

# Optimal Discovery Strategies in White Space Networks

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**Abstract.** The whitespace-discovery problem describes two parties, Alice and Bob, trying to discover one another and establish communication over one of a given large segment of communication channels. Subsets of the channels are occupied in each of the local environments surrounding Alice and Bob, as well as in the global environment (Eve). In the absence of a common clock for the two parties, the goal is to devise time-invariant (stationary) strategies minimizing the discovery time.

We model the problem as follows. There are  $N$  channels, each of which is open (unoccupied) with probability  $p_1, p_2, q$  independently for Alice, Bob and Eve respectively. Further assume that  $N \gg 1/(p_1 p_2 q)$  to allow for sufficiently many open channels. Both Alice and Bob can detect which channels are locally open and every time-slot each of them chooses one such channel for an attempted discovery. One aims for strategies that, with high probability over the environments, guarantee a shortest possible expected discovery time depending only on the  $p_i$ 's and  $q$ .

Here we provide a stationary strategy for Alice and Bob with a guaranteed expected discovery time of  $O(1/(p_1 p_2 q^2))$  given that each party also has knowledge of  $p_1, p_2, q$ . When the parties are oblivious of these probabilities, analogous strategies incur a cost of a poly-log factor, i.e.  $\tilde{O}(1/(p_1 p_2 q^2))$ . Furthermore, this performance guarantee is essentially optimal as we show that any stationary strategies of Alice and Bob have an expected discovery time of at least  $\Omega(1/(p_1 p_2 q^2))$ .

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## 1 Introduction

Consider two parties, Alice and Bob, who wish to establish a communication channel in one out of a segment of  $N$  possible channels. Subsets of these channels may already be occupied in the local environments of either Alice or Bob, as well as in the global environment in between them whose users are denoted by Eve. Furthermore, the two parties do not share a common clock and hence one does not know for how long (if at all) the other party has already been trying to communicate. Motivated by applications in discovery of wireless devices, the goal is thus to devise time-invariant strategies that ensure fast discovery with high probability (w.h.p.) over the environments.

We formalize the above problem as follows. Transmissions between Alice and Bob go over three environments: local ones around Alice and Bob and an additional global one in between them, Eve. Let  $A_i, B_i, E_i$  for  $i = 1, \dots, N$  be the indicators for whether a given channel is open (unoccupied) in the respective environment. Using local diagnostics Alice knows  $A$  yet does not know  $B, E$  and analogously Bob knows  $B$  but is oblivious of  $A, E$ . In each time-slot, each party selects a channel to attempt communication on (the environments do not change between time slots). The parties are said to *discover* one another once they select the same channel  $i$  that happens to be open in all environments (i.e.,  $A_i = B_i = E_i = 1$ ). The objective of Alice and Bob is to devise strategies that would minimize their expected discovery time.

For a concrete setup, let  $A_i, B_i, E_i$  be independent Bernoulli variables with probabilities  $p_1, p_2, q$  respectively for all  $i$ , different channels being independent of each other. (In some applications the two parties have knowledge of the environment densities  $p_1, p_2, q$  while in others these are unknown.) Alice and Bob then seek strategies whose expected discovery time over the environments is minimal.

**Example.** Suppose that  $p_1 = p_2 = 1$  (local environments are fully open) and Alice and Bob use the naive strategy of selecting a channel uniformly over  $[N]$  and independently every round. If there are  $Q \approx qN$  open channels in the global environment Eve then the probability of discovery in a given round is  $Q/N^2 \approx q/N$ , implying an expected discovery time of about  $N/q$  to the very least.

**Example.** Consider again the naive uniform strategy, yet in this example Alice and Bob examine their local environment and each of them selects a channel uniformly over the locally open ones. Suppose for simplicity that  $p_1 = p_2 = p$  for some fixed  $0 < p < 1$  whereas  $q = 1$  (there is no global environment interference), and of-course assume that there *exist* commonly open channels (the probability of not having such channels is exponentially small in  $N$ ). Then each of Alice and Bob has a total of about  $pN$  open channels and the probability that their choice is identical in a given round is about  $1/N$ . In particular, the uniform strategy has an expected discovery time of about  $N$  rounds, diverging with  $N$  despite the clear fact that as  $N$  grows there are more commonly open channels for Alice and Bob. Our main theorem will show in particular that in the above scenario Alice and Bob can have an  $O(1)$  expected discovery time.

