

# An Improved Integrality Gap for Asymmetric TSP Paths

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**Abstract.** The Asymmetric Traveling Salesperson Path (ATSP) problem is one where, given an *asymmetric* metric space  $(V, d)$  with specified vertices  $s$  and  $t$ , the goal is to find an  $s$ - $t$  path of minimum length that visits all the vertices in  $V$ .

This problem is closely related to the Asymmetric TSP (ATSP) problem, which seeks to find a tour (instead of an  $s$ - $t$  path) visiting all the nodes: for ATSP, a  $\rho$ -approximation guarantee implies an  $O(\rho)$ -approximation for ATSP. However, no such connection is known for the *integrality gaps* of the linear programming relaxations for these problems: the current-best approximation algorithm for ATSP is  $O(\log n / \log \log n)$ , whereas the best bound on the integrality gap of the natural LP relaxation (the subtour elimination LP) for ATSP is  $O(\log n)$ .

In this paper, we close this gap, and improve the current best bound on the integrality gap from  $O(\log n)$  to  $O(\log n / \log \log n)$ . The resulting algorithm uses the structure of narrow  $s$ - $t$  cuts in the LP solution to construct a (random) tree witnessing this integrality gap. We also give a simpler family of instances showing the integrality gap of this LP is at least 2.

## 1 Introduction

In the Asymmetric Traveling Salesperson Path (ATSP) problem, we are given an *asymmetric* metric space  $(V, d)$  (i.e., one where the distances satisfy the triangle inequality, but potentially not the symmetry condition), and also specified source and sink vertices  $s$  and  $t$ , and the goal is to find an  $s$ - $t$  Hamilton path of minimum length.

This ATSP problem is a close relative of the Asymmetric TSP problem (ATSP), where the goal is to find a Hamilton tour instead of an  $s$ - $t$  path. For this ATSP problem, the  $\log_2 n$ -approximation of Frieze, Galbiati, and Maffioli [9] from 1982 was improved by constant factors in [4,11,8]. A remarkable breakthrough on this problem was an  $O(\frac{\log n}{\log \log n})$ -approximation result due to Asadpour, Goemans, Mądry, Oveis Gharan, and Saberi [2] where they also bounded the integrality gap of the subtour elimination linear programming relaxation for ATSP by the same factor.

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Surprisingly the study of ATSP has been of a more recent vintage: the first approximations appeared around 2005 [12,6,8]. It is easily seen that the ATSP reduces to ATSP in an approximation preserving fashion (by guessing two consecutive nodes on the tour). In the other direction, [8] showed that a  $\rho$ -approximation to the ATSP problem implies an  $O(\rho)$ -approximation to the ATSP problem. Using the above-mentioned  $O(\frac{\log n}{\log \log n})$ -approximation for ATSP [2], this implies an  $O(\frac{\log n}{\log \log n})$ -approximation for ATSP as well.

The subtour elimination linear program generalizes simply to the ATSP problem and is given in Section 2. However, the best previous integrality gap for this LP for ATSP was  $O(\log n)$  [10]. In this paper we show the following result.

**Theorem 1.** *The integrality gap of the subtour elimination linear program for the ATSP problem is at most  $O(\frac{\log n}{\log \log n})$ .*

We also give a simple construction showing that the integrality gap of this LP is at least 2; this example is simpler than previous known integrality gap instance showing the same lower bound, due to Charikar, Goemans, and Karloff [5].

Given the central nature of linear programs in approximation algorithms, it is useful to understand the integrality gaps for linear programming relaxations of optimization problems. Not only does this study give us a deeper understanding into the underlying problems, but also upper bounds on the integrality gap of LPs are often required for some reductions to go through. For example, the poly-logarithmic approximation guarantees in the work of Nagarajan and Ravi [13] for Directed Orienteering and Minimum Ratio Rooted Cycle, and those in the work of Bateni and Chuzhoy [3] for Directed  $k$ -Stroll and Directed  $k$ -Tour were all improved by a factor of  $\log \log n$  following the improved bound of  $O(\frac{\log n}{\log \log n})$  on the integrality gap of the subtour LP relaxation for ATSP. Note that these improvements do not follow merely from improved approximation guarantees.

## 1.1 Our Approach

Our approach to bound the integrality gap for ATSP is similar to that for ATSP [2], but with some crucial differences. We sample a random spanning tree whose marginals are close to the optimal LP solution  $x^*$  and then augment the directed version of this tree to an integral circulation using Hoffman's circulation theorem while ensuring the  $t$ - $s$  edge is only used once. Following the corresponding Eulerian circuit and deleting the  $t$ - $s$  edge results in a spanning  $s$ - $t$  walk.

