Algorithm Selection for Reinforcement Learning

Romain Laroche  
Microsoft Maluuba, Montréal, QC, Canada  
ROMAIN.LAROCHE@MICROSOFT.COM

Raphaël Féraud  
Orange Labs, Lannion, France  
RAPHAEL.FERAUD@ORANGE.COM

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Abstract

This paper formalises the problem of online algorithm selection in the context of Reinforcement Learning. The setup is as follows: given an episodic task and a finite number of off-policy RL algorithms, a meta-algorithm has to decide which RL algorithm is in control during the next episode so as to maximize the expected return. The article presents a novel meta-algorithm, called Epochal Stochastic Bandit Algorithm Selection (ESBAS). Its principle is to freeze the policy updates at each epoch, and to leave a rebooted stochastic bandit in charge of the algorithm selection. Under some assumptions, a thorough theoretical analysis demonstrates its near-optimality considering the structural sampling budget limitations. ESBAS is first empirically validated on a dialogue task under 32 algorithm configurations. Another experiment on a fruit collection task shows that ESBAS can successfully be adapted to a true online setting where algorithms update their policies after each transition.

1. Introduction

Reinforcement Learning (RL, Sutton and Barto (1998)) is a machine learning framework for optimising the behaviour of an agent interacting with an unknown environment. For the most practical problems, such as dialogue or robotics, trajectory collection is costly and sample efficiency is the main key performance indicator. Consequently, when applying RL to a new problem, one must carefully choose in advance a model, an optimisation technique and their parameters. Facing the complexity of choice, RL and domain expertise is not sufficient. Confronted to the cost of data, the popular trial and error approach shows its limits.

We develop an online learning version (Gagliolo and Schmidhuber, 2006, 2010) of Algorithm Selection (AS, Rice (1976); Smith-Miles (2009); Kotthoff (2012)). It consists in testing several algorithms on the task and in selecting the best one at a given time. For clarity, throughout the whole article, the algorithm selector is called a meta-algorithm, and the set of algorithms available to the meta-algorithm is called a portfolio. The meta-algorithm has for objective to maximise the expected return as defined in the RL task. Beyond the sample efficiency objective, the online AS approach besides addresses three practical problems for online RL-based systems. First, it improves robustness against implementation bugs: if an algorithm fails to terminate, or converges to an aberrant policy, it will be dismissed and others will be selected instead. Second, convergence guarantees and empirical efficiency may be united by covering the empirically efficient algorithms with slower algorithms that have convergence guarantees. Third, it enables staggered learning: shallow models control the policy in the early stages, while deep models discover the best solution and control the policy in late stages.

A fair algorithm selection implies a fair budget allocation between the algorithms, so that they can be equitably evaluated and compared. In order to comply with this requirement, the reinforcement algorithms in the portfolio are assumed to be off-policy, and are trained on every trajectory, regardless which algorithm controls it. Section 2 provides a unifying view of RL algorithms, that allows information sharing between all algorithms of the portfolio, whatever their decision processes, their state representations, and their optimisation techniques. Next, Section 3 formalises the problem of online selection of off-policy RL algorithms. It introduces the absolute pseudo-regret and states two assumptions on the algorithms monotonicity.

Afterwards, Section 4 presents the Epochal Stochastic Bandit AS (ESBAS), a novel meta-algorithm addressing the online off-policy RL AS problem. Its principle is to divide the time-scale into epochs of...
Then, Section 5 leads 32 experiments with various portfolios on a negotiation dialogue game (Laroche and Genevay 2016). An extension of ESBAS to non-epochal training algorithms is also successfully tried out on a fruit collection task. The results show the practical benefits of ESBAS, in most cases even outperforming the best algorithm in the portfolio. Finally, Section 6 concludes the article with prospective ideas of improvement. The appendix includes a glossary of the notations used throughout the article, the extensive numerical results of the experiments, and the proof of all theorems.

2. Unifying view of RL algorithms

The goal of this section is to enable information sharing between algorithms, even though they are considered as black boxes. We propose to share their trajectories expressed in a universal format: the interaction process.

Reinforcement Learning (RL) consists in learning through trial and error to control an agent behaviour in a stochastic environment: at each time step \( t \in \mathbb{N} \), the agent performs an action \( a(t) \in \mathcal{A} \), and then perceives from its environment a signal \( o(t) \in \Omega \) called observation, and receives a reward \( R(t) \in \mathbb{R} \). Figure 1 illustrates the RL framework. This interaction process is not Markovian: the agent may have an internal memory.

In this article, the reward function is assumed to be bounded between \( R_{min} \) and \( R_{max} \), and we define the RL problem as episodic. Let us introduce two time scales with different notations. First, let us define meta-time as the time scale for AS: at one meta-time \( \tau \) corresponds a meta-algorithm decision, i.e. the choice of an algorithm and the generation of a full episode controlled with the policy determined by the chosen algorithm. Its realisation is called a trajectory. Second, RL-time is defined as the time scale inside a trajectory, at one RL-time \( t \) corresponds one triplet composed of an observation, an action, and a reward.

Let \( \mathscr{E} \) denote the space of trajectories. A trajectory \( \varepsilon_\tau \in \mathscr{E} \) collected at meta-time \( \tau \) is formalised as a sequence of (observation, action, reward) triplets: \( \varepsilon_\tau = \{ \langle o_\tau(t), a_\tau(t), R_\tau(t) \rangle \}_{t \in [1,J]} \in \mathscr{E} \), where \( |\varepsilon_\tau| \) is the length of trajectory \( \varepsilon_\tau \). The objective is, given a discount factor \( 0 \leq \gamma < 1 \), to generate trajectories with high discounted cumulative reward, also called return, and noted \( \mu(\varepsilon_\tau) \):

\[
\mu(\varepsilon_\tau) = \sum_{t=1}^{J} \gamma^{t-1} R_\tau(t). \tag{1}
\]

Since \( \gamma < 1 \) and \( R_{min} \leq R \leq R_{max} \), the return is bounded: \( \frac{R_{min}}{1-\gamma} \leq \mu(\varepsilon_\tau) \leq \frac{R_{max}}{1-\gamma} \). The trajectory set at meta-time \( T \) is denoted by \( \mathcal{D}_T = \{ \varepsilon_\tau \}_{\tau \in [1,T]} \in \mathscr{E}^T \). A sub-trajectory of \( \varepsilon_\tau \) until RL-time \( t \) is called the history at RL-time \( t \) and written \( \varepsilon_\tau(t) \) with \( t \leq |\varepsilon_\tau| \). The history records what happened in episode \( \varepsilon_\tau \) until RL-time \( t \): \( \varepsilon_\tau(t) = \langle o_\tau(t'), a_\tau(t'), R_\tau(t') \rangle_{t' \in [1,t]} \in \mathscr{E}. \)

The goal of each RL algorithm \( \alpha \) is to find a policy \( \pi^* : \mathscr{E} \rightarrow \mathcal{A} \) which yields optimal expected returns. Such an algorithm \( \alpha \) is viewed as a black box that takes as an input a trajectory set \( \mathcal{D} \in \mathscr{E}^+ \), where \( \mathscr{E}^+ \) is the ensemble of trajectory sets of undetermined size: \( \mathscr{E}^+ = \bigcup_{T \in \mathbb{N}} \mathscr{E}^T \), and that outputs a policy \( \pi^+_\alpha \). Consequently, a RL algorithm is formalised as follows: \( \alpha : \mathscr{E}^+ \rightarrow (\mathscr{E} \rightarrow \mathcal{A}) \).