In the above framework it could occur that *all* channels are closed, in which case the parties can never discover; as a result, unless this event is excluded the expected discovery time is always infinite. However, since this event has probability at most  $(1 - p_1 p_2 q)^N \leq \exp(-N p_1 p_2 q)$  it poses no real problem for applications (described in further details later) where  $N \gg 1/(p_1 p_2 q)$ . In fact, we aim for performance guarantees that depend only on  $p_1, p_2, q$  rather than on  $N$ , hence a natural way to resolve this issue is to extend the set of channels to be infinite, i.e. define  $A_i, B_i, E_i$  for every  $i \in \mathbb{N}$ . (Our results can easily be translated to the finite setting with the appropriate exponential error probabilities.)

A *strategy* is a sequence of probability measures  $\{\mu^t\}$  over  $\mathbb{N}$ , corresponding to a randomized choice of channel for each time-slot  $t \geq 1$ . Suppose that Alice begins the discovery via the strategy  $\mu_a$  whereas Bob begins the discovery attempt at time  $s$  via the strategy  $\mu_b$ . Let  $X_t$  be the indicator for a successful discovery at time  $t$  and let  $X$  be the first time Alice and Bob discover, that is

$$\mathbb{P}(X_t = 1 \mid A, B, E) = \sum_j \mu_a^t(j) \mu_b^{s+t}(j) A_j B_j E_j, \quad (1)$$

$$X = \min\{t : X_t = 1\}. \quad (2)$$

The choice of  $\mu_a, \mu_b$  aims to minimize  $\mathbb{E}X$  where the expectation is over  $A, B, E$  as well as the randomness of Alice and Bob in applying the strategies  $\mu_a, \mu_b$ .

**Example (*fixed strategies*).** Suppose that both Alice and Bob apply the same pair of strategies independently for all rounds,  $\mu_a$  and  $\mu_b$  respectively. In this special case, given the environments  $A, B, E$  the random variable  $X$  is geometric with success probability  $\sum_j \mu_a(j)\mu_b(j)A_jB_jE_j$ , thus the mappings  $A \mapsto \mu_a$  and  $B \mapsto \mu_b$  should minimize the value of  $\mathbb{E}X = \mathbb{E} \left[ \left( \sum_j \mu_a(j)\mu_b(j)A_jB_jE_j \right)^{-1} \right]$ .

A crucial fact in our setup is that Alice and Bob have no common clock and no means of telling whether or not their peer is already attempting to communicate (until they eventually discover). As such, they are forced to apply a *stationary* strategy, where the law at each time-slot is identical (i.e.  $\mu^t \sim \mu^1$  for all  $t$ ). For instance, Alice may choose a single  $\mu_a$  and apply it independently in each step (cf. above example). Alternatively, strategies of different time-slots can be highly dependent, e.g. Bob may apply a periodic policy given by  $n$  strategies  $\mu_b^1, \dots, \mu_b^n$  and a uniform initial state  $s \in [n]$ .

The following argument demonstrates that stationary strategies are essentially optimal when there is no common clock between the parties. Suppose that Alice has some finite (arbitrarily long) sequence of strategies  $\{\mu_a^t\}_1^{M_a}$  and similarly Bob has a sequence of strategies  $\{\mu_b^t\}_1^{M_b}$ . With no feedback until any actual discovery we may assume that the strategies are non-adaptive, i.e. the sequences are determined in advance. Without loss of generality Alice is joining the transmission after Bob has already attempted some  $\beta$  rounds of communication, in which case the expected discovery time is  $\mathbb{E}_{0,\beta}X$ , where  $\mathbb{E}_{\alpha,\beta}X$  denotes the expectation of  $X$  as defined in (1),(2) using the strategies  $\{\mu_a^{t+\alpha}\}, \{\mu_b^{t+\beta}\}$ . Having no common clock implies that in the worst case scenario (over the state of Bob) the expected time to discover is  $\max_{\beta} \mathbb{E}_{0,\beta}X$  and it now follows that Bob is better off modifying his strategy into a stationary one by selecting  $\beta \in [M_b]$  uniformly at random, leading to an expected discovery time of  $M_b^{-1} \sum_{\beta} \mathbb{E}_{0,\beta}X$ .