However, the non-Eulerian nature of the ATSP problem makes it difficult to satisfy the cut requirements in Hoffman's circulation theorem if we sample the spanning tree directly from the distribution given by the LP solution. It turns out that the problems come from the  $s$ - $t$  cuts  $U$  that are nearly-tight: i.e., which satisfy  $1 < x^*(\partial^+(U)) < 1 + \tau$  for some small constant  $\tau$  — these give rise to problems when the sampled spanning tree includes more than one edge across this cut. Such problems also arise in the symmetric TSP paths case (studied in a recent paper of An, Kleinberg, and Shmoys [1]): their approach is again to take

a random tree directly from the distribution given by the optimal LP solution, but in some cases they need to boost the narrow cuts, and they show that the loss due to this boosting is small.

In our case, the asymmetry in the problem means that boosting the narrow cuts might be prohibitively expensive. Hence, our idea is to preprocess the distribution given by the LP solution to *tighten* the narrow cuts, so that we never pick two edges from a narrow cut. Since the original LP solution lies in the spanning tree polytope, lowering the solution on some edges means we need to raise the fractional value on other edges, which may cause the cost to increase, and technical heart of the paper is to ensure this can be done with little extra loss.

## 1.2 Other Related Work

The first non-trivial approximation for ATSP was an  $O(\sqrt{n})$ -approximation by Lam and Newman [12]. This was improved to  $O(\log n)$  by Chekuri and Pál [6], and the constant was further improved in [8]. The paper [8] also showed that ATSP and ATSP had approximability within a constant factor of each other. All these results are combinatorial and do not bound integrality gap of ATSP. A bound of  $O(\sqrt{n})$  on the integrality gap of ATSP was given by Nagarajan and Ravi [14], and was improved to  $O(\log n)$  by Friggstad, Salavatipour and Svitkina [10]. Note that there is no known result relating the integrality gaps of the ATSP and ATSP problems in a black-box fashion.

In the symmetric case (where the problems become TSP and TSP respectively), constant factor approximations and integrality gaps have long been known. We do not survey the rich body of literature on TSP here, instead pointing the reader to, e.g., the recent paper on graphical TSP by Sebő and Vygen [17]. It is, however, important to mention the recent 1.618-approximation for TSP in a beautiful new result by An, Kleinberg, and Shmoys [1] which has recently improved to a 1.6-approximation by Sebő [16]. They proceed via bounding the integrality gap of the LP relaxation, and their algorithm also proceeds via studying the narrow  $s$ - $t$  cuts; the connections to their work are discussed in Section 1.1.

## 1.3 Notation and Preliminaries

Given a directed graph  $G = (V, A)$ , and two disjoint sets  $U, U' \subseteq V$ , let  $\partial(U; U') = A \cap (U \times U')$ . We use the standard shorthand that  $\partial^+(U) := \partial(U; V \setminus U)$ , and  $\partial^-(U) := \partial(V \setminus U; U)$ . When the set  $U$  is a singleton (say  $U = \{u\}$ ), we use  $\partial^+(u)$  or  $\partial^-(u)$  instead of  $\partial^+(\{u\})$  or  $\partial^-(\{u\})$ . For undirected graph  $H = (V, E)$ , we use  $\partial(U; U')$  to denote edges crossing between  $U$  and  $U'$ , and  $\partial(U)$  to denote the edges with exactly one endpoint in  $U$  (which is the same as  $\partial(V \setminus U)$ ).

For a digraph  $G = (V, A)$ , a set of arcs  $B \subseteq A$  is *weakly connected* if the undirected version of  $B$  forms a connected graph that spans all vertices in  $A$ .

For values  $x_a \in \mathbb{R}$  for all  $a \in A$ , and a set of arcs  $B \subseteq A$ , we let  $x(B)$  denote the sum  $\sum_{a \in B} x_a$ .

Given an undirected graph  $H = (V, E)$ , we let  $\chi_T \in \{0, 1\}^{|E|}$  denote the characteristic vector of a spanning tree  $T$ , then the spanning tree polytope is the convex hull of  $\{\chi_T \mid T \text{ spanning tree of } H\}$ . See, e.g., [15, Chapter 50] for several equivalent linear programming formulations of this polytope. We sometimes abuse notation and call a set of directed arcs  $T$  a tree if the undirected version of  $T$  is a tree in the usual sense.

