Tutorial on Bandits Games

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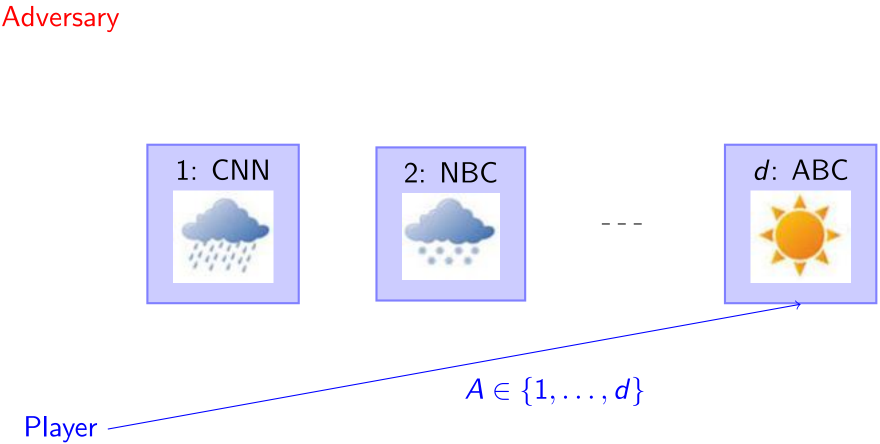
Online Learning with Full Information

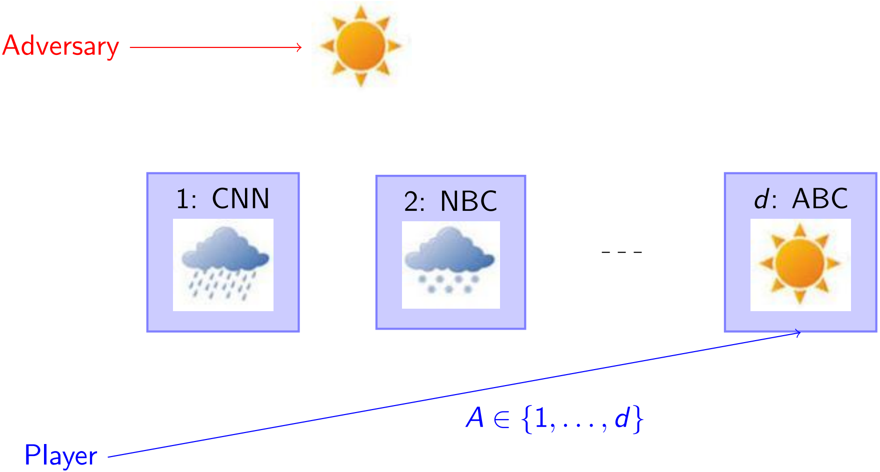
Adversary

Player

Online Learning with Full Information



Online Learning with Full Information



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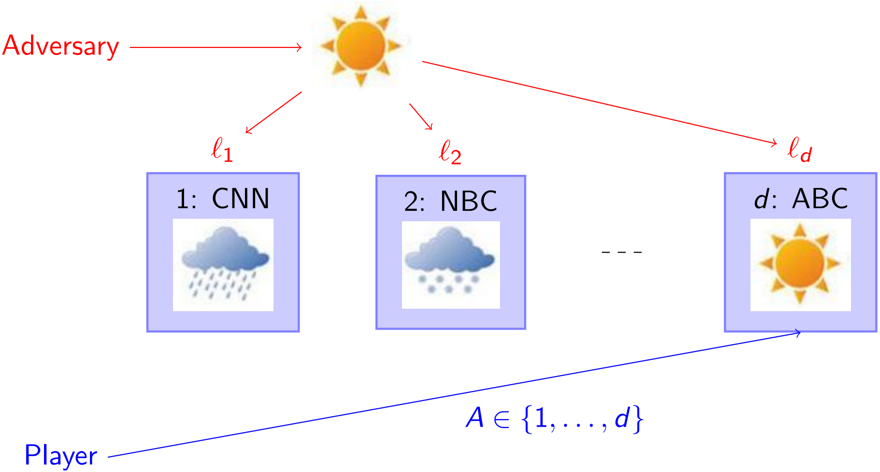

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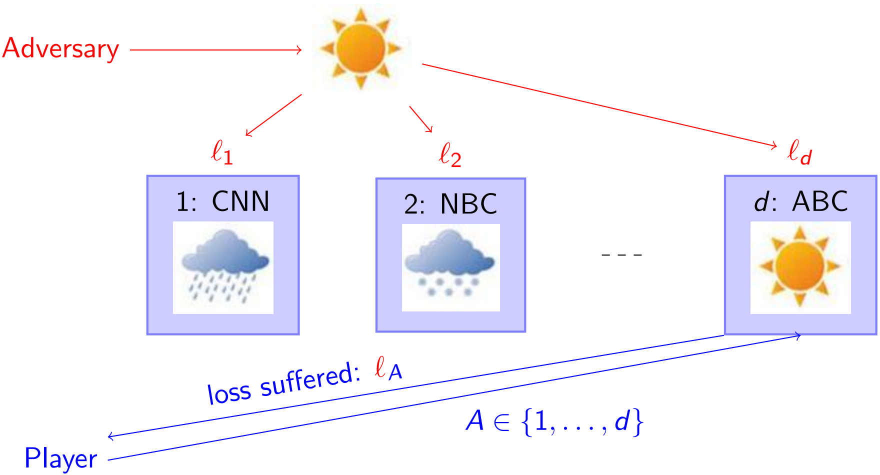

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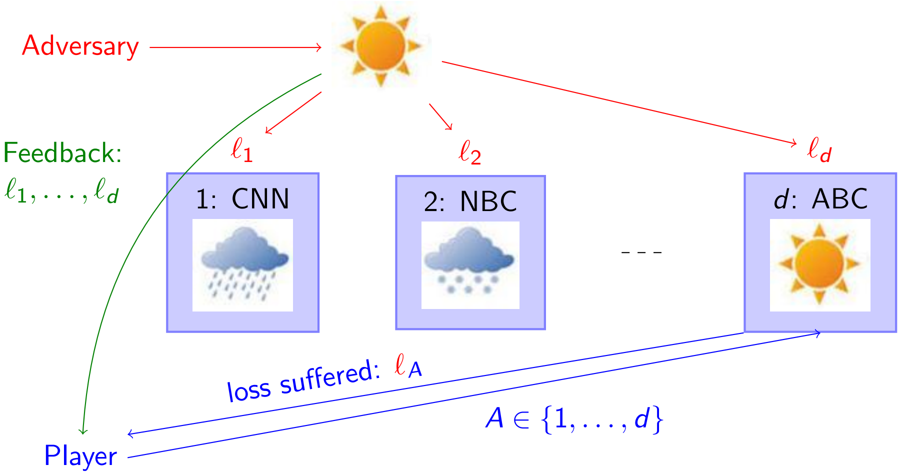

Adversary

1

2

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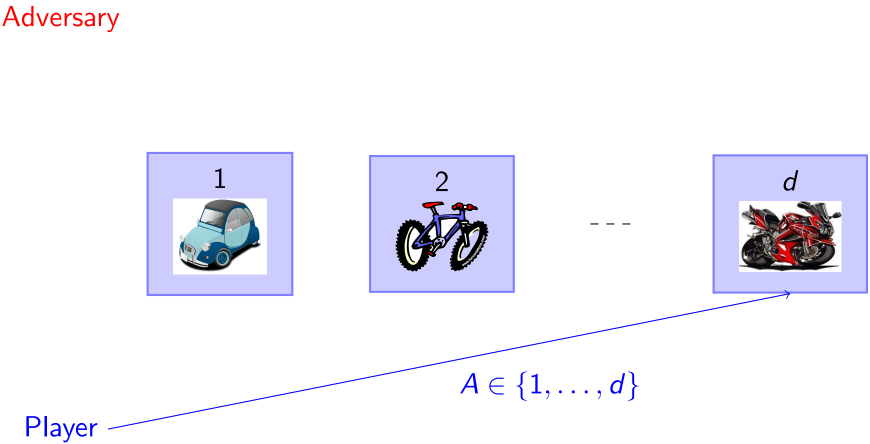
Online Learning with Bandit Feedback