Such a high level definition of the RL algorithms allows to share trajectories between algorithms: a trajectory as a sequence of observations, actions, and rewards can be interpreted by any algorithm in its own decision process and state representation. For instance, RL algorithms classically rely on an MDP defined on a state space representation \( S^D_\mathcal{D} \) thanks to a projection \( \Phi^D_\mathcal{D} : \mathscr{E} \rightarrow S^D_\mathcal{D} \).

The state representation may be built dynamically as the trajectories are collected. Many algorithms doing so can be found in the literature, for instance Legenstein et al. (2010); Böhmer et al. (2015). Then, \( \alpha \) trains its
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policy $π_{Dα}$ on the trajectories projected on its state space representation. Off-policy RL optimisation techniques compatible with this approach are numerous in the literature (Maei et al., 2010): Q-Learning (Watkins, 1989), Fitted-Q Iteration (Ernst et al., 2005), Kalman Temporal Difference (Geist and Pietquin, 2010), DQN (Mnih et al., 2013), etc. As well, any post-treatment of the state set, any alternative decision process model, such as POMDPs (Lovejoy, 1991), and any off-policy technique for control optimisation may be used. The algorithms are defined here as black boxes and the considered meta-algorithms will be indifferent to how the algorithms compute their policies, granted they satisfy the assumptions stated in the following section.

3. Algorithm selection model

3.1 Online algorithm selection

The online learning approach is tackled in this article: different algorithms are experienced and evaluated during the data collection. Since it boils down to a classical exploration/exploitation trade-off, multi-armed bandit (Bubeck and Cesa-Bianchi, 2012) have been used for combinatorial search AS (Gagliolo and Schmidhuber, 2006–2010) and evolutionary algorithm meta-learning (Fialho et al., 2010). The online AS problem for off-policy RL is novel and we define it here: $D ∈ B^+$ is the trajectory set; $P = \{α^k\}_{k∈[1,K]}$ is the portfolio of off-policy RL algorithms; and $μ : B^+ → ℝ$ is the objective function defined in Equation 1.

Pseudo-code 1 formalises the online AS setting. A meta-algorithm is defined as a function from a trajectory set to the selection of an algorithm: $σ : B^+ → P$. The meta-algorithm is queried at each meta-time $τ = |D_{τ−1}| + 1$, with input $D_{τ−1}$, and it outputs algorithm $σ(D_{τ−1}) = σ(τ) ∈ P$ controlling with its policy $π_{Dσ(τ)}$, the generation of the trajectory $ε_{σ(τ)}$ in the stochastic environment. The final goal is to optimise the cumulative expected return. It is the expectation of the sum of rewards obtained after a run of $T$ trajectories:

$$E_σ \left[\sum_{τ=1}^{T} μ(ε_{σ(τ)})\right] = E_σ \left[\sum_{τ=1}^{T} E_μ[σ(τ)]\right],$$

with $E_μ[σ(τ)] = E_μ[σ(ε)]$ as a condensed notation for the expected return of policy $π_{Dσ}$, trained on trajectory set $D$ by algorithm $α$. Equation 2 transforms the cumulative expected return into two nested expectations. The outside expectation $E_σ$ assumes the AS fixed and averages over the trajectory set generation and the corresponding algorithms policies. The inside expectation $E_μ$ assumes the policy fixed and averages over its possible trajectories in the stochastic environment. Nota bene: there are three levels of decision: meta-algorithm $σ$ selects algorithm $α$ that computes policy $π$ that is in control. The focus is at the meta-algorithm level.

3.2 Meta-algorithm evaluation

In order to evaluate the meta-algorithms, let us formulate two additional notations. First, the optimal expected return $E_μ^{∞}$ is defined as the highest expected return achievable by a policy of an algorithm in portfolio $P$:

$$\forall D ∈ B^+, \forall α ∈ P, \ E_μ^{Dα} ≤ E_μ^{∞},$$

$$\forall ε > 0, \exists D ∈ B^+, \exists α ∈ P, \ E_μ^{∞} − E_μ^{Dα} < ε.$$

Second, for every algorithm $α$ in the portfolio, let us define $σ^α$ as its canonical meta-algorithm, i.e. the meta-algorithm that always selects algorithm $α$: $∀τ, σ^α(τ) = α$. The absolute pseudo-regret $p^α_{abs}(T)$ defines the regret as the loss for not having controlled the trajectory with an optimal policy.

<table>
<thead>
<tr>
<th>Pseudo-code 1: Online AS setting</th>
</tr>
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<tbody>
<tr>
<td><strong>Data:</strong> $P ← {α^k}_{k∈[1,K]}$: algorithm portfolio</td>
</tr>
<tr>
<td><strong>Data:</strong> $D_0 ← ∅$: trajectory set</td>
</tr>
<tr>
<td><strong>for</strong> $τ ← 1$ <strong>to</strong> $∞$ <strong>do</strong></td>
</tr>
<tr>
<td>&gt; Select $σ(D_{τ−1}) = σ(τ) ∈ P$;</td>
</tr>
<tr>
<td>&gt; Generate trajectory $ε_τ$ with policy $π_{Dσ(τ)}$;</td>
</tr>
<tr>
<td>&gt; Get return $μ(ε_τ)$;</td>
</tr>
<tr>
<td>&gt; $D_τ ← D_{τ−1} ∪ {ε_τ}$;</td>
</tr>
<tr>
<td><strong>end</strong></td>
</tr>
</tbody>
</table>
Definition 1 (Absolute pseudo-regret) The absolute pseudo-regret $\rho_{\text{abs}}^\sigma(T)$ compares the meta-algorithm’s expected return with the optimal expected return:

$$
\rho_{\text{abs}}^\sigma(T) = T\mathbb{E}\mu_\infty^\sigma - \mathbb{E}_\sigma \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\sigma_{\tau-1}^\sigma(\tau)} \right].
$$

(4)

It is worth noting that an optimal meta-algorithm will not yield a null regret because a large part of the absolute pseudo-regret is caused by the sub-optimality of the algorithm policies when the trajectory set is still of limited size. Indeed, the absolute pseudo-regret considers the regret for not selecting an optimal policy: it takes into account both the pseudo-regret of not selecting the best algorithm and the pseudo-regret of the algorithms for not finding an optimal policy. Since the meta-algorithm does not interfere with the training of policies, it cannot account for the pseudo-regret related to the latter.