## 2 The Rounding Algorithm

In this section, we give the linear programming relaxation for the Asymmetric TSP Path problem, and show how to round it to get a path of cost at most  $O(\frac{\log n}{\log \log n})$  times the cost of the optimal LP solution. We then give the proof, with some of the details being deferred to the following sections.

Given a directed metric graph  $G = (V, A)$  with arc costs  $\{c_a\}_{a \in A}$ , we use the following standard linear programming relaxation for ATSP which is also known as the subtour elimination linear program.

$$\begin{aligned}
 & \text{minimize : } \sum_{a \in E} c_a x_a && (ATSP) \\
 \text{s.t. : } & x(\partial^+(s)) = x(\partial^-(t)) = 1 && (1) \\
 & x(\partial^-(s)) = x(\partial^+(t)) = 0 && (2) \\
 & x(\partial^+(v)) = x(\partial^-(v)) = 1 && \forall v \in V \setminus \{s, t\} \quad (3) \\
 & x(\partial^+(U)) \geq 1 && \forall \{s\} \subseteq U \subsetneq V \quad (4) \\
 & x_a \geq 0 && \forall a \in E
 \end{aligned}$$

We begin by solving the above LP to obtain an optimal solution  $x^*$ . Consider the undirected (multi)graph  $H = (V, E)$  obtained by removing the orientation of the arcs of  $G$ . That is, create precisely two edges between every two nodes  $u, v \in V$  in  $H$ , one having cost  $c_{uv}$  and the other having cost  $c_{vu}$ . (Hence,  $|E| = |A|$ .) For a point  $w \in \mathbb{R}_+^A$ , let  $\kappa(w)$  denote the corresponding point in  $\mathbb{R}_+^E$ , and view  $\kappa(w)$  as the “undirected” version of  $w$ .

We will use the following definition: An  $s$ - $t$  cut is a subset  $U \subset V$  such that  $\{s\} \subseteq U \subseteq V \setminus \{t\}$ . The LP constraints imply that  $x^*(\partial^+(U)) - x^*(\partial^-(U)) = 1$  for every  $s$ - $t$  cut  $U$ . Also,  $x^*(\partial^+(U)) = x^*(\partial^-(U)) \geq 1$  for every nonempty  $U \subseteq V \setminus \{s, t\}$ .

**Definition 1 (Narrow cuts).** *Let  $\tau \geq 0$ . An  $s$ - $t$  cut  $U$  is  $\tau$ -narrow if  $x^*(\partial^+(U)) < 1 + \tau$  (or equivalently,  $x^*(\partial^-(U)) < \tau$ ).*

The main technical lemma is the following:

**Lemma 1.** *For any  $\tau \in [0, 1/4]$ , one can find, in polynomial-time, a vector  $z \in [0, 1]^A$  (over the directed arcs) such that:*

- (a) *its undirected version  $\kappa(z)$  lies in the spanning tree polytope for  $H$ ,*
- (b)  *$z \leq \frac{1}{1-3\tau} x^*$  (where the inequality denotes component-wise dominance), and*
- (c)  *$z(\partial^+(U)) = 1$  and  $z(\partial^-(U)) = 0$  for every  $\tau$ -narrow  $s$ - $t$  cut  $U$ .*

Before we prove the lemma (in Section 2.1), let us sketch how it will be useful to get a cheap solution to the ATSP. Since  $z$  (or more correctly, its undirected version  $\kappa(z)$ ) lies in the spanning tree polytope, it can be represented as a convex combination of spanning trees. Using some recently-developed algorithms (e.g., those due to [2,7]) one can choose a spanning tree that crosses each cut only  $O(\frac{\log n}{\log \log n})$  times more than the LP solution. Finally, we can use  $O(\frac{\log n}{\log \log n})$  times the LP solution to patch this tree to get an  $s$ - $t$  path. Since the LP solution is “weak” on the narrow cuts and may contribute very little to this patching (at most  $\tau$ ), it is crucial that by property (c) above, this tree will cross the narrow cuts *only once*, and that too, it crosses in the “right” direction, so we never need to use the LP when verifying the cut conditions of Hoffman’s circulation theorem on narrow cuts. The details of these operations appear in Section 3.

### 2.1 The Structure of Narrow Cuts

We now prove Lemma 1: it says that we can take the LP solution  $x^*$  and find another vector  $z$  such that if a  $s$ - $t$  cut is narrow in  $x^*$  (i.e., the total  $x^*$  value crossing the cut lies in  $[1, 1 + \tau)$ , then  $z$  crosses it to an extent precisely 1. Moreover, the undirected version of  $z$  can be written as a convex combination of spanning trees, and  $z_a$  is not much larger than  $x_a^*$  for any arc  $a$ .