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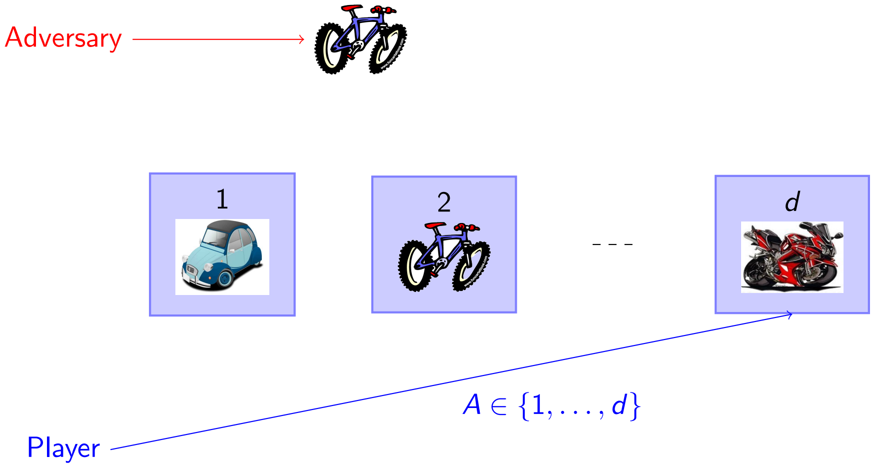

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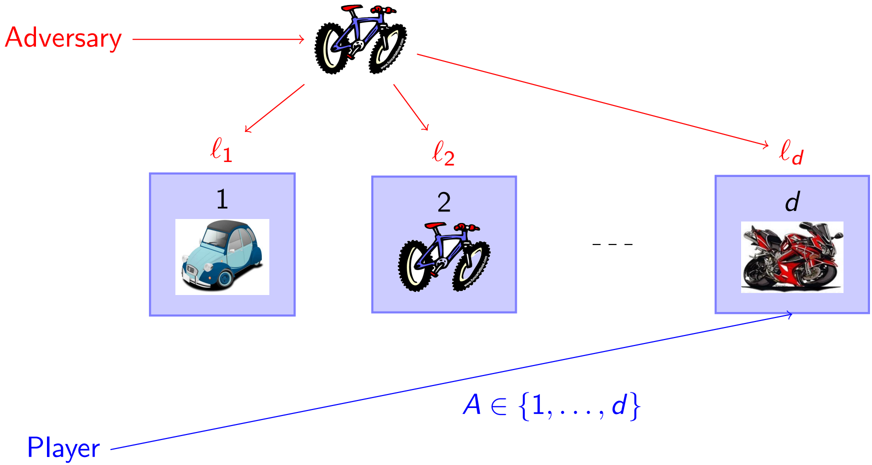

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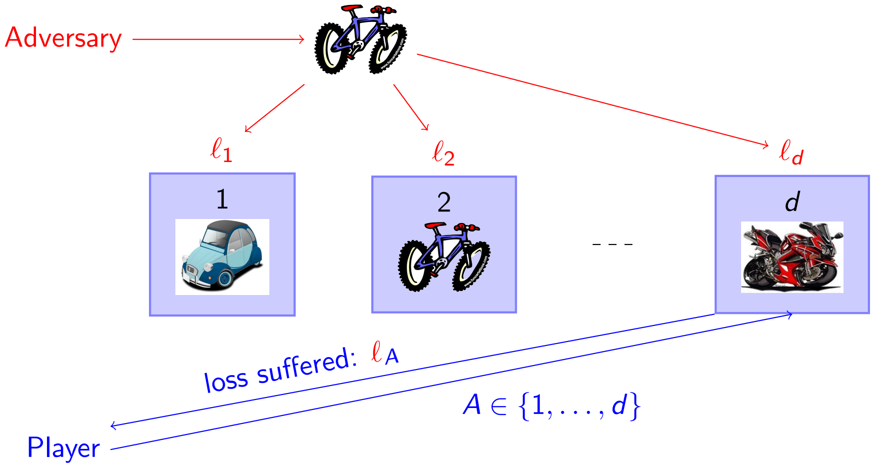

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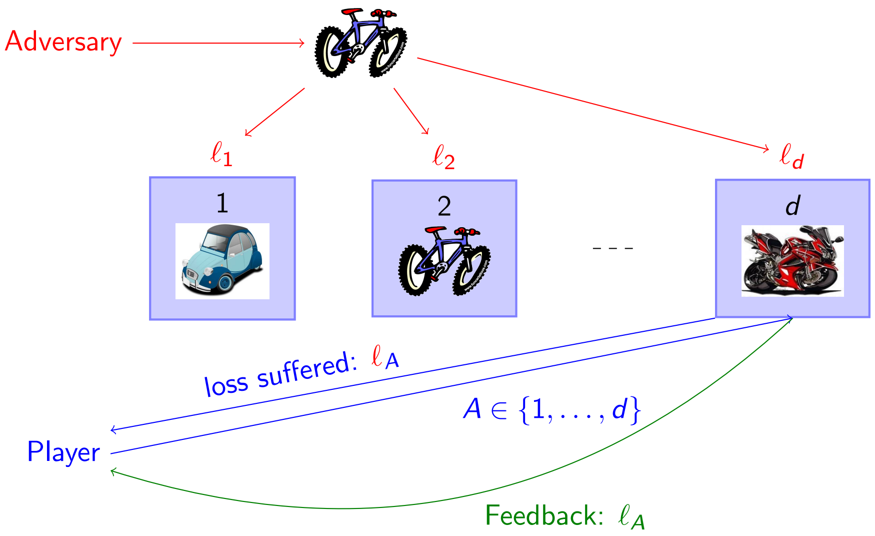
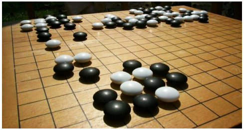


Some Applications

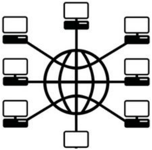
Computer Go



Brain computer interface

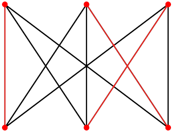
Medical trials

Packets routing



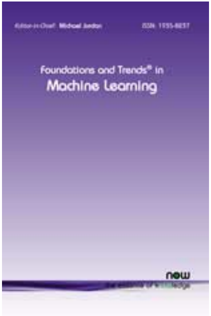
Ads placement

Dynamic allocation



A little bit of advertising

Survey on multi-armed bandits to appear in



S. Bubeck and N. Cesa-Bianchi.
Regret Analysis of Stochastic and Nonstochastic Multi-armed Bandit Problems.
To appear in *Foundations and Trends in Machine Learning*, 2012. (Draft available on my webpage.)

Notation

For each round $t = 1, 2, \ldots, n$;

1. The player chooses an arm $I_t \in \{1, \ldots, d\}$, possibly with the help of an external randomization.
2. Simultaneously the adversary chooses a loss vector $\ell_t = (\ell_{1,t}, \ldots, \ell_{d,t}) \in [0, 1]^d$.
3. The player incurs the loss $\ell_{I_t,t}$, and observes:
   - The loss vector ℓ_t in the full information setting.
   - Only the loss incured $\ell_{I_t,t}$ in the bandit setting.

Goal: Minimize the cumulative loss incured. We consider the regret:

$$R_n = \mathbb{E} \sum_{t=1}^{n} \ell_{I_t,t} - \min_{i=1,\ldots,d} \mathbb{E} \sum_{t=1}^{n} \ell_{i,t}.$$

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Exponential Weights (EW, EWA, MW, Hedge, ect)

Draw I_t at random from p_t where

$$p_t(i) = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \ell_{i,s}\right)}{\sum_{j=1}^{d} \exp\left(-\eta \sum_{s=1}^{t-1} \ell_{j,s}\right)}$$

Theorem (Cesa-Bianchi, Freund , Haussler, Helmbold, Schapire and Warmuth [1997])

Exp satisfies

$$R_n \leq \sqrt{\frac{n \log d}{2}}.$$

Moreover for any strategy,

$$\sup_{\text{adversaries}} R_n \geq \sqrt{\frac{n \log d}{2}} + o(\sqrt{n \log d}).$$

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The one-slide-proof

$$w_t(i) = \exp\left(-\eta \sum_{s=1}^{t-1} \ell_{i,s}\right), \quad W_t = \sum_{i=1}^{d} w_t(i), \quad p_t(i) = \frac{w_t(i)}{W_t}$$

$$\log \frac{W_{n+1}}{W_1} = \log\left(\frac{1}{d}\sum_{i=1}^{d} w_{n+1}(i)\right) \geq \log\left(\frac{1}{d}\max_i w_{n+1}(i)\right)$$

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$$= \sum_{t=1}^{n} \log\left(\mathbb{E}\exp(-\eta \ell_{I_t,t})\right)$$

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Magic trick for bandit feedback

$$\tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{p_t(i)} \mathbb{1}_{I_t = i},$$

is an unbiased estimate of $\ell_{i,t}$. We call Exp3 the Exp strategy run on the estimated losses.