3.3 Assumptions

Theoretical analysis is hindered by the fact that AS, not only directly influences the return distribution, but also the trajectory set distribution and therefore the policies learnt by algorithms for next trajectories, which will indirectly affect the future expected returns. In order to allow policy comparison, based on relation on trajectory sets they are derived from, our analysis relies on two assumptions.

Assumption 1 (More data is better data) The algorithms train better policies with a larger trajectory set on average, whatever the algorithm that controlled the additional trajectory:

$$
\forall D \in \mathcal{S}^+, \forall \alpha, \alpha' \in \mathcal{P}, \quad \mathbb{E}\mu_D^\alpha \leq \mathbb{E}_{\alpha'} \left[ \mathbb{E}\mu_{D \cup \varepsilon_{\alpha'}}^\alpha \right].
$$

(5)

Assumption 1 states that algorithms are off-policy learners and that additional data cannot lead to performance degradation on average. An algorithm that is not off-policy could be biased by a specific behavioural policy and would therefore transgress this assumption.

Assumption 2 (Order compatibility) If an algorithm trains a better policy with one trajectory set than with another, then it remains the same, on average, after collecting an additional trajectory from any algorithm:

$$
\forall D, D' \in \mathcal{S}^+, \forall \alpha, \alpha' \in \mathcal{P}, \quad \mathbb{E}\mu_D^\alpha < \mathbb{E}_{\mu_{D'}}^\alpha \Rightarrow \mathbb{E}_{\alpha'} \left[ \mathbb{E}\mu_{D \cup \varepsilon_{\alpha'}}^\alpha \right] \leq \mathbb{E}_{\alpha'} \left[ \mathbb{E}\mu_{\alpha'_{D' \cup \varepsilon_{\alpha'}}}^\alpha \right].
$$

(6)

Assumption 2 states that a performance relation between two policies trained on two trajectory sets is preserved on average after adding another trajectory, whatever the behavioural policy used to generate it. From these two assumptions, Theorem 1 provides an upper bound in order of magnitude in function of the worst algorithm in the portfolio. It is verified for any meta-algorithm $\sigma$.

Theorem 1 (Not worse than the worst) The absolute pseudo-regret is bounded by the worst algorithm absolute pseudo-regret in order of magnitude:

$$
\forall \sigma, \quad \rho_{\text{abs}}^\sigma(T) \in O \left( \max_{\alpha \in \mathcal{P}} \rho_{\text{abs}}^\alpha(T) \right).
$$

(7)

Contrarily to what the name of Theorem 1 suggests, a meta-algorithm might be worse than the worst algorithm (similarly, it can be better than the best algorithm), but not in order of magnitude. Its proof is rather complex for such an intuitive and loose result because, in order to control all the possible outcomes, one needs to translate the selections of algorithm $\alpha$ with meta-algorithm $\sigma$ into the canonical meta-algorithm $\sigma^\alpha$’s view, in order to be comparable with it.
4. Epochal stochastic bandit AS

4.1 Related work

Related to AS for RL, Schweighofer and Doya (2003) use meta-learning to tune a fixed RL algorithm in order to fit observed animal behaviour, which is a very different problem to ours. In Cauwet et al. (2014); Liu and Teytaud (2014), the RL AS problem is solved with a portfolio composed of online RL algorithms. In those articles, the core problem is to balance the budget allocated to the sampling of old policies through a lag function, and budget allocated to the exploitation of the up-to-date algorithms. The main limitation from these works relies on the fact that on-policy algorithms were used, which prevents them from sharing trajectories among algorithms (Cauwet et al., 2015). Meta-learning specifically for the eligibility trace parameter has also been studied in White and White (2016). A recent work (Wang et al., 2016) studies the learning process of RL algorithms and selects the best one for learning faster on a new task. This work is related to batch AS.

AS for RL can also be related to ensemble RL. Wiering and Van Hasselt (2008) and Laroche et al. (2017) use combinations of a set of RL algorithms to build its online control such as policy voting or value function averaging. This approach shows good results when all the algorithms are efficient, but not when some of them are underachieving. Hence, no convergence bound has been proven with this family of meta-algorithms. HORDE (Sutton et al., 2011) and multi-objective ensemble RL (Brys et al., 2014; van Seijen et al., 2016) are algorithms for hierarchical RL and do not directly compare with AS.

Regarding policy selection, ESBAS advantageously compares with the RL with Policy Advice’s regret bounds on static policies Azar et al. (2013) (on non-episodic RL tasks).

An intuitive way to solve the AS problem is to consider algorithms as arms in a multi-armed bandit setting. The bandit meta-algorithm selects the algorithm controlling the next trajectory $\varepsilon$ and the trajectory return $\mu(\varepsilon)$ constitutes the reward of the bandit. The aim of prediction with expert advice is to minimise the regret against the best expert of a set of predefined experts. When the experts learn during time, their performances evolve and hence the sequence of expert rewards is non-stationary. The exponential weight algorithms (Auer et al., 2002b; Cesa-Bianchi and Lugosi, 2006) are designed for prediction with expert advice when the sequence of rewards of experts is generated by an oblivious adversary. This approach has been extended for competing against the best sequence of experts by adding in the update of weights a forgetting factor proportional to the mean reward (see Exp3.S in Auer et al. (2002b)), or by combining Exp3 with a concept drift detector Allesiardo and Féraud (2015). The exponential weight algorithms have been extended to the case where the rewards are generated by any sequence of stochastic processes of unknown means (Besbes et al., 2014). On one hand, the exponential weight algorithms can be applied to any individual sequence when no knowledge is available about the experts. On the other hand, the exponential weight algorithms suffer from a constant exploration factor, which is an issue for learning experts for which we know that the mean reward tends to be constant after an unknown time horizon.