Note that the undirected version of  $x^*$  itself can be written as a convex combination of spanning trees. Thus if we force  $z$  to cross the narrow cuts to an extent less than  $x^*$  (loosely, this reduces the connectivity), we must increase the fractional value on other arcs. To show we can perform this operation without changing any of the coordinates by very much, we need to study the structure of narrow cuts more closely. (Such a study is done in the *symmetric* TSP path paper of An et al. [1], but our goals and theorems are somewhat different.)

First, say two  $s$ - $t$  cuts  $U$  and  $W$  *cross* if  $U \setminus W$  and  $W \setminus U$  are non-empty.

**Lemma 2.** *For  $\tau \leq 1/4$ , no two  $\tau$ -narrow  $s$ - $t$  cuts cross.*

Lemma 2 says that the  $\tau$ -narrow cuts form a chain  $\{s\} = U_1 \subset U_2 \subset \dots \subset U_k = V \setminus \{t\}$  with  $k \geq 2$ . For  $1 < i \leq k$ , let  $L_i := U_i \setminus U_{i-1}$ . We also define  $L_1 = \{s\}$  and  $L_{k+1} = \{t\}$ . Let  $L_{\leq i} := \bigcup_{j=1}^i L_j$  and  $L_{\geq i} := \bigcup_{j=i}^{k+1} L_j$ . For the rest of this paper, we will use  $\tau$  to denote a value in the range  $[0, 1/4]$ . Ultimately, we will set  $\tau := 1/4$  for the final bound but we state the lemmas in their full generality for  $\tau \leq 1/4$ .

Next, we show that out of the (at most)  $1 + \tau$  mass of  $x^*$  across each  $\tau$ -narrow cut  $U_i$ , most of it comes from the “local” arcs in  $\partial(L_i; L_{i+1})$ .

**Lemma 3.** *For each  $1 \leq i \leq k$ ;  $x^*(\partial(L_i, L_{i+1})) \geq 1 - 3\tau$ .*

Now, recall that  $\kappa(x^*)$  denotes the assignment of arc weights to the graph  $H = (V, E)$  from the previous section obtained by “removing” the directions from arcs in  $A$ . We prove that the restriction of  $\kappa(x^*)$  to any  $L_i$  almost satisfies the partition inequalities that characterize the convex hull of connected graphs. For a partition  $\pi = \{W_1, \dots, W_\ell\}$ , we let  $\partial(\pi)$  denote the set of edges whose endpoints lie in two different sets in the partition.

**Lemma 4.** *For any  $1 \leq i \leq k + 1$  and any partition  $\pi = \{W_1, \dots, W_\ell\}$  of  $L_i$ , we have  $\kappa(x^*)(\partial(\pi)) \geq \ell - 1 - 2\tau$ .*

The following corollary will be useful.

**Corollary 1.** *For any partition  $\pi$  of  $L_i$ , we have  $\frac{\kappa(x^*)(\partial(\pi))}{1-2\tau} \geq |\pi| - 1$ .*

Finally, to efficiently implement the arguments in the proof of Lemma 1, we need to be able to efficiently find all  $\tau$ -narrow cuts  $U_i$ . This is done by a standard recursive algorithm that exploits the fact that the cuts are nested.

**Lemma 5.** *There is a polynomial-time algorithm to find all  $\tau$ -narrow  $s-t$  cuts.*

We are now in a position to prove Lemma 1, the main result of this section.

*Proof (Proof of Lemma 1).* The claimed vector  $z$  can be described by linear constraints: indeed, consider the following LP on the variables  $z$  where constraints (5) imply that  $\kappa(z)$  is in the convex hull of spanning connected graphs [15, Corollary 50.8a].<sup>1</sup>

$$\kappa(z)(\partial(\pi)) \geq |\pi| - 1 \quad \forall \text{ partitions } \pi \text{ of } V \quad (5)$$

$$z_a \leq \frac{1}{1-3\tau} x_a^* \quad \forall a \in A \quad (6)$$

$$z(\partial^+(U_i)) = 1 \quad \forall \tau\text{-narrow } s\text{-}t \text{ cuts } U_i \quad (7)$$

$$z(\partial^-(U_i)) = 0 \quad \forall \tau\text{-narrow } s\text{-}t \text{ cuts } U_i \quad (8)$$

$$z_a \geq 0 \quad \forall a \in A \quad (9)$$

We demonstrate a feasible  $z$  as follows.