Theorem (Auer, Cesa-Bianchi, Freund and Schapire [2003])

Exp3 satisfies:

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High probability bounds

What about bounds directly on the *true* regret

$$\sum_{t=1}^{n} \ell_{I_t,t} - \min_{i=1,\dots,d} \sum_{t=1}^{n} \ell_{i,t} \;?$$

Auer et al. [2003] proposed Exp3.P:

$$p_t(i) = (1-\gamma)\frac{\exp\left(-\eta \sum_{s=1}^{t-1} \tilde{\ell}_{i,s}\right)}{\sum_{j=1}^{d} \exp\left(-\eta \sum_{s=1}^{t-1} \tilde{\ell}_{j,s}\right)} + \frac{\gamma}{d},$$

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Theorem (Auer et al. [2003], Audibert and Bubeck [2011])

Let $\delta \in (0,1)$, with $\beta = \sqrt{\frac{\log(d\delta^{-1})}{nd}}$, $\eta = 0.95\sqrt{\frac{\log d}{nd}}$ and $\gamma = 1.05\sqrt{\frac{d \log d}{n}}$, Exp3.P satisfies with probability at least $1-\delta$:

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Other types of normalization

- INF (Implicitly Normalized Forecaster) is based on a potential function $\psi : \mathbb{R}_-^* \to \mathbb{R}_+^*$ increasing, convex, twice continuously differentiable, and such that $(0,1] \subset \psi(\mathbb{R}_-^*)$.
- At each time step INF computes the new probability distribution as follows:

$$p_t(i) = \psi\left(C_t - \sum_{s=1}^{t-1} \tilde{\ell}_{i,s}\right),$$

where C_t is the unique real number such that $\sum_{i=1}^{d} p_t(i) = 1$.
- $\psi(x) = \exp(\eta x) + \frac{\gamma}{d}$ corresponds exactly to the Exp3 strategy.
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Minimax optimal regret bound

Theorem (Audibert and Bubeck [2009], Audibert and Bubeck [2010], Audibert, Bubeck and Lugosi [2011])

Quadratic INF satisfies:

$$R_n \leq 2\sqrt{2nd}.$$

Extension: partial monitoring

- Partial monitoring: the received feedback at time t is some signal $S(I_t, \ell_t)$, see Cesa-Bianchi and Lugosi [2006].
- A simple interpolation between full info. and bandit feedback is the partial monitoring setting of Mannor and Shamir [2011]: $S(I_t, \ell_t) = \{\ell_{i,t}, i \in \mathcal{N}(I_t)\}$ where $\mathcal{N} : \{1, \dots, d\} \to \mathcal{P}(\{1, \dots, d\})$ is some known neighboorhood mapping. A natural loss estimate in that case is

$$\tilde{\ell}_{i,t} = \frac{\ell_{i,t} \mathbb{1}_{i \in \mathcal{N}(I_t)}}{\sum_{j \in \mathcal{N}(i)} p_t(j)}.$$

Mannor and Shamir [2011] proved that Exp with the above estimate has a regret of order $\sqrt{\alpha n}$ where α is the independence number of the graph associated to $\mathcal{N}$.

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- Partial monitoring: the received feedback at time t is some signal $S(I_t, \ell_t)$, see Cesa-Bianchi and Lugosi [2006].
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- Contextual bandits: at each time step t one receives a context $s_t \in \mathcal{S}$, and one wants to perform as well as the best mapping from contexts to arms:

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- A related problem is bandit with experts advice: N experts are playing the game, and the player observes their actions ξ_t^k, $k = 1, \dots, N$. One wants to compete with the best expert:

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Stochastic Assumption

Assumption (Robbins [1952])

The sequence of losses $(\ell_t)_{1 \leq t \leq n}$ is a sequence of i.i.d random variables.

For historical reasons in this setting we consider gains rather than losses and we introduce different notation:

- Let ν_i be the unknown reward distribution underlying arm i, μ_i the mean of ν_i, $\mu^* = \max_{1 \leq i \leq d} \mu_i$ and $\Delta_i = \mu^* - \mu_i$.
- Let $X_{i,s} \sim \nu_i$ be the reward obtained when pulling arm i for the s^{th} time, and $T_i(t) = \sum_{s=1}^{t} \mathbb{1}_{I_s = i}$ the number of times arm i was pulled up to time t.
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Optimism in face of uncertainty

General principle: given some observations from an unknown environment, build (with some probabilistic argument) a set of *possible* environments Ω, then act as if the real environment was the most favorable one in Ω.

Application to stochastic bandits: given the past rewards, build confidence intervals for the means (μ_i) (in particular build upper confidence bounds), then play the arm with the highest upper confidence bound.

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UCB (Upper Confidence Bounds)

Theorem (Hoeffding [1963])

Let $X, X_1, \dots, X_t$ be i.i.d random variables in $[0,1]$, then with probability at least $1-\delta$,

$$\mathbb{E}X \leq \frac{1}{t}\sum_{s=1}^{t} X_s + \sqrt{\frac{\log \delta^{-1}}{2t}}.$$

This directly suggests the famous UCB strategy of Auer, Cesa-Bianchi and Fischer [2002]:

$$I_t \in \operatorname*{argmax}_{1 \leq i \leq d} \frac{1}{T_i(t-1)} \sum_{s=1}^{T_i(t-1)} X_{i,s} + \sqrt{\frac{2\log t}{T_i(t-1)}}.$$

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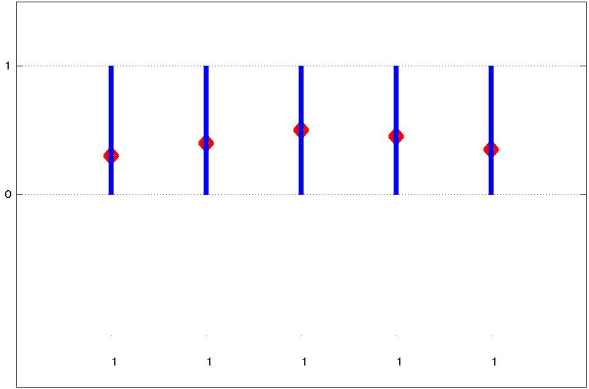

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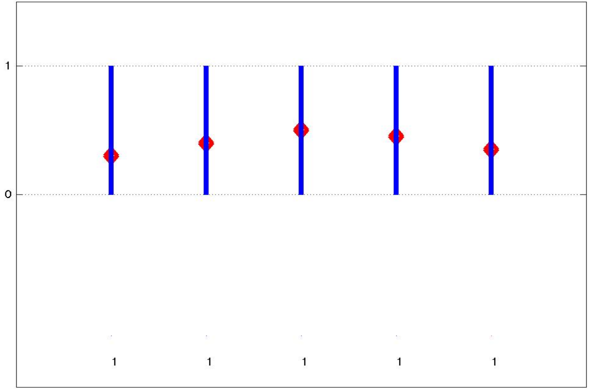

MOSS (Minimax Optimal Stochastic Strategy)

In a distribution-free sense one can show that UCB has a regret always bounded as $R_n \leq c\sqrt{nd \log n}$. Furthermore one can prove that for any strategy there exists a set of distributions such that $R_n \geq \frac{1}{20}\sqrt{nd}$. To close this gap Audibert and Bubeck (2009) introduced MOSS:

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For any $p, q \in [0,1]$, let

$$\mathrm{kl}(p,q) = p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q}.$$

Theorem (Lai and Robbins [1985])

Consider a consistent strategy, i.e. s.t. $\forall a > 0$, we have $\mathbb{E} T_i(n) = o(n^a)$ if $\Delta_i > 0$. Then for any Bernoulli reward distributions,

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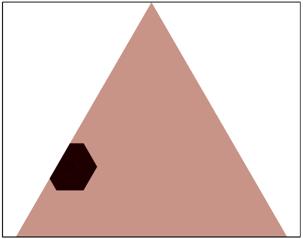

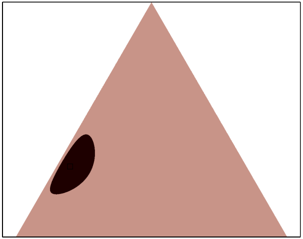

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A non-UCB strategy: Thompson's sampling

In Thompson [1933] the following strategy was proposed for the case of Bernoulli distributions:

- Assume a uniform prior on the parameters $\mu_i \in [0, 1]$.
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Heavy-tailed distributions

The standard UCB works for all σ^2 - subgaussian distributions (not only bounded distributions), i.e. such that

$$\mathbb{E}\exp\left(\lambda(X - \mathbb{E}X)\right) \leq \frac{\sigma^2\lambda^2}{2}, \forall \lambda \in \mathbb{R}.$$

It is easy to see that this is equivalent to

$$\exists \alpha > 0 \text{ s.t. } \mathbb{E}\exp(\alpha X^2) < +\infty.$$

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Median of means, Alon, Gibbons, Matias and Szegedy [2002]

