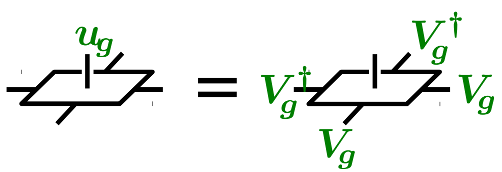
- **Matrix Product States** and **PEPS**:
  build entanglement locally



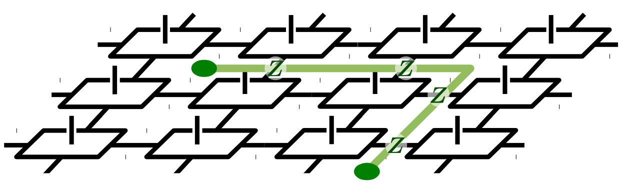
- Efficient approximation: powerful **numerical tool**



- Framework to **study structure** of many-body systems



- **Topological order** $\leftrightarrow$ entanglement symmetry



- Explicit 1D Hilbert space for entanglement
  $\rightarrow$ study of **entanglement spectra & edge physics**