The stochastic bandit algorithm such as UCB can be extended to the case of switching bandits using a discount factor or a window to forget the past Garivier and Moulines (2011). This class of algorithms performs well on the switching bandit problem but does not apply when experts learn and hence evolve at each time step. The ideas behind the proposed algorithm ESBAS consist in reducing a non-stationary problem into several stationary problems, and in using our knowledge on common reinforcement learning algorithms.

Alternatively, one might consider using policy search methods (Ng and Jordan, 2000), but they rely on a state space representation in order to apply the policy gradient. And in our case, neither policies do share any, nor the meta-algorithm does have at disposal any other than the intractable histories $\varepsilon_\tau(t)$ of Section 2.

4.2 ESBAS description

To solve the off-policy RL AS problem, we propose a novel meta-algorithm called Epochal Stochastic Bandit AS (ESBAS). Because of the non-stationarity induced by the algorithm learning, the stochastic bandit cannot directly select algorithms. Instead, the stochastic bandit can choose fixed policies. To comply to this constraint, the meta-time scale is divided into epochs inside which the algorithms policies cannot be updated: the algorithms optimise their policies only when epochs start, in such a way that the policies are constant inside
each epoch. As a consequence and since the returns are bounded, at each new epoch, the problem can be cast into an independent stochastic $K$-armed bandit $\Xi$, with $K = |\mathcal{P}|$.

The ESBAS meta-algorithm is formally sketched in Pseudo-code 2 embedding UCB1. Auer et al. (2002a) as the stochastic $K$-armed bandit $\Xi$. The meta-algorithm takes as an input the set of algorithms in the portfolio. Meta-time scale is fragmented into epochs of exponential size. The $\beta$th epoch lasts $2^\beta$ meta-time steps, so that, at meta-time $\tau = 2^\beta$, epoch $\beta$ starts. At the beginning of each epoch, the ESBAS meta-algorithm asks each algorithm in the portfolio to update their current policy. Inside an epoch, the policy is never updated anymore. At the beginning of each epoch, a new $\Xi$ instance is reset and run. During the whole epoch, $\Xi$ selects at each meta-time step the algorithm in control of the next trajectory.

### 4.3 Short-sighted pseudo-regret analysis

ESBAS intends to minimise the regret for not choosing the best algorithm at a given meta-time $\tau$. It is short-sighted: it does not intend to optimise the algorithms learning.

**Definition 2 (Short-sighted pseudo-regret)** The short-sighted pseudo-regret $p_{\text{ss}}^\alpha(T)$ is the difference between the immediate best expected return algorithm and the one selected:

$$p_{\text{ss}}^\alpha(T) = \mathbb{E}_\sigma\left[\sum_{\tau=1}^T \left(\max_{\alpha \in \mathcal{P}} \mathbb{E}\mu^\alpha_{D_{\tau-1}^\sigma} - \mathbb{E}\mu^\alpha_{D_{\tau-1}^\sigma}\right)\right].$$

(8)

**Theorem 2 (ESBAS short-sighted pseudo-regret)** If the stochastic multi-armed bandit $\Xi$ guarantees a regret of order of magnitude $\mathcal{O}(\log(T)/\Delta^1_\beta)$, then:

$$p_{\text{ss}}^\text{ESBAS}(T) \in \mathcal{O}\left(\sum_{\beta=0}^{\log(T)} \frac{\beta}{\Delta^1_\beta}\right).$$

(9)

Theorem 2 expresses in order of magnitude an upper bound for the short-sighted pseudo-regret of ESBAS. But first, let define the gaps: $\Delta^1_\beta = \max_{\alpha' \in \mathcal{P}} \mathbb{E}\mu^\alpha'_{D_{2^\beta-1}^\sigma} - \mathbb{E}\mu^\alpha_{D_{2^\beta-1}^\sigma}$. It is the difference of expected return between the best algorithm during epoch $\beta$ and algorithm $\alpha$. The smallest non null gap at epoch $\beta$ is noted: $\Delta^1_\beta = \min_{\alpha \in \mathcal{P}, \Delta^1_\alpha > 0} \Delta^1_\beta$. If $\Delta^1_\beta$ does not exist, i.e., if there is no non-null gap, the regret is null.

Several upper bounds in order of magnitude on $\mathbb{E}_\sigma p_{\text{ss}}^\alpha(T)$ can be easily deduced from Theorem 2, depending on an order of magnitude of $\Delta^1_\beta$ (see corollaries in the appendices). Table 1 reports some of them for a two-fold portfolio. It must be read by line. According to the first column: the order of magnitude of $\Delta^1_\beta$, the ESBAS short-sighted pseudo-regret bounds are displayed in the second column. The main result comes from the last line, where the algorithms converge to policies with different performance: ESBAS logarithmically

<table>
<thead>
<tr>
<th>$\Delta^1_\beta$</th>
<th>$p_{\text{ss}}^\text{ESBAS}(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta(1/T)$</td>
<td>$\mathcal{O}(\log(T))$</td>
</tr>
<tr>
<td>$\Theta(T^{-c_1})$, $c_1 \geq 0.5$</td>
<td>$\mathcal{O}(T^{1-c_1})$</td>
</tr>
<tr>
<td>$\Theta(T^{-c_1})$, $c_1 &lt; 0.5$</td>
<td>$\mathcal{O}(T^{c_1}\log(T))$</td>
</tr>
<tr>
<td>$\Theta(1)$</td>
<td>$\mathcal{O}(\log^2(T)/\Delta^1_\infty)$</td>
</tr>
</tbody>
</table>

Table 1: Bounds on $p_{\text{ss}}^\alpha(T)$ obtained with

Theorem 2 for the ESBAS meta-
algorithm with a two-fold portfolio.
converges to the best algorithm with a regret in $\mathcal{O}\left(\log^2(T)/\Delta^1_\beta\right)$. One should notice that the first two bounds are obtained by summing the gaps. This means that the algorithms are equally good and their gap goes beyond the threshold of distinguishability. This threshold is structurally at $\Delta^1_\beta \in \mathcal{O}(1/\sqrt{T})$. Even in the worst case, we have the guarantee of $\overline{p}_{abs}(T) \in \mathcal{O}(\sqrt{T})$, still better than the generic adversarial bounds. The impossibility to determine which is the better algorithm is interpreted in Cauwet et al. (2014) as a budget issue. The meta-time necessary to distinguish through evaluation arms that are $\Delta^1_\beta$ apart takes $\Theta(1/\Delta^1_\beta)$ meta-time steps. As a consequence, if $\Delta^1_\beta \in \mathcal{O}(1/\sqrt{T})$, then $1/\Delta^2_\beta \in \Omega(T)$. However, the budget, i.e. the length of epoch $\beta$ starting at meta-time $T = 2^\beta$, equals $T$.