Lemma

Let $X, X_1, \ldots, X_n$ be i.i.d random variables such that $\mathbb{E}(X - \mathbb{E}X)^2 \leq 1$. Let $\delta \in (0,1)$, $k = 8 \log \delta^{-1}$ and $N = \frac{n}{8 \log \delta^{-1}}$. Then with probability at least $1 - \delta$,

$$\mathbb{E}X \leq \textit{median}\left(\frac{1}{N}\sum_{s=1}^{N} X_s, \ldots, \frac{1}{N}\sum_{s=(k-1)N+1}^{kN} X_s\right) + 8\sqrt{\frac{8\log(\delta^{-1})}{n}}.$$

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Robust UCB

This suggests a Robust UCB strategy, Bubeck, Cesa-Bianchi and Lugosi [2012]:

$$I_t \in \operatorname*{argmax}_{1 \le i \le d} \text{median}\left(\frac{1}{N_{i,t}} \sum_{s=1}^{N_{i,t}} X_{i,s}, \dots, \frac{1}{N_{i,t}} \sum_{s=(k_t-1)N_{i,t}+1}^{k_t N_{i,t}} X_{i,s} \right) + 32\sqrt{\frac{\log t}{T_i(t-1)}},$$

with $k_t = 16 \log t$ and $N_{i,t} = \frac{T_i(t-1)}{16 \log t}$. The following regret bound can be proved for any set of distributions with variance bounded by 1:

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Markovian rewards

Assumption

The sequence $(X_{i,t})_{t\geq 1}$ forms an aperiodic irreducible finite-state Markov chain with unknown transition matrix P_i.

Again in this framework it is possible to design a UCB strategy with logarithmic regret (Tekin and Liu, [2011]), using the following result:

Theorem (Lezaud [1998])

Let $X_1, \ldots, X_t$ be an aperiodic irreducible finite-state Markov chain with transition matrix P. Let λ_2 be the second largest eigenvalue of the multiplicative symmetrization of P and $\epsilon = 1 - \lambda_2$. Let μ be the expectation of X_1 under the stationary distribution. There exists $C > 0$ such that for any $\gamma \in (0, 1]$,

$$\mathbb{P}\left(\frac{1}{t}\sum_{s=1}^{t} X_s \geq \mu + \gamma\right) \leq C \exp\left(-\frac{t\gamma^2\epsilon}{28}\right).$$

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Online Lipschitz and Stochastic Optimization

Stochastic multi-armed bandit where $\{1, \dots, K\}$ is replaced by $\mathcal{X}$.
At time t, select $x_t \in \mathcal{X}$, then receive a random variable $r_t \in [0, 1]$ such that $\mathbb{E}[r_t | x_t] = f(x_t)$.

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$\mathcal{X}$ is equipped with a symmetric function $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ such that $\rho(x, x) = 0$. f is Lipschitz with respect to ρ, that is

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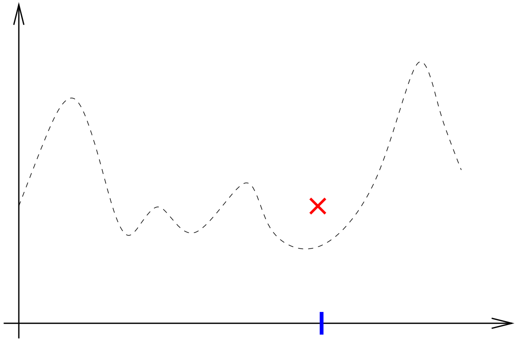
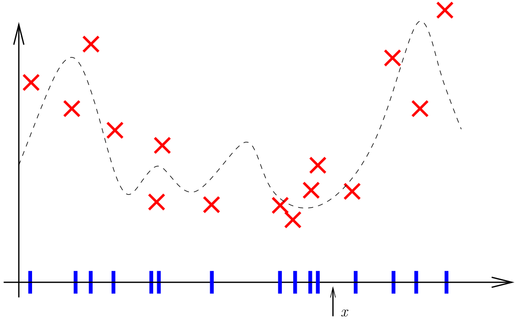
Example in 1d

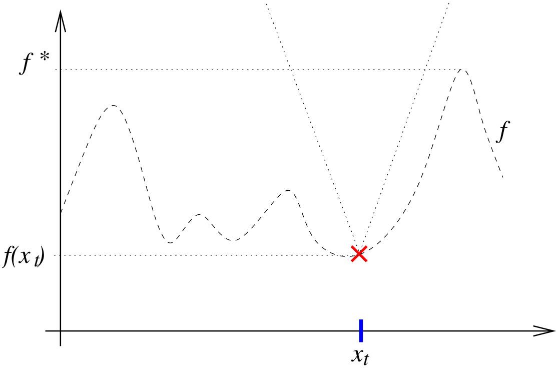


Where should one sample next?



How to define a high probability upper bound at any state x?

Noiseless case, $r_t = f(x_t)$



Lipschitz property $\rightarrow$ the evaluation of f at x_t provides a first upper-bound on f.

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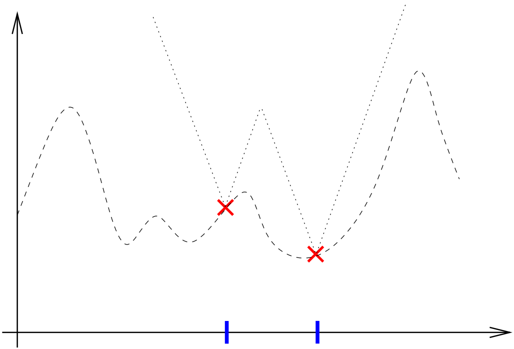
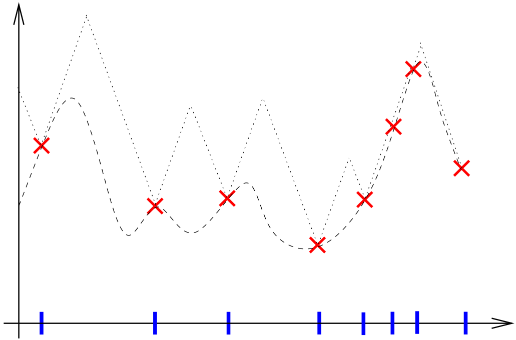
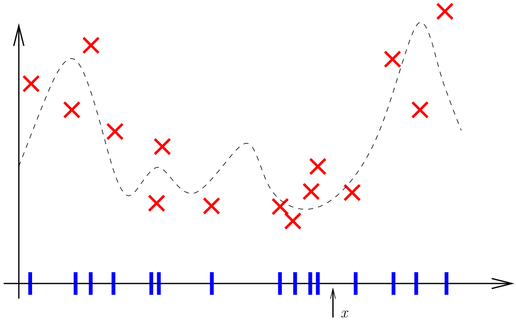


New point $\rightarrow$ refined upper-bound on f.

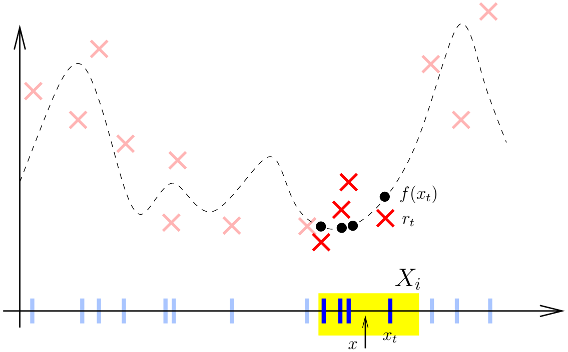
Noiseless case, $r_t = f(x_t)$



Back to the noisy case



UCB in a given domain



For a fixed domain $X_i \ni x$ containing n_i points $\{x_t\} \in X_i$, we have that $\sum_{t=1}^{n_i} r_t - f(x_t)$ is a martingale. Thus by Azuma's inequality,

$$\frac{1}{n_i}\sum_{t=1}^{n_i} r_t + \sqrt{\frac{\log 1/\delta}{2n_i}} \geq \frac{1}{n_i}\sum_{t=1}^{n_i} f(x_t) \geq f(x) - diam(X_i),$$

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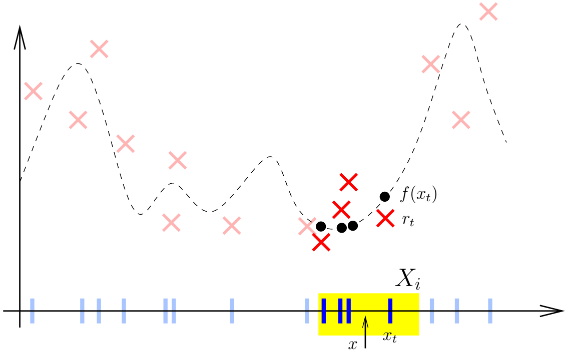

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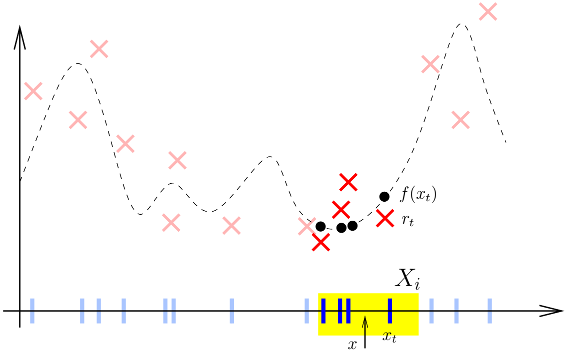

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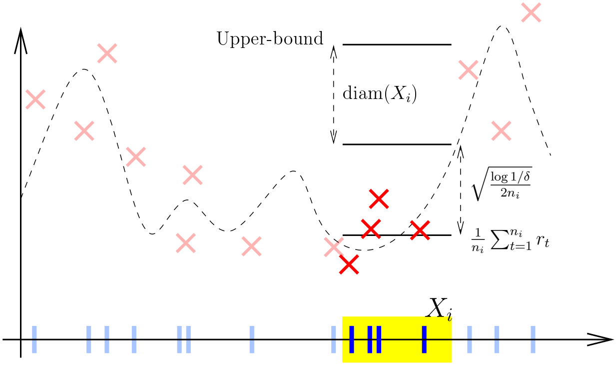
High probability upper bound



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Tradeoff between number of points in a domain and size of the domain.