$$z_a = \begin{cases} \frac{x_a^*}{x^*(\partial(L_i; L_{i+1}))} & \text{if } a \in \partial(L_i; L_{i+1}) \text{ for some } i; \\ \frac{x_a^*}{1-2\tau} & \text{if } a \in E[L_i] \text{ for some } i; \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

We claim that this solution  $z$  satisfies the above constraints. Constraints (8) and (9) are satisfied by construction. Constraint (6) follows from Lemma 3 for edges in  $\partial(L_i; L_{i+1})$  and by construction for rest of the edges. For constraint (7), note that

$$z(\partial^+(U_i)) = z(\partial(L_i; L_{i+1})) + z(\partial^+(U_i) \setminus \partial(L_i; L_{i+1})) = \frac{x^*(\partial(L_i; L_{i+1}))}{x^*(\partial(L_i; L_{i+1}))} + 0 = 1.$$

To complete the proof, we now show constraints (5) holds. It suffices to show that  $\kappa(z)$  can be decomposed as a convex combination of characteristic vectors of

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<sup>1</sup> The statement of Lemma 1 makes a claim about  $\kappa(z)$  being in the convex hull of spanning *trees* and not spanning *connected graphs*. However, the equivalent statement for spanning trees will follow by dropping some edges from the connected subgraphs in the decomposition of  $z$  to get spanning trees. Constraints (7) and (8) will still be satisfied by  $y$  since we retain connectivity.

connected graphs. For  $1 \leq i \leq k + 1$ , let  $z^i$  denote the restriction of  $\kappa(z)$  to edges whose both endpoints are contained in  $L_i$ . Then Corollary 1, constraints (9), and [15, Corollary 50.8a] imply that  $z^i$  can be decomposed as a convex combination of integral vectors, each of which corresponds to an edge set that is connected on  $L_i$ . Next, let  $z'$  denote the restriction of  $\kappa(z)$  to edges whose both endpoints are contained in some common  $L_i$  for some  $i$ . Since the sets  $E(L_1), \dots, E(L_{k+1})$  are disjoint, we have that  $z' = \sum_i z^i$  (where the addition is component-wise). Furthermore,  $z'$ , being the sum of the  $z^i$  vectors, can be decomposed as a convex combination of integral vectors corresponding to edge sets  $E'$  such that the connected components of the graph  $H' = (V, E')$  are precisely the sets  $\{L_i\}_{i=1}^{k+1}$ .

Next, let  $z''$  denote the restriction of  $\kappa(z)$  to edges contained in one such  $\partial(L_i; L_{i+1})$ . We also note that the sets  $\partial(L_1; L_2), \dots, \partial(L_k; L_{k+1})$  are disjoint. By construction, we have  $z''(\partial(L_i; L_{i+1})) = 1$  for each  $1 \leq i \leq k$  so we may decompose  $z''$  as a convex-combination of integral vectors, each of which includes precisely one edge across each  $\partial(L_i; L_{i+1})$ .

Now, adding any integral point  $y'$  in the decomposition of  $z'$  to any integral point  $y''$  in the decomposition of  $z''$  results in an integral vector that corresponds to a connected graph: each  $L_i$  is connected by  $y'$  and consecutive  $L_i$  are connected by  $y''$ . By construction of  $z$ , we have  $\kappa(z) = z' + z''$  so we may write  $z$  as a convex combination of characteristic vectors of connected graphs, each of which satisfies constraints (5).

To see why  $z$  can be found efficiently, we first compute all  $\tau$ -narrow cuts using Lemma 5. Then  $z$  is easy to compute in equation 10. Finally, [15, Corollary 51.6a] implies the decomposition of  $\kappa(z)$  into a convex combination of connected graphs can be done efficiently. Thus the arguments in the footnote to reduce  $z$  such that  $\kappa(z)$  is in the spanning tree polytope can be implemented efficiently.

### 3 Obtaining an $s$ - $t$ Path

Having transformed the optimal LP solution  $x^*$  into the new vector  $z$  (as in Lemma 1) without increasing it too much in any coordinate, we now sample a random tree such that it has a small total cost, and that the tree does not cross any cut much more than prescribed by  $x^*$ . Finally we add some arcs to this tree (without increasing its cost much) so that it is Eulerian at all nodes except  $\{s, t\}$ , and hence gives us an Eulerian  $s$ - $t$  walk. By the triangle inequality, shortcutting this walk past repeated nodes yields a Hamiltonian  $s - t$  path of no greater cost. While this general approach is similar to that used in [2], some new ideas are required because we are working with the LP for ATSP—in particular, only one unit of flow is guaranteed to cross  $s$ - $t$  cuts, which is why we needed to deal with narrow cuts in the first place. The details appear in the rest of this section.