### 4.4 ESBAS absolute pseudo-regret analysis

The short-sighted pseudo-regret optimality depends on the meta-algorithm itself. For instance, a poor deterministic algorithm might be optimal at meta-time $\tau$ but yield no new information, implying the same situation at meta-time $\tau + 1$, and so on. Thus, a meta-algorithm that exclusively selects the deterministic algorithm would achieve a short-sighted pseudo-regret equal to 0, but selecting other algorithms are, in the long run, more efficient. Theorem 2 is a necessary step towards the absolute pseudo-regret analysis.

The absolute pseudo-regret can be decomposed between the absolute pseudo-regret of the best canonical meta-algorithm (i.e. the algorithm that finds the best policy), the regret for not always selecting the best algorithm, and potentially not learning as fast, and the short-sighted regret: the regret for not gaining the returns granted by the best algorithm. This decomposition leads to Theorem 3 that provides an upper bound of the absolute pseudo-regret in function of the best canonical meta-algorithm, and the short-sighted pseudo-regret.

But first let us introduce the fairness assumption. The *fairness of budget distribution* has been formalised in Cauwet et al. (2015). It is the property stating that every algorithm in the portfolio has as much resources as the others, in terms of computational time and data. It is an issue in most online AS problems, since the algorithm that has been the most selected has the most data, and therefore must be the most advanced one. A way to circumvent this issue is to select them equally, but, in an online setting, the goal of AS is precisely to select the best algorithm as often as possible. Our answer is to require that all algorithms in the portfolio are learning off-policy, i.e. without bias induced by the behavioural policy used in the learning dataset. By assuming that all algorithms learn off-policy, we allow information sharing Cauwet et al. (2015) between algorithms. They share the trajectories they generate. As a consequence, we can assume that every algorithm, the least or the most selected ones, will learn from the same trajectory set. Therefore, the control unbalance does not directly lead to unfairness in algorithms performances: all algorithms learn equally from all trajectories. However, unbalance might still remain in the exploration strategy if, for instance, an algorithm takes more benefit from the exploration it has chosen than the one chosen by another algorithm. For analysis purposes, Theorem 3 assumes the fairness of AS:

| $\Delta^1_\beta$ | $\overline{p}_{abs}^\text{ESBAS}(T)$ in function of $\overline{p}_{abs}^\ast(T)$ | $\overline{p}_{abs}(T) \in \mathcal{O}(\log(T))$ | $\overline{p}_{abs}^\ast(T) \in \mathcal{O}(T^{1-c^\ast})$
|---|---|---|
| $\Theta(1/T)$ | $\mathcal{O}(\log(T))$ | $\mathcal{O}(T^{1-c^\ast})$
| $\Theta(T^{-c^\ast})$, $c^\ast \geq 0.5$ | $\mathcal{O}(T^{1-c^\ast})$ | $\mathcal{O}(T^{1-c^\ast})$
| $\Theta(T^{-c^\ast})$, $c^\ast < 0.5$ | $\mathcal{O}(T^{c^\ast}\log(T))$ | $\mathcal{O}(T^{1-c^\ast})$, if $c^\ast < 1 - c^\ast$
| $\Theta(1)$ | $\mathcal{O}(\log^2(T)/\Delta^1_\beta)$ | $\mathcal{O}(T^{1-c^\ast})$, if $c^\ast \geq 1 - c^\ast$

Table 2: Bounds on $\overline{p}_{abs}(T)$ given various configurations of settings, for a two-fold portfolio AS with ESBAS.
Assumption 3 (Learning is fair) If one trajectory set is better than another for training one given algorithm, it is the same for other algorithms.

\[ \forall \alpha, \alpha' \in P, \ \forall D, D' \in \mathcal{B}', \ \mathbb{E} \mu_D \prec \mathbb{E} \mu_{D'}, \ \Rightarrow \ \mathbb{E} \mu_D' \leq \mathbb{E} \mu_{D'}'. \]  \tag{10}

Theorem 3 (ESBAS absolute pseudo-regret upper bound) Under assumption 3, f the stochastic multi-armed bandit \( \Xi \) guarantees that the best arm has been selected in the \( T \) first episodes at least \( T/K \) times, with high probability \( \delta_T \in \mathcal{O}(1/T) \), then:

\[ \exists \kappa > 0, \ \forall T \geq 9K^2, \ \bar{p}_{\text{abs}}\text{ESBAS}(T) \leq (3K + 1)\bar{p}_{\text{abs}}^\alpha(\frac{T}{3K}) + \bar{p}_{\text{abs}}\text{ESBAS}(T) + \kappa \log(T), \]  \tag{11}

where meta-algorithm \( \sigma^* \) selects exclusively algorithm \( \alpha^* = \arg\min_{\alpha \in P} \bar{p}_{\text{abs}}^\alpha(T) \).

Successive and Median Elimination (Even-Dar et al., 2002) and Upper Confidence Bound (Auer et al., 2002a) under some conditions (Audibert and Bubeck, 2010) are examples of appropriate \( \Xi \) satisfying both conditions stated in Theorems 2 and 3.

Table 2 reports an overview of the absolute pseudo-regret bounds in order of magnitude of a two-fold portfolio in function of the asymptotic behaviour of the gap \( \Delta^\beta \) and the absolute pseudo-regret of the meta-algorithm of the the best algorithm \( \bar{p}_{\text{abs}}^\beta(T) \), obtained with Theorems 1, 2 and 3. Table 2 is interpreted by line. According the order of magnitude of \( \Delta^\beta \) in the first column, the second and third columns display the ESBAS absolute pseudo-regret bounds cross depending on the order of magnitude of \( \bar{p}_{\text{abs}}^\alpha(T) \). Obtained bounds are logarithmic in the best case and inferior to \( \mathcal{O}(\sqrt{T}) \) in the worst case, which compares favorably with those of discounted UCB and Exp3.S in \( \mathcal{O}(\sqrt{T \log(T)}) \) and Rexp3 in \( \mathcal{O}(T^{2/3}) \).