By considering several domains we can derive a tigther upper bound.

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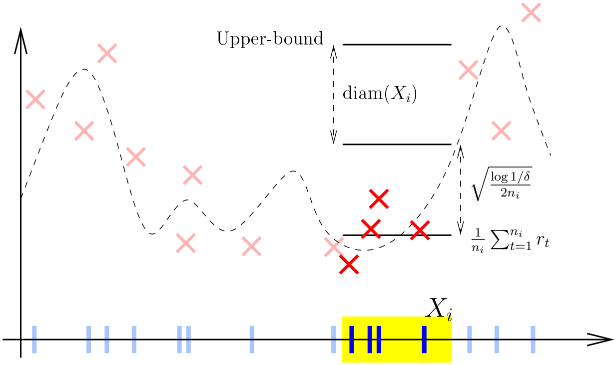

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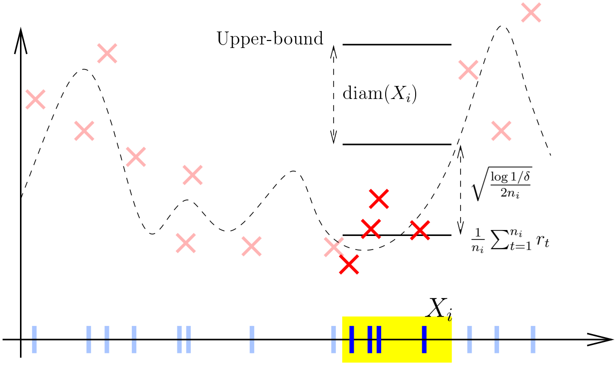

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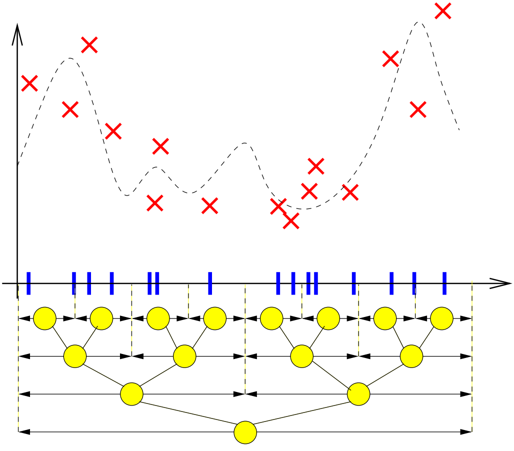
A hierarchical decomposition

Use a tree of partitions at all scales:

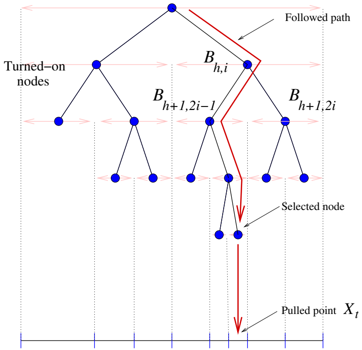


$$B_i(t) \stackrel{\mathrm{def}}{=} \min\left\{ \hat{\mu}_i(t) + \sqrt{\frac{2\log(t)}{T_i(t)}} + \mathrm{diam}(i), \max_{j \in \mathcal{C}(i)} B_j(t) \right\}$$

Hierarchical Optimistic Optimization (HOO)

[Bubeck, Munos, Stoltz, Szepesvári, 2008, 2011]: Consider a tree of partitions of $\mathcal{X}$, each node i corresponds to a subdomain X_i.

HOO Algorithm:
Let $\mathcal{T}_t$ be the set of expanded nodes at round t.
- $\mathcal{T}_1 = \{\text{root}\}$ (space $\mathcal{X}$)
- At t, select a leaf I_t of $\mathcal{T}_t$ by maximizing the B-values,
- $\mathcal{T}_{t+1} = \mathcal{T}_t \cup \{I_t\}$
- Select $x_t \in X_{I_t}$
- Observe reward r_t and update the B-values:



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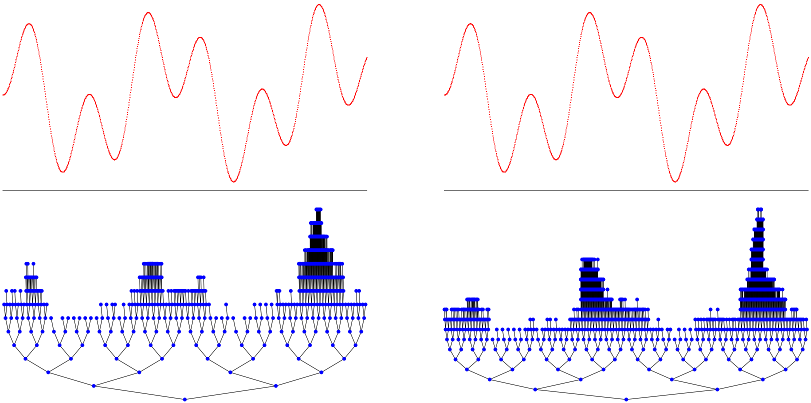
Example in 1d

$r_t \sim \mathcal{B}(f(x_t))$ a Bernoulli distribution with parameter $f(x_t)$



Resulting tree at time $n = 1000$ and at $n = 10000$.

Analysis of HOO

The near-optimality dimension d of f is defined as follows: Let

$$X_\epsilon \stackrel{\text{def}}{=} \{x \in \mathcal{X}, f(x) \geq f^* - \epsilon\}$$

be the set of ϵ-optimal points. Then X_ϵ can be covered by $O(\epsilon^{-d})$ balls of radius ϵ. A similar notion was introduced in [Kleinberg, Slivkins, Upfal, 2008].

Theorem (Bubeck, Munos, Stoltz, Szepesvári, 2008)

HOO satisfies:

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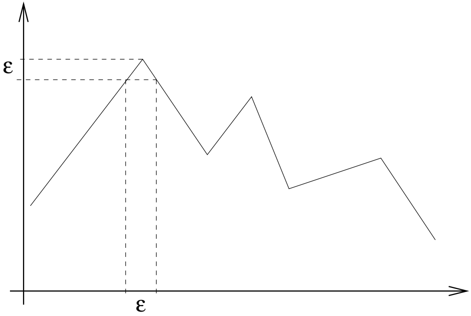
Assume the function is locally peaky around its maximum:

$$f(x^*) - f(x) = \Theta(||x^* - x||).$$



It takes $O(\epsilon^0)$ balls of radius ϵ to cover X_ϵ with $\rho(x,y) = ||x-y||$.
Thus $d = 0$ and the regret is $\widetilde{O}(\sqrt{n})$.

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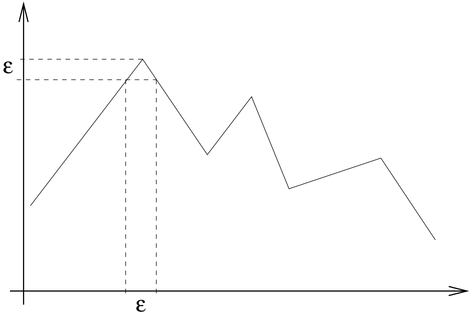

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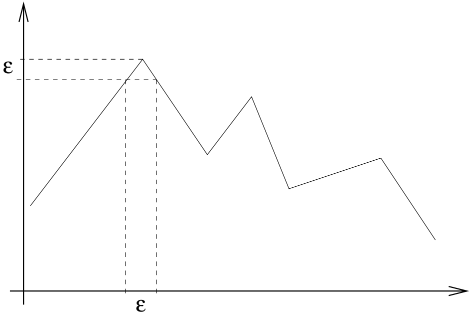

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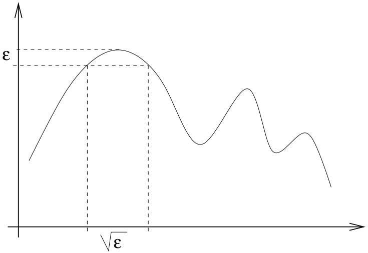

- For $\rho(x, y) = ||x - y||$, it takes $O(\epsilon^{-D/2})$ balls of radius ϵ to cover X_ϵ. Thus $d = D/2$ and $R_n = \widetilde{O}(n^{\frac{D+2}{D+4}})$.
- For $\rho(x, y) = ||x - y||^2$, it takes $O(\epsilon^0)$ ρ-balls of radius ϵ to cover X_ϵ. Thus $d = 0$ and $R_n = \widetilde{O}(\sqrt{n})$.