#### 3.1 Sampling a Tree

For a collection of arcs  $\mathcal{A} \subset A$ , we say  $\mathcal{A}$  is  $\alpha$ -thin with respect to  $x^*$  if  $|\mathcal{A} \cap \partial^+(U)| \leq \alpha x^*(\partial^+(U))$  for every  $\emptyset \subsetneq U \subsetneq V$ . The set  $\mathcal{A}$  is also  $\beta$ -approximate

with respect to  $x^*$  if the total cost of all arcs in  $\mathcal{A}$  is at most  $\beta$  times the cost of  $x^*$ —i.e.,  $\sum_{a \in \mathcal{A}} c_a \leq \beta \sum_{a \in \mathcal{A}} c_a x_a^*$ . The reason we are deviating from the undirected to the directed setting is that the orientation of the arcs across each  $\tau$ -narrow cut will be important when we sample a random “tree”.

**Lemma 6.** *Let  $\tau \in [0, 1/4]$ . Let  $\beta = \frac{3}{1-3\tau}$  and  $\alpha = \Theta(\frac{\log n}{\tau \log \log n})$ . There is a randomized, polynomial time algorithm that, with probability at least  $1/2$ , finds an  $\alpha$ -thin and  $\beta$ -approximate (with respect to  $x^*$ ) collection of arcs  $\mathcal{A}$  that is weakly connected and satisfies  $|\mathcal{A} \cap (\partial^+(U))| = 1$  and  $|\mathcal{A} \cap (\partial^-(U))| = 0$  for each  $\tau$ -narrow  $s$ - $t$  cut  $U$ .*

*Proof.* Let  $z$  be a vector as promised by Lemma 1. From  $\kappa(z)$ , randomly sample a set of arcs  $\mathcal{A}$  whose undirected version  $\mathcal{T}$  is a spanning tree on  $V$ . This should be done from any distribution with the following two properties:

- (i) (Correct Marginals)  $\Pr[e \in \mathcal{T}] = \kappa(z)_e$
- (ii) (Negative Correlation) For any subset of edges  $F \subseteq E$ ,  $\Pr[F \subseteq \mathcal{T}] \leq \prod_{e \in F} \Pr[e \in \mathcal{T}]$

This can be obtained using, for example, the swap rounding approach for the spanning tree polytope given by Chekuri et al. [7]. As in [2], the negative correlation property implies the following theorem.

**Theorem 2.** *The tree  $\mathcal{T}$  is  $\alpha$ -thin with high probability.*

By Lemma 1(b), property (i) of the random sampling, and Markov’s inequality, we get that  $\mathcal{A}$  (from Lemma 6) is  $\frac{3}{1-3\tau}$ -approximate with respect to  $x^*$  with probability at least  $2/3$ . By a trivial union bound, for large enough  $n$  we have with probability at least  $1/2$  that  $\mathcal{A}$  is both  $\alpha$ -thin and  $\beta$ -approximate with respect to  $x^*$ . It is also weakly connected—i.e., the undirected version of  $\mathcal{A}$  (namely,  $\mathcal{T}$ ) connects all vertices in  $V$ .

The statement for  $\tau$ -narrow  $s$ - $t$  cuts follows from the fact that  $z$  satisfies Lemma 1(c). That is,  $\mathcal{A}$  contains no arcs of  $\partial^-(U)$ , since  $z(\partial^-(U)) = 0$  (for  $U$  being a  $\tau$ -narrow  $s$ - $t$  cut). But since  $\mathcal{T}$  is a spanning tree,  $\mathcal{A}$  must contain at least one arc from  $\partial^+(U)$ . Finally, since  $z(\partial^+(U))$  is exactly 1, then any set of arcs supported by this distribution we use must have precisely one arc from  $\partial^+(U)$ .

### 3.2 Augmenting to an Eulerian $s$ - $t$ Walk

Finally, we wrap up by augmenting the set of arcs  $\mathcal{A}$  to an Eulerian  $s$ - $t$  walk. For this, we use Hoffman’s circulation theorem, as in [2], which we recall here for convenience (see, e.g. [15, Theorem 11.2]):

**Theorem 3.** *Given a directed flow network  $D = (V, A)$ , with each arc having a lower bound  $\ell_a$  and an upper bound  $u_a$  (and  $0 \leq \ell_a \leq u_a$ ), there exists a circulation  $f : A \rightarrow \mathbb{R}_+$  satisfying  $\ell_a \leq f(a) \leq u_a$  for all arcs  $a$  if and only if  $\ell(\partial^+(U)) \leq u(\partial^-(U))$  for all  $U \subseteq V$ . Moreover, if the  $\ell$  and  $u$  are integral, then the circulation  $f$  can be taken to be integral.*