### 1.1 Optimal Discovery Strategies

Our main result is a recipe for Alice and Bob to devise stationary strategies guaranteeing an optimal expected discovery time up to an absolute constant factor, assuming they know the environment densities  $p_1, p_2, q$  (otherwise the expected discovery time is optimal up to a poly-log factor).

**Theorem 1.** Consider the discovery problem with probabilities  $p_1, p_2, q$  for the environments  $A, B, E$  respectively and let  $X$  denote the expected discovery time. The following then holds:

- (i) There are fixed strategies for Alice and Bob guaranteeing an expected discovery time of  $\mathbb{E}X = O(1/(p_1 p_2 q^2))$ , namely:
  - Alice takes  $\mu_a \sim \text{Geom}(p_2 q/6)$  over her open channels  $\{i : A_i = 1\}$ ,
  - Bob takes  $\mu_b \sim \text{Geom}(p_1 q/6)$  over his open channels  $\{i : B_i = 1\}$ .
Furthermore, for any fixed  $\varepsilon > 0$  there are fixed strategies for Alice and Bob that do not require knowledge of  $p_1, p_2, q$  and guarantee  $\mathbb{E}X = O\left(\frac{1}{p_1 p_2 q^2} \log^{2+\varepsilon}\left(\frac{1}{p_1 p_2 q}\right)\right) = \tilde{O}\left(\frac{1}{p_1 p_2 q^2}\right)$ , obtained by taking  $\mu_a(j\text{-th open } A \text{ channel}) = \mu_b(j\text{-th open } B \text{ channel}) \propto (j \log^{1+\varepsilon/2} j)^{-1}$ .
- (ii) The above strategies are essentially optimal as every possible choice of stationary strategies by Alice and Bob satisfies  $\mathbb{E}X = \Omega(1/(p_1 p_2 q^2))$ .

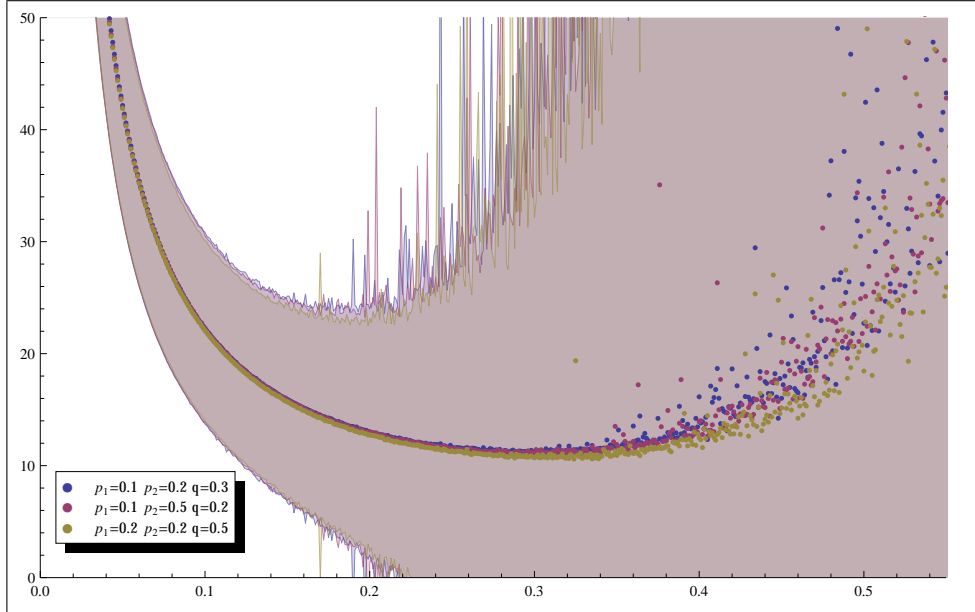
**Remark.** The factor  $1/6$  in the parameters of the geometric distributions can be fine-tuned to any smaller (or even slightly larger) fixed  $\alpha > 0$  affecting the expected discovery time  $\mathbb{E}X$  by a multiplicative constant. See Fig. 1 for a numerical evaluation of  $\mathbb{E}X$  for various values of  $\alpha$ .