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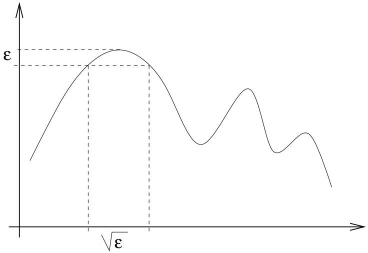

- For $\rho(x, y) = ||x - y||$, it takes $O(\epsilon^{-D/2})$ balls of radius ϵ to cover X_ϵ. Thus $d = D/2$ and $R_n = \widetilde{O}(n^{\frac{D+2}{D+4}})$.
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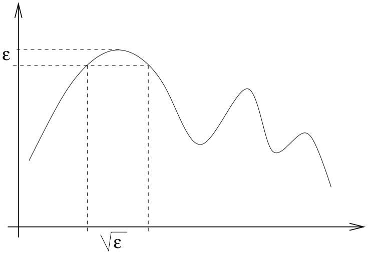

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Example

$\mathcal{X} = [0,1]^D$, $\alpha \geq 0$ and mean-payoff function f locally "α-smooth" around (any of) its maximum x^* (in finite number):

$$f(x^*) - f(x) = \Theta(||x - x^*||^\alpha) \text{ as } x \to x^*.$$

Theorem

Assume that we run HOO using $\rho(x,y) = ||x-y||^\beta$.

- **Known smoothness:** $\beta = \alpha$. $R_n = \tilde{O}(\sqrt{n})$, **i.e., the rate is independent of the dimension** D.
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  $R_n = \tilde{O}(n^{(d+1)/(d+2)})$ *where* $d = D\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)$.
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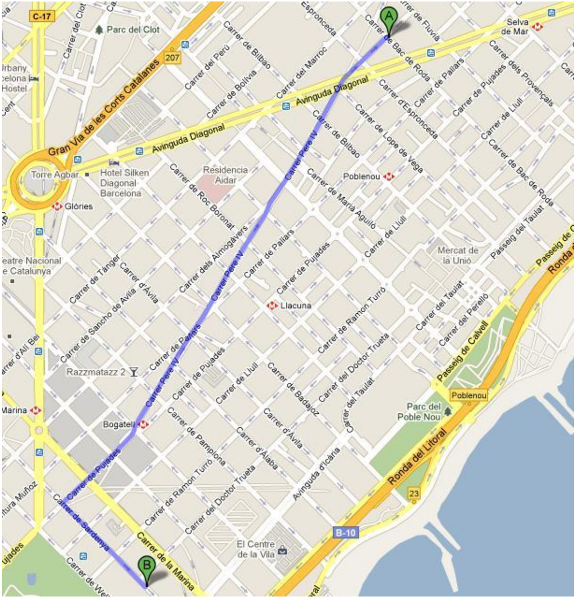
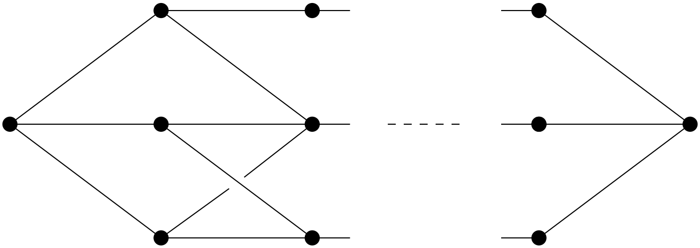
Path planning



Combinatorial prediction game

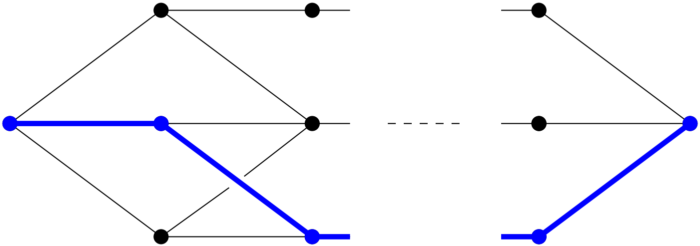
Adversary

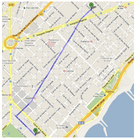


Player

Combinatorial prediction game

Adversary



Player ⟶ 

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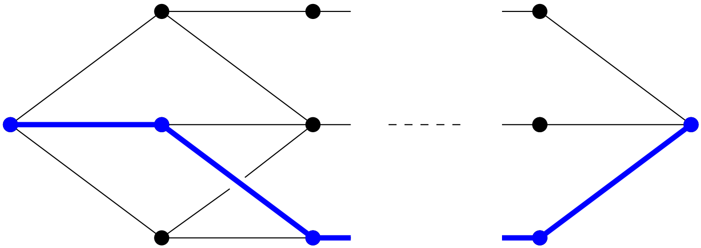
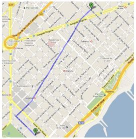


Player ⟶ 

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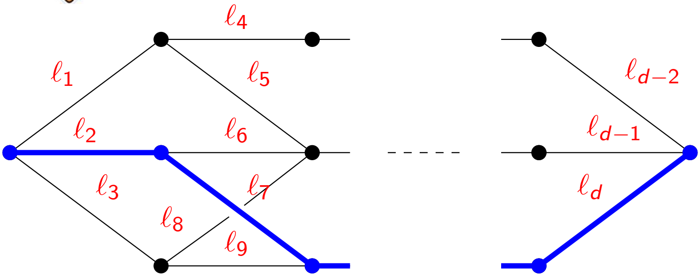

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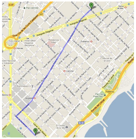

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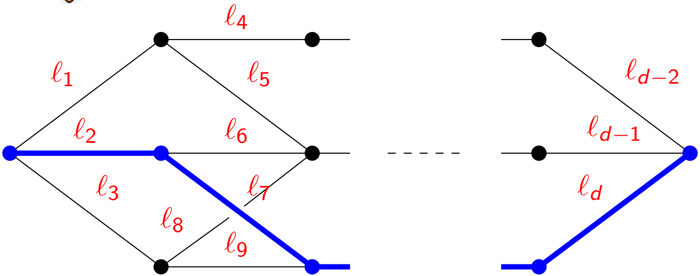

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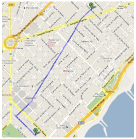
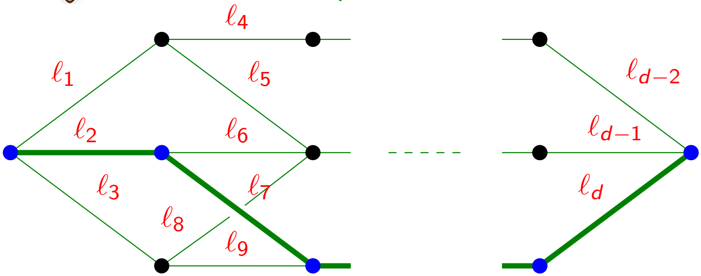


loss suffered: $\ell_2 + \ell_7 + \ldots + \ell_d$

Combinatorial prediction game

Adversary ⟶

Feedback: Full Info: $\ell_1, \ell_2, \dots, \ell_d$



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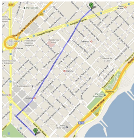

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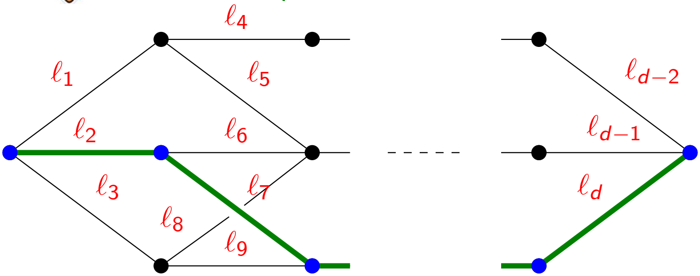
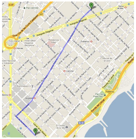
Combinatorial prediction game

Adversary ⟶ 

Feedback:

- Full Info: $\ell_1, \ell_2, \ldots, \ell_d$
- Semi-Bandit: $\ell_2, \ell_7, \ldots, \ell_d$
- Bandit: $\ell_2 + \ell_7 + \ldots + \ell_d$



Player ⟶ 

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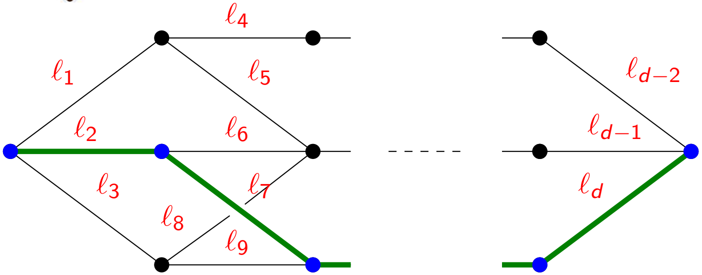
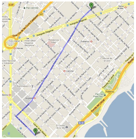


Player $\longrightarrow$ 

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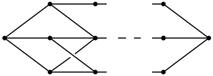
Notation



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$\rightsquigarrow$ $\ell_t \in \mathbb{R}_+^d$

$\rightsquigarrow$ $V_t \in \mathcal{S}$, loss suffered: $\ell_t^T V_t$

$$R_n = \mathbb{E} \sum_{t=1}^{n} \ell_t^T V_t - \min_{u \in S} \mathbb{E} \sum_{t=1}^{n} \ell_t^T u$$

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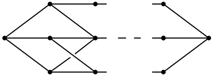

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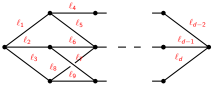

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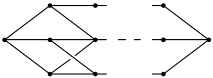

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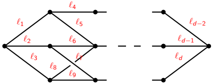

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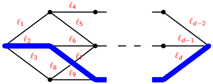

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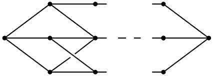

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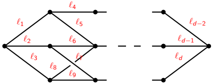

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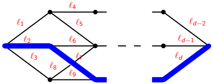

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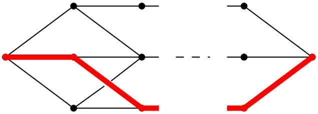
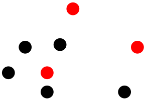
Set of concepts $S \subset \{0,1\}^d$

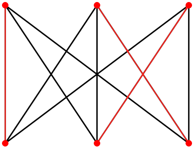
Paths



k-sets

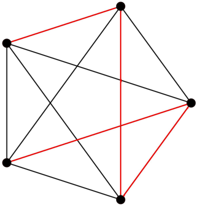


Matchings



k-sized intervals

Spanning trees



Parallel bandits

Key idea

$$V_t \sim p_t, \quad p_t \in \Delta(\mathcal{S})$$

Then, unbiased estimate $\tilde{\ell}_t$ of the loss ℓ_t:

- $\tilde{\ell}_t = \ell_t$ in the full information game,
- $\tilde{\ell}_{i,t} = \frac{\ell_{i,t}}{\sum_{V \in \mathcal{S}: V_i = 1} p_t(V)} V_{i,t}$ in the semi-bandit game,
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Loss assumptions

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We say that the adversary satisfies the L_∞ **assumption:** if $\|\ell_t\|_\infty \leq 1$ for all $t = 1, \ldots, n$.

Definition (L_2)

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$$p_t(v) = \frac{\exp\left(-\eta \sum_{s=1}^{t-1} \tilde{\ell}_s^T v\right)}{\sum_{u \in \mathcal{S}} \exp\left(-\eta \sum_{s=1}^{t-1} \tilde{\ell}_s^T u\right)}$$

- In the full information game, against L_2 adversaries, we have (for some η)

$$R_n \leq \sqrt{2dn},$$

  which is the optimal rate, Dani, Hayes and Kakade [2008].
- Thus against L_∞ adversaries we have

$$R_n \leq d^{3/2}\sqrt{2n}.$$

  But this is suboptimal, Koolen, Warmuth and Kivinen [2010].
- Audibert, Bubeck and Lugosi [2011] showed that, for any η, there exists a subset $S \subset \{0,1\}^d$ and an L_∞ adversary such that:

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Legendre function

Definition

Let $\mathcal{D}$ be a convex subset of $\mathbb{R}^d$ with nonempty interior $\text{int}(\mathcal{D})$ and boundary $\partial\mathcal{D}$. We call Legendre any function $F : \mathcal{D} \to \mathbb{R}$ such that

- F is strictly convex and admits continuous first partial derivatives on $\text{int}(\mathcal{D})$,
- For any $u \in \partial\mathcal{D}$, for any $v \in \text{int}(\mathcal{D})$, we have

$$\lim_{s \to 0, s > 0} (u - v)^T \nabla F\big((1-s)u + sv\big) = +\infty.$$

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Bregman divergence

Definition

The Bregman divergence $D_F : \mathcal{D} \times \text{int}(\mathcal{D})$ associated to a Legendre function F is defined by

$$D_F(u, v) = F(u) - F(v) - (u - v)^T \nabla F(v).$$

Definition

The Legendre transform of F is defined by

$$F^*(u) = \sup_{x \in \mathcal{D}} x^T u - F(x).$$

Key property for Legendre functions: $\nabla F^* = (\nabla F)^{-1}$.

Online Stochastic Mirror Descent (OSMD)

Parameter: F Legendre on $\mathcal{D} \supset Conv(\mathcal{S})$

(1) $w'_{t+1} \in \mathcal{D}$:

$$w'_{t+1} = \nabla F^* \left(\nabla F(w_t) - \tilde{\ell}_t \right)$$

(2) $w_{t+1} \in \underset{w \in Conv(\mathcal{S})}{\operatorname{argmin}} D_F(w, w'_{t+1})$

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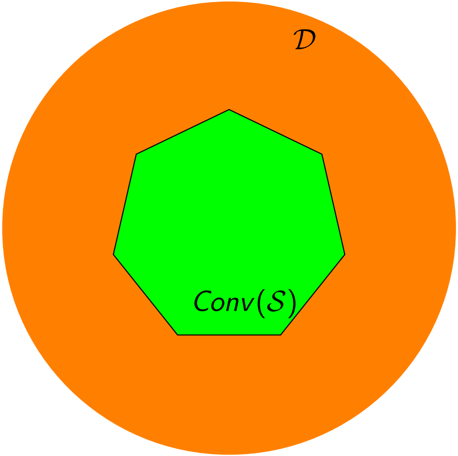

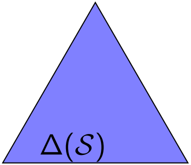

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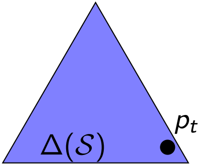

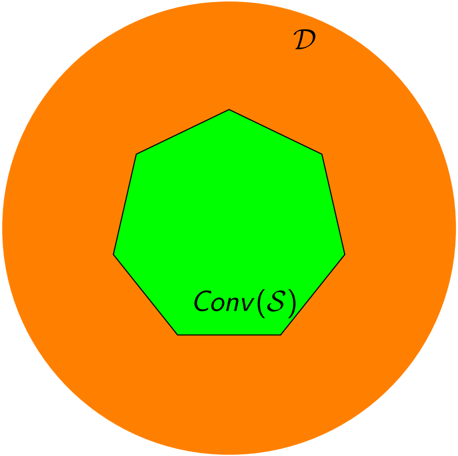

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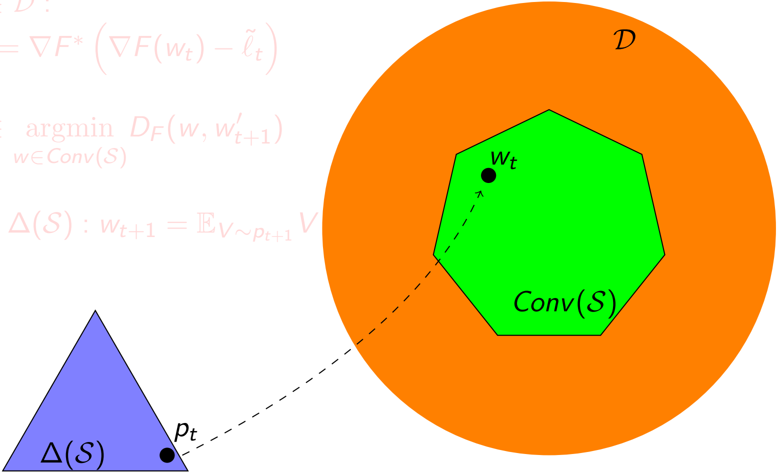

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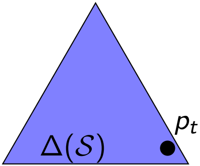

