The Optimal Mechanism for Selling to a Budget-Constrained Buyer: The General Case

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We consider a revenue-maximizing seller with a single item facing a single buyer with a private budget. The (value, budget) pair is drawn from an *arbitrary* and possibly correlated distribution. We characterize the optimal mechanism in such cases, and quantify the amount of price discrimination that might be present. For example, there could be up to $3 \cdot 2^{k-1} - 1$ distinct non-trivial menu options in the optimal mechanism for such a buyer with k distinct possible budgets (compared to k if the marginal distribution of values conditioned on each budget has decreasing marginal revenue [CG00], or 2 if there is an arbitrary distribution and one possible budget [CMM11]).

Our approach makes use of the duality framework of [CDW16], and duality techniques related to the "FedEx Problem" of [FGKK16]. In contrast to [FGKK16] and other prior work, we characterize the optimal primal/dual without nailing down an explicit closed form.

CCS Concepts: •**Theory of computation → Algorithmic mechanism design;**

Additional Key Words and Phrases: Revenue maximization; budgets

The theory of optimal auction often equates willingness to pay with the ability to pay. While this leads to clean formulations and elegant solutions, there are many instances where this assumption is violated (see [CG00] for more discussion). In this paper we consider selling a single item to a single bidder, who has a quasi-linear utility function as well as a hard upper bound on her payment. In other words, if the buyer obtains the item and pays p, then her utility is $v - p$, as long as $p < b$, for some two numbers v and b. We call v her valuation and b her budget, and both of them are private information. We consider the problem in the Bayesian setting, where there is a joint probability distribution over types (v, b), which is known to the seller. In such settings, it is known that the optimal mechanism necessitates selling *lotteries* (e.g. awarding the item with probability in $(0, 1)$).

We study the optimal *conditional* mechanism, where the seller prevents the buyer from exaggerating her budget (but not under-reporting her budget). This can be enforced, for example, by requiring a cash bond or by requiring the buyer to pay her full budget with some small probability.

The core of the matter is how to use the budget constraints to do price discrimination. Buyers can feasibly report a lower budget, so the ones with higher budgets extract higher information rents. The economic intuition gained from previous work on this subject is restricted to cases where the budget is fixed/known ([LR96, CMM11]), or the marginal distribution of values conditioned on each

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possible budget satisfies *declining marginal revenues* (DMR) [CG00].[1] In all cases previously studied, the number of non-trivial menu items is shown to be at most the number of distinct budgets.

Our main contribution in this paper is to characterize the optimal mechanism in the private budget setting without any assumption on the distributions. The main economic intuition gained from our results is a quantitative understanding of the degree of price discrimination that might arise in the general case. For example:

> If the buyer's value is drawn from an arbitrary distribution and there are possible budgets $b_1 < \ldots < b_k$, private to the buyer, the optimal mechanism has the following format: the menu offered to buyers with budget b_1 has at most two non-trivial options. There is a cuttoff budget b_i, below which the budget binds and above which the menu contains an option to receive the item with probability 1. Every option on the menu presented to a buyer with budget b_{j-1} *splits* into at most two options to be included on the menu presented to a buyer with budget b_j. In addition, there is at most one additional option at the top (i.e. higher allocation probability than all other options). If $j < i$, the additional option offers the item at probability $q < 1$ at price $b_j q$. If $j \geq i$, the additional option offers the item at probability 1 ($\leq 3 \cdot 2^{k-1} - 1$ non-trivial menu options in total).

A key aspect of our approach is the idea of Lagrangifying only the incentive/budget constraints, as in the duality framework of [CDW16]. A particularly nice property of this is that the duals give a virtual valuation for each type. The optimal mechanism is then such that all the types with positive virtual values are allocated with probability 1, and all the types with negative virtual values are not allocated at all. For the types with a zero virtual value the allocation probability is typically in $(0, 1)$; and we need to use additional structure of the dual to pin down these probabilities (specifically, complementary slackness). The optimal dual is such that the sign of the virtual value is monotone in the buyer's value.

We emphasize one key technical departure from previous works. In prior works, an explicit optimal dual solution and explicit optimal primal solution are proposed, and then complementary slackness is proved. In our setting, this would be a complete nightmare - mostly due to the multiple budget constraints, so instead we characterize the optimal dual solution without nailing down a closed form. A little more specifically, we provide three elementary operations on a candidate dual solution that can only *improve* the candidate's quality. We can then conclude that there exists an optimal dual such that no more of these operations are possible, thereby characterizing what optimal duals might look like. This structure turns out to be enough to show that there exists an optimal primal solution of the desired form which satisfies complementary slackness. All this is done without excessive algebraic calculation to nail down a closed form.

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[1] A one-dimensional distribution with CDF F and density f satisfies DMR if $v(1 - F(v))$ is concave.