5. Experiment and results

5.1 The Negotiation Dialogue Game

ESBAS practical efficiency is illustrated on a dialogue negotiation game (Laroche and Genevay, 2016) that involves two players: the system \( p_s \) and a user \( p_u \). Their goal is to find an agreement among 4 alternative options. At each dialogue, for each option \( \eta \), players have a private uniformly drawn cost \( \nu^p_{\eta} \sim \mathcal{U}[0, 1] \) to agree on it. Each player is considered fully empathetic to the other one. As a result, if the players come to an agreement, the system’s immediate reward at the end of the dialogue is \( R^{p_s}(s_f) = 2 - \nu^p_{\eta} - \nu^p_{\eta'} \), where \( s_f \) is the state reached by player \( p_s \) at the end of the dialogue, and \( \eta \) is the agreed option; if the players fail to agree, the final immediate reward is \( R^{p_s}(s_f) = 0 \), and finally, if one player misunderstands and agrees on a wrong option, the system gets the cost of selecting option \( \eta \) without the reward of successfully reaching an agreement: \( R^{p_s}(s_f) = -\nu^p_{\eta} - \nu^p_{\eta'} \).

Players act each one in turn, starting randomly by one or the other. They have four possible actions. First, **REFPROP**(\( \eta \)): the player makes a proposition: option \( \eta \). If there was any option previously proposed by the other player, the player refuses it. Second, **ASKREPEAT**: the player asks the other player to repeat its proposition. Third, **ACCEPT**(\( \eta \)): the player accepts option \( \eta \) that was understood to be proposed by the other player. This act ends the dialogue either way: whether the understood proposition was the right one or not. Four, **ENDDIAL**: the player does not want to negotiate anymore and ends the dialogue with a null reward.

Understanding through speech recognition of system \( p_s \) is assumed to be noisy: with a sentence error rate of probability \( SER^s_\eta = 0.3 \), an error is made, and the system understands a random option instead of the one that was actually pronounced. In order to reflect human-machine dialogue asymmetry, the simulated user always understands what the system says: \( SER^u_\eta = 0 \). We adopt the way Khouzaimi et al. (2015) generate
All learning algorithms are using Fitted-\(\gamma\) \(\text{40,920\ meta-time\ steps\ and\ as\ many\ trajectories.\ The\ performance\ is\ the\ average\ return\ of\ the\ reinforcement\ learning\ problem\ defined\ in\ Equation\ 1:\ it\ equals\ algorithms\ actually\ induced\ a\ strongly\ negative\ relative\ pseudo-regret\ (see\ Table\ 3\ in\ the\ appendix).\}

obtains a negative relative pseudo-regret of -90. More generally, most of such two-fold portfolios with learning algorithms actually induced a strongly negative relative pseudo-regret (see Table 3 in the appendix).

5.2 Learning algorithms

All learning algorithms are using Fitted-\(Q\) iteration (Ernst et al. 2005), with a linear parametrisation and an \(\epsilon\beta\)-greedy exploration : \(\epsilon_\beta = 0.6^\beta\), \(\beta\) being the epoch number. Six algorithms differing by their state space representation \(\Phi^\alpha\) are considered:

- **simple**: state space representation of four features: the constant feature \(\phi_0 = 1\), the last recognition score feature \(\phi_{\text{asr}}\), the difference between the cost of the proposed option and the next best option \(\phi_{\text{diff}}\), and finally an RL-time feature \(\phi_t = \frac{0.2t}{0.2t+1}\). \(\Phi^\alpha = \{\phi_0, \phi_{\text{asr}}, \phi_{\text{diff}}, \phi_t\}\).
- **fast**: \(\Phi^\alpha = \{\phi_0, \phi_{\text{asr}}, \phi_{\text{diff}}, \phi_t\}\).
- **simple-2**: state space representation of ten second order polynomials of simple features. \(\Phi^\alpha = \{\phi_0, \phi_{\text{asr}}, \phi_{\text{diff}}, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2\}\).
- **fast-2**: state space representation of six second order polynomials of fast features. \(\Phi^\alpha = \{\phi_0, \phi_{\text{asr}}, \phi_{\text{diff}}, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2, \phi_{\text{asr}}^2\}\).
- \(n\cdot\zeta\cdot\{\text{simple/fast/simpler-2/fast-2}\}\): Versions of previous algorithms with \(\zeta\) additional features of noise, randomly drawn from the uniform distribution in [0, 1].
- **constant-\(\mu\)**: the algorithm follows a deterministic policy of average performance \(\mu\) without exploration nor learning. Those constant policies are generated with simple-2 learning from a predefined batch of limited size.

5.3 Results

In all our experiments, ESBAS has been run with UCB parameter \(\xi = 1/4\). We consider 12 epochs. The first and second epochs last 20 meta-time steps, then their lengths double at each new epoch, for a total of 40,920 meta-time steps and as many trajectories. \(\gamma\) is set to 0.9. The algorithms and ESBAS are playing with a stationary user simulator built through Imitation Learning from real-human data. All the results are averaged over 1000 runs. The performance figures plot the curves of algorithms individual performance \(\sigma^\alpha\) against the ESBAS portfolio control \(\sigma^{\text{ESBAS}}\) in function of the epoch (the scale is therefore logarithmic in meta-time). The performance is the average return of the reinforcement learning problem defined in Equation 1: it equals \(\gamma^{\epsilon}\mathbb{E}\{R^\alpha(s_f)\}\) in the negotiation game. The ratio figures plot the average algorithm selection proportions of ESBAS at each epoch. We define the relative pseudo regret as the difference between the ESBAS absolute pseudo-regret and the absolute pseudo-regret of the best canonical meta-algorithm. All relative pseudo-regrets, as well as the gain for not having chosen the worst algorithm in the portfolio, are provided in the appendix. Relative pseudo-regrets have a 95\% confidence interval about \(\pm 6 \approx \pm 1.5 \times 10^{-4}\) per trajectory.

Figures 2a and 2b plot the typical curves obtained with ESBAS selecting from a portfolio of two learning algorithms. On Figure 2a, the ESBAS curve tends to reach more or less the best algorithm in each point as expected. Surprisingly, Figure 2b reveals that the algorithm selection ratios are not very strong in favour of one or another at any time. Indeed, the variance in trajectory set collection makes simple better on some runs until the end. ESBAS proves to be efficient at selecting the best algorithm for each run and unexpectedly obtains a negative relative pseudo-regret of -90. More generally, most of such two-fold portfolios with learning algorithms actually induced a strongly negative relative pseudo-regret (see Table 3 in the appendix).
Figure 2: The figures on the left plot the performance over time. The learning curves display the average performance of each algorithm/portfolio over the epochs. The figures on the right show the ESBAS selection ratios over the epochs. Figures 2b and 2d also show the selection ratio standard deviation from one run to another. *Nota bene:* since the abscissae are in epochs, all figure are actually logarithmic in meta-time. As a consequence, Epoch 12 is 1024 times larger than Epoch 2, and this explains for instance that simple-2 has actually a higher cumulative returns than simple.