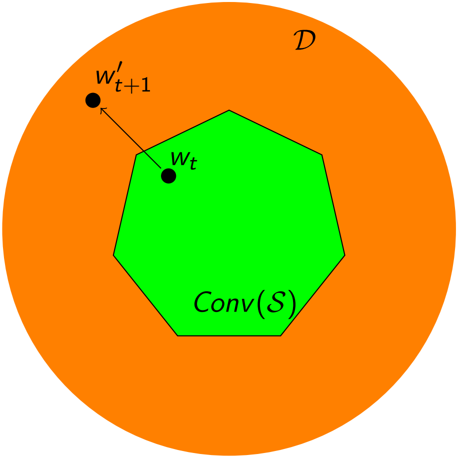

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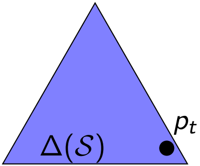

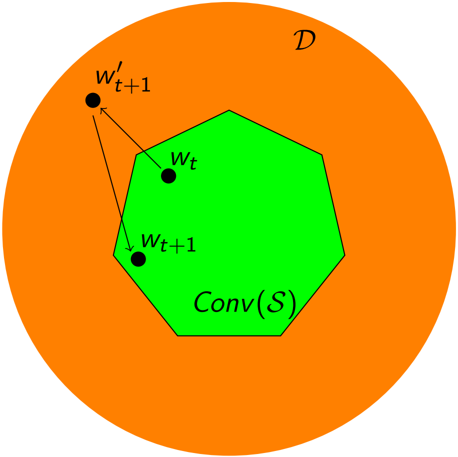

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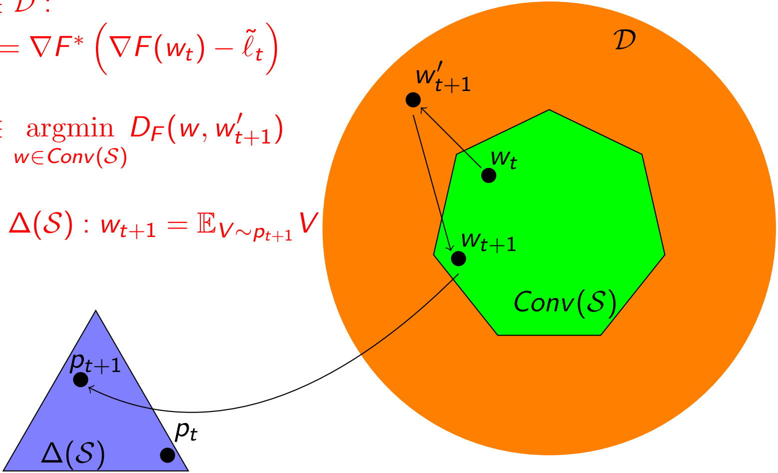

General regret bound for OSMD

Theorem

If F admits a Hessian $\nabla^2 F$ always invertible then,

$$R_n \lessapprox diam_{D_F}(\mathcal{S}) + \mathbb{E} \sum_{t=1}^{n} \tilde{\ell}_t^T \left(\nabla^2 F(w_t)\right)^{-1} \tilde{\ell}_t .$$

Different instances of OSMD: LinExp (Entropy Function)

$\mathcal{D} = [0, +\infty)^d$, $F(x) = \frac{1}{\eta} \sum_{i=1}^{d} x_i \log x_i$

Full Info: Exp

Semi-Bandit=Bandit: Exp3
Auer et al. [2002]

Full Info: Component Hedge
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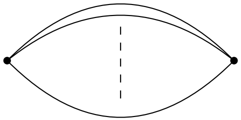

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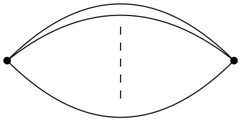

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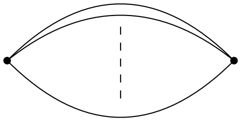

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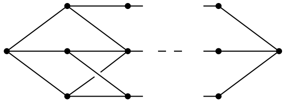

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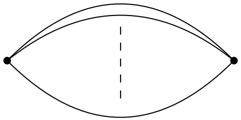

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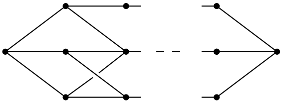

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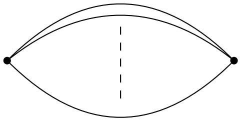

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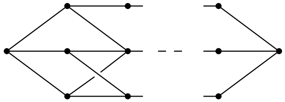

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Different instances of OSMD: LinINF (Exchangeable Hessian)

$\mathcal{D} = [0, +\infty)^d$, $F(x) = \sum_{i=1}^{d} \int_0^{x_i} \psi^{-1}(s)ds$

INF, Audibert and Bubeck [2009]

$$\begin{cases} \psi(x) = \exp(\eta x) : \text{LinExp} \\ \psi(x) = (-\eta x)^{-q}, q > 1 : \text{LinPoly} \end{cases}$$

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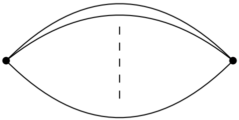

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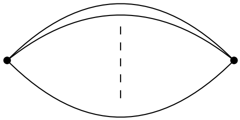

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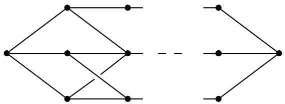

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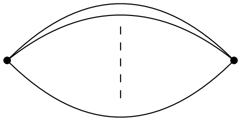

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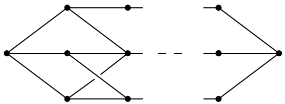

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Different instances of OSMD: Follow the regularized leader

$\mathcal{D} = Conv(\mathcal{S})$, then

$$w_{t+1} \in \operatorname*{argmin}_{w \in \mathcal{D}} \left(\sum_{s=1}^{t} \tilde{\ell}_s^T w + F(w) \right)$$

Particularly interesting choice: F self-concordant barrier function, Abernethy, Hazan and Rakhlin [2008]

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Minimax regret for the full information game

Theorem (Koolen, Warmuth and Kivinen [2010])

In the full information game, the LinExp strategy (with well-chosen parameters) satisfies for any concept class $S \subset \{0,1\}^d$ and any L_∞-adversary:

$$R_n \leq d\sqrt{2n}.$$

Moreover for any strategy, there exists a subset $S \subset \{0,1\}^d$ and an L_∞-adversary such that:

$$R_n \geq 0.008\ d\sqrt{n}.$$

Minimax regret for the semi-bandit game

Theorem (Audibert, Bubeck and Lugosi [2011])

In the semi-bandit game, the LinExp strategy (with well-chosen parameters) satisfies for any concept class $S \subset \{0,1\}^d$ and any L_∞-adversary:

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Minimax regret for the bandit game

For the bandit game the situation becomes trickier.

- First it appears necessary to add some sort of forced exploration on S to control third order error terms in the regret bound.
- Second, the control of the quadratic term $\tilde{\ell}_t^T \left(\nabla^2 F(w_t)\right)^{-1} \tilde{\ell}_t$ is much more involved than previously.

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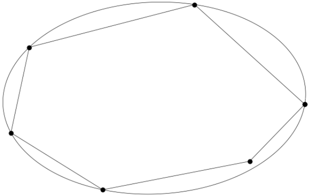
John's distribution

Theorem (John's Theorem)

Let $\mathcal{K} \subset \mathbb{R}^d$ be a convex set. If the ellipsoid $\mathcal{E}$ of minimal volume enclosing $\mathcal{K}$ is the unit ball in some norm derived from a scalar product $\langle \cdot, \cdot \rangle$, then there exists $M \leq d(d+1)/2 + 1$ contact points $u_1, \ldots, u_M$ between $\mathcal{E}$ and $\mathcal{K}$, and $\mu \in \Delta_M$ (the simplex of dimension $M - 1$), such that

$$x = d \sum_{i=1}^{M} \mu_i \langle x, u_i \rangle u_i, \forall\, x \in \mathbb{R}^d.$$



Minimax regret for the bandit game

Theorem (Audibert, Bubeck and Lugosi [2011], Bubeck, Cesa-Bianchi and Kakade [2012])

In the bandit game, the Exp2 strategy with John's exploration satisfies for any concept class $S \subset \{0,1\}^d$ and any L_∞-adversary:

$$R_n \leq 4d^2\sqrt{n},$$

and respectively $R_n \leq 4d\sqrt{n}$ for an L_2-adversary.
Moreover for any strategy, there exists a subset $S \subset \{0,1\}^d$ and an L_∞-adversary such that:

$$R_n \geq 0.01\ d^{3/2}\sqrt{n}.$$

For L_2-adversaries the lower bound is $0.05 \min(n, d\sqrt{n})$.

Conjecture: for an L_∞-adversary the correct order of magnitude is $d^{3/2}\sqrt{n}$ and it can be attained with OSMD.