Set lower bounds  $\ell : A \rightarrow \{0, 1\}$  on the arcs by:

$$\ell_a = \begin{cases} 1 & \text{if } a \in \mathcal{A} \text{ or } a = ts \\ 0 & \text{otherwise} \end{cases}$$

For now, we set an upper bound of 1 on arc  $ts$  and leave all other arc upper bounds at  $\infty$ . We compute the minimum cost circulation satisfying these bounds (we will soon see why one must exist). Since the bounds are integral and since  $\mathcal{A}$  is weakly connected, this circulation gives us a directed Eulerian graph. Furthermore, since  $u_{ta} = \ell_{ta} = 1$ , the  $ts$  arc must appear exactly once in this Eulerian graph. Our final Hamiltonian  $s$ - $t$  path is obtained by following an Eulerian circuit, removing the single  $ts$  arc from this circuit to get an Eulerian  $s$ - $t$  walk, and finally shortcutting this walk past repeated nodes. The cost of this Hamiltonian path will be, by the triangle inequality, at most the cost of the circulation minus the cost of the  $ts$  arc.

Finally, we need to bound the cost of the circulation (and also to prove one exists). To this end, we will impose further upper bounds  $u : A \rightarrow \mathbb{R}_{\geq 0}$  as follows:

$$u_a = \begin{cases} 1 & \text{if } a = ts \\ 1 + (1 + \tau^{-1})\alpha x_a^* & \text{if } a \in \mathcal{A} \\ (1 + \tau^{-1})\alpha x_a^* & \text{otherwise} \end{cases}$$

We use Hoffman’s circulation theorem to show that a circulation  $f$  exists satisfying these bounds  $\ell$  and  $u$  (The calculations appear in the next paragraph.) Since  $u$  is no longer integral, the circulation  $f$  might not be integral, but it does demonstrate that a circulation exists where each arc  $a \neq ts$  is assigned at most  $(1 + \tau^{-1})\alpha x_a^*$  more flow in the circulation than the number of times it appears in  $\mathcal{A}$ . Consequently, it shows that the minimum cost circulation  $g$  in the setting where we only had a non-trivial upper bound of 1 on the arc  $ts$  can be no more expensive (since there are fewer constraints), and that circulation  $g$  can be chosen to be integral. The cost of circulation  $g$  is at most the cost of  $f$ , which is at most

$$\sum_{a \in A} c_a u_a = \sum_{a \in \mathcal{A}} c_a + (1 + \tau^{-1})\alpha \sum_{a \in A} c_a x_a^* + c_{ts}.$$

Subtracting the cost of the  $ts$  arc (since we drop it to get the Hamilton path) and recalling that  $\mathcal{A}$  is  $\frac{3}{1-3\tau}$ -approximate with respect to  $x^*$  (and hence  $\sum_{a \in \mathcal{A}} c_a \leq \frac{3}{1-3\tau} \sum_{a \in A} c_a x_a^*$ ), we get that the final Hamiltonian path has cost at most

$$\left( \frac{3}{1-3\tau} + (1 + \tau^{-1})\alpha \right) \sum_{a \in A} c_a x_a^*,$$

and hence  $O(\frac{\log n}{\log \log n})$  times the cost of the LP relaxation for  $\tau = 1/4$ . This proves the claim that the cost of the  $s$ - $t$  path we found is  $O(\frac{\log n}{\log \log n})$  times the LP value, with constant probability, and completes the proof of [Theorem 1](#).

One detail remains: we need to verify the conditions of [Theorem 3](#) for the bounds  $\ell$  and  $u$ . Firstly, it is clear by definition that  $\ell_a \leq u_a$  for each arc  $a$ . Now we need to check  $\ell(\partial^+(U)) \leq u(\partial^-(U))$  for each cut  $U$ . This is broken into four cases (where saying  $U$  is a  $u$ - $v$  cut means  $u \in U, v \notin U$ ).