Figures 2c and 2d plot the typical curves obtained with ESBAS selecting from a portfolio constituted of a learning algorithm and an algorithm with a deterministic and stationary policy. ESBAS succeeds in remaining close to the best algorithm at each epoch and saves 5361 return value for not selecting the worse algorithm. One can also observe a nice property: ESBAS even dominates both algorithm curves at some point, because the constant algorithm helps the learning algorithm to explore close to a reasonable policy. However, when the deterministic strategy is not so good, the reset of the stochastic bandits generally prevents from achieving negative relative pseudo-regrets: $+169$ in Figure 2c. Subsection 5.5 offers a straightforward improvement of ESBAS when one or several algorithm are known to be constant.

ESBAS also performs well on larger portfolios of 8 learners (see Figure 2e) with negative relative pseudo-regrets: $-10$ (and $-280$ against the worst algorithm), even if the algorithms are, on average, almost selected uniformly as Figure 2f reveals. It is important to recall here that Figure 2f is the average of 1000 independent runs. Each individual run may present different ratios, depending on the quality of the trained policies. ESBAS also offers some staggered learning, but more importantly, early bad policy accidents in learners are avoided. The same kind of results are obtained with 4-learner portfolios. See Table 3 in the appendix for more extensive numerical results.

5.4 Reasons of ESBAS’s empirical success

We interpret ESBAS’s success at reliably outperforming the best algorithm in the portfolio as the result of the four following potential added values. First, calibrated learning: ESBAS selects the algorithm that is the most fitted with the data size. This property allows for instance to use shallow algorithms when having only a few data and deep algorithms once collected a lot. Second, diversified policies: ESBAS computes and experiments several policies. Those diversified policies generate trajectories that are less redundant, and therefore more
informational. As a result, the policies trained on these trajectories should be more efficient. Third, robustness: if one algorithm learns a terrible policy, it will soon be left apart until the next policy update. This property prevents the agent from repeating again and again the same obvious mistakes. Four and last, run adaptation: of course, there has to be an algorithm that is the best on average for one given task at one given meta-time. But depending on the variance in the trajectory collection, it did not necessarily train the best policy for each run. The ESBAS meta-algorithm tries and selects the algorithm that is the best at each run. All these properties are inherited by algorithm selection similarity with ensemble learning (Dietterich, 2002). Simply, instead of a vote amongst the algorithms to decide the control of the next transition (Wiering and Van Hasselt, 2008), ESBAS selects the best performing algorithm.

In order to look deeper into the variance control effect of algorithm selection, in a similar fashion to ensemble learning, we tested two portfolios: four times the same algorithm \( n-2\text{-simple} \), and four times the same algorithm \( n-3\text{-simple} \). The results show that they both outperform the \( \text{simple} \) algorithm baseline, but only slightly (respectively \(-20\) and \(-13\)). Our interpretation is that, in order to control variance, adding randomness is not as good as changing hypotheses, i.e. state space representations.

### 5.5 No arm reset for constant algorithms

Algorithms with a constant policy do not improve over time and the full reset of the \( K \)-multi armed bandit urges ESBAS to unnecessarily explore again and again the same underachieving algorithm. One easy way to circumvent this drawback is to use the knowledge that these constant algorithms do not change and prevent their arm from resetting. By operating this way, when the learning algorithm(s) start(s) outperforming the constant one, ESBAS simply neither exploits nor explores the constant algorithm anymore.

Figure 3 displays the learning curve in the no-arm-reset configuration for the constant algorithm. One can notice that ESBAS’s learning curve follows perfectly the learning algorithm’s learning curve when this one outperforms the constant algorithm and achieves a strong negative relative pseudo-regret of \(-125\).

### 5.6 Assumptions transgressions

Several results show that, in practice, the assumptions are transgressed. Firstly, we observe that Assumption 3 is transgressed. Indeed, it states that if a trajectory set is better than another for a given algorithm, then it’s the same for the other algorithms. This assumption does not prevent calibrated learning, but it prevents the run adaptation property introduced in Subsection 5.4 that states that an algorithm might be the best on some run and another one on other runs. Still, this assumption infringement does not seem to harm the experimental results. It even seems to help in general with run adaptation.

And secondly, off-policy reinforcement learning algorithms exist, but in practice, we use state space representations that distort their off-policy property (Munos et al., 2016). However, experiments do not reveal any obvious bias related to the off/on-policiness of the trajectory set the algorithms train on.

### 5.7 N-ESBAS: Non-Epochal Stochastic Bandit AS

In this section, we propose to adapt ESBAS to a true online setting where algorithms update their policies after each transition. Similarly to Garivier and Moulines (2011), the stochastic bandit is now trained on a sliding window with the last \( \tau/2 \) selections. Even though the arms are not stationary over this window, there is the guarantee of eventually forgetting the oldest arm pulls. The goal here is to show the hyper-parameter optimisation of N-ESBAS on a simple tabular domain.

Our domain is a 5x5 gridworld problem (see Figure 4), where the goal is to collect the fruits placed at each corner as fast as possible. The episodes terminate when all fruits have been collected or after 100 transitions. The objective function \( \mu \) used to
optimise the stochastic bandit $\Psi$ is no longer the RL return, but the time spent to collect all the fruits (200 in case of it did not). The agent has 18 possible positions and there are $2^4 - 1 = 15$ non-terminal fruits configurations, resulting in 270 states. The action set is $\mathcal{A} = \{N, E, S, W\}$. The reward function mean is 1 when eating a fruit, 0 otherwise. The reward function also incorporates a strong Gaussian white noise of variance $\zeta = 1$. The portfolio is composed of 4 $Q$-learning algorithms varying from each other by their learning rates: $\{0.001, 0.01, 0.1, 0.5\}$. They all have the same linearly annealing $\epsilon$-greedy exploration.

The results displayed in Figure 5 show that N-ESBAS efficiently selected the algorithm with the best learning rate at each time step. First the highest (0.5) learning rate enables to propagate efficiently the reward signal through the visited states, but it is too sensible to the reward noise and overtime N-ESBAS preferentially chooses the algorithm using a learning rate of 0.01. This algorithm is in turn eventually slowly discarded by N-ESBAS in favour of the algorithm with the smallest learning rate (0.001), because it is the least perturbed by noise. The selection ratio after 1 million episodes is 87% in favor of algorithm (0.001). After 1 million episodes, N-ESBAS enables to save half a transition per episode on average as compared to the best constant learning rate value (0.1), and two transitions against the worst learning rate in the portfolio (0.001). We also evaluate the efficiency of a linearly annealing learning rate: $1/(1 + 0.0001 \tau)$.