Recall that Alice and Bob must apply stationary strategies in the absence of any common clock or external synchronization device shared by them, a restriction which is essential in many of the applications of wireless discovery protocols. However, whenever a common external clock does happen to be available there may be strategies that achieve improved performance. The next theorem, whose short proof appears in the full version of the paper, establishes the optimal strategies in this simpler scenario.

**Theorem 2.** Consider the discovery problem with probabilities  $p_1, p_2, q$  for the environments  $A, B, E$  respectively and let  $X$  denote the expected discovery time. If Alice and Bob have access to a common clock then there are non-stationary strategies for them giving  $\mathbb{E}X = O(1/(\min\{p_1, p_2\}q))$ . Furthermore, this is tight as the expected discovery time for any strategies always satisfies  $\mathbb{E}X = \Omega(1/(\min\{p_1, p_2\}q))$ .

## 1.2 Applications in wireless networking and related work

The motivating application for this work comes from recent developments in wireless networking. In late 2008, the FCC issued a historic



**Fig. 1.** discovery time  $\mathbb{E}X$  as in (2) normalized by a factor of  $p_1 p_2 q^2$  for a protocol using geometric distributions with parameters  $\alpha p_i q$  for various values of  $0 < \alpha < 1$ . Markers represent the average of the expected discovery time  $\mathbb{E}X$  over  $10^5$  random environments with  $n = 10^4$  channels; surrounding envelopes represent a window of one standard deviation around the mean.

ruling permitting the unlicensed use of unused portions of the wireless RF spectrum (mainly the part between 512Mhz and 698Mhz, i.e., the UHF spectrum), popularly referred to as “White Spaces” [7]. Due to the potential for substantial bandwidth and long transmission ranges, whitespace networks (which are sometimes also called cognitive radio networks) represent a tremendous opportunity for mobile and wireless communication, and consequently, there has recently been significant interest on white space networking in the networking research community, e.g. [5, 6] as well as industry. One critical rule imposed by the FCC in its ruling is that wireless devices operating over white spaces must not interfere with incumbents, i.e., the current users of this spectrum (specifically, in the UHF bands, these are TV broadcasters as well as licensed wireless microphones). These incumbents are considered “primary users” of the spectrum, while whitespace devices are secondary users and are allowed to use the spectrum only opportunistically, whenever no primary user is

using it (The FCC originally mandated whitespace devices to detect the presence of primary users using a combination of sensing techniques and a geo-location database, but in a recent amendment requires only the geo-location database approach [8]). At any given time, each whitespace device thus has a spectrum map on which some parts are blocked off while others are free to use.

The problem studied in this paper captures (and in fact even generalizes) the situation in whitespace networks when two nodes  $A$  and  $B$  seek to discover one another to establish a connection. Each node knows its own free channels on which it can transmit, but it does not know which of these channels may be available at the other node, too. Furthermore, given the larger transmission range in whitespace networks (up to a mile at Wi-Fi transmission power levels), it is likely that the spectrum maps at  $A$  and  $B$  are similar yet different. For example, a TV broadcast tower is likely to block off a channel for both  $A$  and  $B$ , but a wireless microphone — due to its small transmission power — will prevent only one of the nodes from using a channel.

Thus far, the problem of synchronizing/discovery of whitespace nodes has only been addressed when one of the nodes is a fixed access point (AP) and the other node is a client. Namely, in the framework studied in [5] the AP broadcasts on a fixed channel and the client node wishes to scan its local environment and locate this channel efficiently. That setting thus calls for technological solutions (e.g. based on scanning wider channel widths) to allow the client to find the AP channel faster than the approach of searching all possible channels one by one.