1.  $U$  is a  $\tau$ -narrow  $s$ - $t$  cut. Then  $\ell(\partial^+(U)) = 1$ , since  $\mathcal{A}$  contains only one arc in  $\partial^+(U)$ . But  $1 = u_{ts} \leq u(\partial^-(U))$ .
2.  $U$  is an  $s$ - $t$  cut, but not  $\tau$ -narrow. Then by the  $\alpha$ -thinness of  $\mathcal{A}$ ,

$$\ell(\partial^+(U)) \leq \alpha x^*(\partial^+(U)) = \alpha x^*(\partial^-(U)) + \alpha.$$

On the other hand,

$$u(\partial^-(U)) \geq (1 + \tau^{-1})\alpha x^*(\partial^-(U)) = \alpha x^*(\partial^-(U)) + \tau^{-1}\alpha x^*(\partial^-(U)) \geq \alpha x^*(\partial^-(U)) + \alpha$$

where the last inequality used the fact that  $x^*(\partial^-(U)) \geq \tau$ .

3.  $U$  is a  $t$ - $s$  cut. Then

$$\ell(\partial^+(U)) \leq 1 + \alpha x^*(\partial^+(U)) = 1 + \alpha x^*(\partial^-(U)) - \alpha \leq \alpha x^*(\partial^-(U)),$$

the last inequality using that  $\alpha \geq 1$ . Moreover

$$u(\partial^-(U)) \geq (1 + \tau^{-1})\alpha x^*(\partial^-(U)) \geq \alpha x^*(\partial^-(U)).$$

Then  $\ell(\partial^+(U)) \leq u(\partial^-(U))$ .

4.  $U$  does not separate  $s$  from  $t$ . Then

$$\ell(\partial^+(U)) \leq \alpha x^*(\partial^+(U)) = \alpha x^*(\partial^-(U)) \leq (1 + \tau^{-1})\alpha x^*(\partial^-(U)) \leq u(\partial^-(U))$$

## 4 A Simple Integrality Gap Example

In this section, we show that the integrality gap of the subtour elimination LP *ATSP* is at least 2. This result can also be inferred from the integrality gap of 2 for the ATSP tour problem [5], but our construction is relatively simpler.

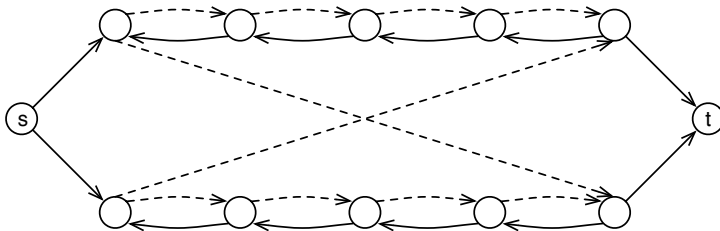
For a fixed integer  $r \geq 1$ , consider the directed graph  $G_r$  defined below (and illustrated in Figure 1). The vertices of  $G_r$  are  $\{s, t\} \cup \{u_1, \dots, u_r\} \cup \{v_1, \dots, v_r\}$ ; the edges are as follows:

- $\{su_1, sv_1, u_r t, v_r t\}$ , each with cost 1,
- $\{u_1 v_r, v_1 u_r\}$ , each with cost 0,
- $\{u_{i+1} u_i \mid 1 \leq i < r\} \cup \{v_{i+1} v_i \mid 1 \leq i < r\}$ , each with cost 1,
- and  $\{u_i u_{i+1} \mid 1 \leq i < r\} \cup \{v_i v_{i+1} \mid 1 \leq i < r\}$ , each with cost 0.

Let  $F_r$  denote the ATSP instance obtained from the metric completion of  $G_r$ .

**Lemma 7.** *The integrality gap of the LP ATSP on the instance  $F_r$  is at least  $2 - o(1)$ .*

*Proof (sketch).* The assignment  $x_a^* = \frac{1}{2}$  for every edge  $a$  of  $F_r$  corresponding to an edge of  $G_r$  is feasible for LP *ATSP* with cost  $k + 1$ . However, any spanning  $s - t$  walk in  $G_r$  has length at least  $2k - O(1)$ , so the optimum ATSP solution in  $F_r$  also has cost at least  $2k - O(1)$ .



**Fig. 1.** The graph  $G_r$  with  $r = 5$ . The solid edges have cost 1 and the dashed edges have cost 0.

## 5 Conclusion

In this paper we showed that the integrality gap for the ATSP problem is  $O(\frac{\log n}{\log \log n})$ . In fact, our proof also shows an integrality gap of  $\alpha$  for ATSP whenever we can construct a procedure which takes a point  $y \in \mathbb{R}^{|E|}$  in the spanning tree polytope of an undirected (multi)graph  $H = (V, E)$  and outputs a tree  $T$  that is (a)  $\alpha$ -thin, and (b) also satisfies  $|T \cap \partial(U)| = 1$  for any cut  $U$  where  $y(\partial(U)) = 1$ . We also showed a simpler construction achieving a lower bound of 2 for the subtour elimination LP.

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