To the best of our knowledge, the results in this paper are the first to provide an efficient discovery scheme in the setting where both nodes are remote clients that may broadcast on any given channel in the whitespace region.

### 1.3 Related work on Rendezvous games

From a mathematical standpoint, the discovery problems considered in this paper seem to belong to the field of Rendezvous Search Games. The most familiar problem of this type is known as The Telephone Problem or The Telephone Coordination Game. In the telephone problem each of two players is placed in a distinct room with  $n$  telephone lines connecting

the rooms. The lines are not labeled and so the players, who wish to communicate with each other, cannot simply use the first line (note that, in comparison, in our setting the channels are labeled and the difficulty in discovery is due to the local and global noise).

The optimal strategy in this case, achieving an expectation of  $n/2$ , is for the first player to pick a random line and continue using it, whereas the second player picks a uniformly random permutation on the lines and try them one by one. However, this strategy requires the players to determine which is the first and which is the second. It is very plausible that such coordination is not possible, in which case we require both players to employ the same strategy.

The obvious solution is for each of them to pick a random line at each turn, which gives an expectation of  $n$  turns. It turns out, however, that there are better solutions: Anderson and Weber [4] give a solution yielding an expectation of  $\approx 0.8288497n$  and conjecture it's optimality.

To our knowledge, the two most prominent aspects of our setting, the presence of asymmetric information and the stationarity requirement (stemming from unknown start times) have not been considered in the literature. For example, the Anderson-Weber strategy for the telephone problem is not stationary — it has a period of  $n-1$ . It would be interesting to see what can be said about the optimal stationary strategies for this and other rendezvous problems. The interested reader is referred to [2, 3] and the references therein for more information on rendezvous search games.

## 2 Analysis of Discovery Strategies

### 2.1 Proof of Theorem 1, upper bound on the discovery time

Let  $\mu_a$  be geometric with mean  $(\alpha p_2 q)^{-1}$  over the open channels for Alice  $\{i : A_i = 1\}$  and analogously let  $\mu_b$  be geometric with mean  $(\alpha p_1 q)^{-1}$  over the open channels for Bob  $\{i : B_i = 1\}$ , where  $0 < \alpha < 1$  will be determined later.

Let  $J = \min\{j : A_j = B_j = E_j = 1\}$  be the minimal channel open in all three environments. Further let  $J_a, J_b$  denote the number of locally open channels prior to channel  $J$  for Alice and Bob resp., that is

$$J_a = \#\{j < J : A_j = 1\}, \quad J_b = \#\{j < J : B_j = 1\}.$$



Finally, for some integer  $k \geq 0$  let  $M_k$  denote the event

$$k \leq \max \{J_a p_2 q, J_b p_1 q\} < k + 1. \quad (3)$$

Notice that, by definition, Alice gives probability  $(1 - \alpha p_2 q)^{j-1} \alpha p_2 q$  to her  $j$ -th open channel while Bob gives probability  $(1 - \alpha p_1 q)^{j-1} \alpha p_1 q$  to his  $j$ -th open channel. Therefore, on the event  $M_k$  we have that in any specific round, channel  $J$  is chosen by both players with probability at least

$$(1 - \alpha p_1 q)^{\frac{k+1}{p_1 q}} (1 - \alpha p_2 q)^{\frac{k+1}{p_2 q}} \alpha^2 p_1 p_2 q^2 \geq e^{-4\alpha(k+1)} \alpha^2 p_1 p_2 q^2,$$

where in the last inequality we used the fact that  $(1 - x) \geq \exp(-2x)$  for all  $0 \leq x \leq \frac{1}{2}$ , which will be justified by later choosing  $\alpha < \frac{1}{2}$ . Therefore, if  $X$  denotes the expected number of rounds required for discovery, then

$$\mathbb{E}[X \mid M_k] \leq e^{4\alpha(k+1)} (\alpha^2 p_1 p_2 q^2)^{-1}. \quad (4)$$

On the other hand,  $J_a$  is precisely a geometric variable with the rule  $\mathbb{P}(J_a = j) = (1 - p_2 q)^j p_2 q$  and similarly  $\mathbb{P}(J_b = j) = (1 - p_1 q)^j p_1 q$ . Hence,

$$\mathbb{P}(M_k) \leq (1 - p_2 q)^{k/(p_2 q)} + (1 - p_1 q)^{k/(p_1 q)} \leq 2e^{-k}.$$

Combining this with (4) we deduce that

$$\begin{aligned} \mathbb{E}X &\leq 2 \sum_k e^{-k} \mathbb{E}[X \mid M_k] \leq 2e^{4\alpha} (\alpha^2 p_1 p_2 q^2)^{-1} \sum_k e^{(4\alpha-1)k} \\ &\leq \frac{2e}{\alpha^2 (e^{1-4\alpha} - 1)} (p_1 p_2 q^2)^{-1} \end{aligned} \quad (5)$$

where the last inequality holds for any fixed  $\alpha < \frac{1}{4}$ . In particular, a choice of  $\alpha = \frac{1}{6}$  implies that  $\mathbb{E}X \leq 500 / (p_1 p_2 q^2)$ , as required.  $\square$

**Remark.** In the special case where  $p_1 = p_2$  (denoting this probability simply by  $p$ ) one can optimize the choice of constants in the proof above to obtain an upper bound of  $\mathbb{E}X \leq 27 / (pq)^2$ .

Due to space constraints, we postpone the argument establishing discovery strategies oblivious of the environment densities to the full version of the paper.

## 2.2 Proof of Theorem 1, lower bound on the discovery time

**Theorem 3.** *Let  $\mu_a, \mu_b$  be the stationary distribution of the strategies of Alice and Bob resp., and let  $R = \sum_j \mu_a(j)\mu_b(j)A_jB_jE_j$  be the probability of successfully discovering in any specific round. Then there exists some absolute constant  $C > 0$  such that  $\mathbb{P}(R < Cp_0p_1q^2) \geq \frac{1}{2}$ .*

*Proof.* Given the environments  $A, B$  define

$$S_k^a = \{j : 2^{-k} < \mu_a(j) \leq 2^{-k+1}\}, \quad S_k^b = \{j : 2^{-k} < \mu_b(j) \leq 2^{-k+1}\}.$$

Notice that the variables  $S_k^a$  are a function of the strategy of Alice which in turn depends on her local environment  $A$  (an analogous statement holds for  $S_k^b$  and  $B$ ). Further note that clearly  $|S_k^a| < 2^k$  and  $|S_k^b| < 2^k$  for any  $k$ . Let  $T_k^a$  denote all the channels where the environments excluding Alice's (i.e., both of the other environments  $B, E$ ) are open, and similarly let  $T_k^b$  denote the analogous quantity for Bob:

$$T_k^a = \{j \in S_k^a : B_j = E_j = 1\}, \quad T_k^b = \{j \in S_k^b : A_j = E_j = 1\}.$$

Obviously,  $\mathbb{E}|T_k^a| < 2^k p_2 q$  and  $\mathbb{E}|T_k^b| < 2^k p_1 q$ .

Since  $\{B_j\}_{j \in \mathbb{N}}$  and  $\{E_j\}_{j \in \mathbb{N}}$  are independent of  $S_k^a$  (and of each other), for any  $\beta > 0$  we can use the Chernoff bound (see, e.g., [9, Theorem 2.1] and [1, Appendix A]) with a deviation of  $t = (\beta - 1)2^k p_2 q$  from the expectation to get

$$\mathbb{P}\left(|T_k^a| > \beta 2^k p_2 q\right) < \exp\left(-\frac{3(\beta - 1)^2}{2(\beta + 2)} 2^k p_2 q\right),$$

and analogously for Bob we have

$$\mathbb{P}\left(|T_k^b| > \beta 2^k p_1 q\right) < \exp\left(-\frac{3(\beta - 1)^2}{2(\beta + 2)} 2^k p_1 q\right).$$

Clearly, setting  $K_a = \log_2(1/(p_2 q)) - 3$  and  $K_b = \log_2(1/(p_1 q)) - 3$  and taking  $\beta$  large enough (e.g.,  $\beta = 20$  would suffice) we get

$$\mathbb{P}\left(\bigcup_{k \geq K_a} \left\{|T_k^a| > \beta 2^k p_2 q\right\}\right) \leq 2 \mathbb{P}\left(|T_{K_a}^a| > \beta 2^{K_a} p_2 q\right) < \frac{1}{8} \quad (6)$$

and

$$\mathbb{P}\left(\bigcup_{k \geq K_b} \left\{|T_k^b| > \beta 2^k p_1 q\right\}\right) < \frac{1}{8}. \quad (7)$$

Also, since  $\sum_{k < K_a} |S_k^a| < 2^{K_a} \leq (8p_2q)^{-1}$  and similarly  $\sum_{k < K_b} |S_k^b| < 2^{K_b} \leq (8p_1q)^{-1}$ , we have by Markov's inequality that

$$\mathbb{P}\left(\bigcup_{k < K_a} \{|T_k^a| > 0\}\right) \leq \sum_{k < K_a} \mathbb{E}|T_k^a| = p_2q \sum_{k < K_a} \mathbb{E}|S_k^a| < \frac{1}{8} \quad (8)$$

and similarly

$$\mathbb{P}\left(\bigcup_{k < K_b} \{|T_k^b| > 0\}\right) < \frac{1}{8}. \quad (9)$$

Putting together (6),(7),(8),(9), with probability at least  $\frac{1}{2}$  the following holds:

$$|T_k^a| \leq \begin{cases} \beta 2^k p_2 q & k \geq K_a \\ 0 & k < K_a \end{cases}, \quad |T_k^b| \leq \begin{cases} \beta 2^k p_1 q & k \geq K_b \\ 0 & k < K_b \end{cases} \quad \text{for all } k. \quad (10)$$

When (10) holds we can bound  $R$  as follows:

$$\begin{aligned} R &= \sum_j \mu_a(j) \mu_b(j) A_j B_j E_j = \sum_k \sum_\ell \sum_{j \in T_k^a \cap T_\ell^b} \mu_a(j) \mu_b(j) \\ &\leq \sum_k \sum_\ell |T_k^a \cap T_\ell^b| 2^{-k+1} 2^{-\ell+1} \\ &\leq \sum_k \sum_\ell \sqrt{|T_k^a| |T_\ell^b|} 2^{-k+1} 2^{-\ell+1} = 4 \left( \sum_k \sqrt{|T_k^a|} 2^{-k} \right) \left( \sum_\ell \sqrt{|T_\ell^b|} 2^{-\ell} \right) \\ &\leq 4\beta(p_1 p_2)^{1/2} q \left( \sum_{k \geq K_a} 2^{-k/2} \right) \left( \sum_{\ell \geq K_b} 2^{-\ell/2} \right), \end{aligned}$$

where the second inequality used the fact that  $|F_1 \cap F_2| \leq \min\{|F_1|, |F_2|\} \leq \sqrt{|F_1| |F_2|}$  for any two finite sets  $F_1, F_2$  and the last inequality applied (10). From here the proof is concluded by observing that

$$R \leq 16(p_1 p_2)^{1/2} q 2^{-K_a/2} 2^{-K_b/2} = 128\beta p_1 p_2 q^2. \quad \square$$

**Corollary 4.** *There exists some absolute  $c > 0$  such that for any pair of stationary strategies, the expected number of rounds required for a successful discovery is at least  $c/(p_1 p_2 q^2)$ .*

*Proof.* Conditioned on the value of  $R$ , the probability of discovery in one of the first  $1/(2R)$  rounds is at most  $\frac{1}{2}$ . Theorem 3 established that with probability at least  $\frac{1}{2}$  we have  $R < Cp_1 p_2 q^2$ , therefore altogether with probability at least  $\frac{1}{4}$  there is no discovery before time  $(2Cp_1 p_2 q^2)^{-1}$ . We conclude that the statement of the corollary holds with  $c = 1/(8C)$ .  $\square$

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