We found that the linearly annealing learning rate deals better with the early steps (it finds faster a first policy that succeeds in collecting all the fruits), but it converges more slowly to a very efficient solution: N-ESBAS performs under 21 steps on average after $10^5$, while the linearly annealing learning rate algorithm still performs a bit over 21 steps after $10^6$ steps. The choice of using constant learning rate was made for pedagogical purposes, but N-ESBAS can be used the same way with any learning rate annealing scheme in order to get the best of both. It is possible to optimise the same way the $\gamma$ hyper-parameter when, as in this gridworld domain, the objective function cannot be cast into a RL return.

6. Conclusion

In this article, we tackle the problem of selecting online off-policy RL algorithms. The problem is formalised as follows: from a fixed portfolio of algorithms, a meta-algorithm learns which one performs the best on the task at hand. Fairness of algorithm evaluation is granted by the fact that the RL algorithms learn off-policy. ESBAS, a novel meta-algorithm, is proposed. Its principle is to divide the meta-time scale into epochs. Algorithms are allowed to update their policies only at the start each epoch. As the policies are constant inside each epoch, the problem can be cast into a stochastic multi-armed bandit. An implementation is detailed and a theoretical analysis leads to upper bounds on the regrets.

Most of the experiments are led on a negotiation dialogue game, interacting with a human data-built simulated user. Several space representations are considered to constitute a large set of algorithms. ESABS has been evaluated on 32 portfolios and, in most cases, not only ESABS demonstrates its efficiency to select the best algorithm, but it also outperforms the best algorithm in the portfolio thanks to staggered learning, and variance reduction similar to that of Ensemble Learning. In practice, Assumptions 2 and 3 may be transgressed, but without compromising ESABS’s empirical robustness and efficiency. A quick experiment on a fruit collection task proves that it is possible in practice to let the algorithms update their policies after every transition (or episode) and still benefit from the AS.

Regarding the portfolio design, it mostly depends on the available computational power per sample ratio. We recommend to limit the use of 2 highly demanding algorithms, paired with several faster algorithms that can take care of first learning stages, and to use algorithms that are diverse regarding models, hypotheses, etc. Adding two algorithms that are too similar adds inertia, while they are likely to not be distinguishable by ESBAS. More precise recommendations for building an efficient RL portfolio are left for future work.
Algorithm Selection for Reinforcement Learning

References


Jialin Liu and Olivier Teytaud. Meta online learning: experiments on a unit commitment problem. In Proceedings of the 22nd European Symposium on Artificial Neural Networks, Computational Intelligence and Machine Learning (ESANN), 2014.


# Appendix A. Glossary

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Designation</th>
<th>First use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>Reinforcement learning time aka RL-time</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\tau, T$</td>
<td>Meta-algorithm time aka meta-time</td>
<td>Section 2</td>
</tr>
<tr>
<td>$a(t)$</td>
<td>Action taken at RL-time $t$</td>
<td>Figure 1</td>
</tr>
<tr>
<td>$o(t)$</td>
<td>Observation made at RL-time $t$</td>
<td>Figure 1</td>
</tr>
<tr>
<td>$R(t)$</td>
<td>Reward received at RL-time $t$</td>
<td>Figure 1</td>
</tr>
<tr>
<td>$A$</td>
<td>Action set</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Observation set</td>
<td>Section 2</td>
</tr>
<tr>
<td>$R_{\text{min}}$</td>
<td>Lower bound of values taken by $R(t)$</td>
<td>Section 2</td>
</tr>
<tr>
<td>$R_{\text{max}}$</td>
<td>Upper bound of values taken by $R(t)$</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\varepsilon_\tau$</td>
<td>Trajectory collected at meta-time $\tau$</td>
<td>Section 2</td>
</tr>
<tr>
<td>$</td>
<td>X</td>
<td>$</td>
</tr>
<tr>
<td>$[a, b]$</td>
<td>Ensemble of integers comprised between $a$ and $b$</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\mathcal{G}$</td>
<td>Space of trajectories</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Discount factor of the decision process</td>
<td>Equation 1</td>
</tr>
<tr>
<td>$\mu(\varepsilon_\tau)$</td>
<td>Return of trajectory $\varepsilon_\tau$ aka objective function</td>
<td>Equation 1</td>
</tr>
<tr>
<td>$\mathcal{D}_{\varepsilon}$</td>
<td>Trajectory set collected until meta-time $T$</td>
<td>Section 2</td>
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<tr>
<td>$\varepsilon_\tau(t)$</td>
<td>History of $\varepsilon_\tau$ until RL-time $t$</td>
<td>Section 2</td>
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<tr>
<td>$\pi$</td>
<td>Policy</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\pi^*$</td>
<td>Optimal policy</td>
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<tr>
<td>$\alpha$</td>
<td>Algorithm</td>
<td>Section 2</td>
</tr>
<tr>
<td>$\mathcal{G}_\alpha^+$</td>
<td>Ensemble of trajectory sets</td>
<td>Section 2</td>
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<td>$\mathcal{S}_\alpha^+$</td>
<td>State space of algorithm $\alpha$ from trajectory set $\mathcal{D}$</td>
<td>Section 2</td>
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<tr>
<td>$\Phi_{\alpha}^+$</td>
<td>State space projection of algorithm $\alpha$ from trajectory set $\mathcal{D}$</td>
<td>Equation Section 2</td>
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<tr>
<td>$\pi_{\alpha}^+$</td>
<td>Policy learnt by algorithm $\alpha$ from trajectory set $\mathcal{D}$</td>
<td>Section 2</td>
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<tr>
<td>$\mathcal{P}$</td>
<td>Algorithm set aka portfolio</td>
<td>Section 3.1</td>
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<tr>
<td>$K$</td>
<td>Size of the portfolio</td>
<td>Section 3.1</td>
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<tr>
<td>$\sigma$</td>
<td>Meta-algorithm</td>
<td>Section 3.1</td>
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<tr>
<td>$\sigma(\tau)$</td>
<td>Algorithm selected by meta-algorithm $\sigma$ at meta-time $\tau$</td>
<td>Section 3.1</td>
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<tr>
<td>$\mathbb{E}_x[f(x_0)]$</td>
<td>Expected value of $f(x)$ conditionally to $x = x_0$</td>
<td>Equation ??</td>
</tr>
<tr>
<td>$\mathbb{E}<em>F(\varepsilon</em>\tau)$</td>
<td>Expected return of trajectories controlled by policy $\pi_{\alpha}^+$</td>
<td>Equation ??</td>
</tr>
<tr>
<td>$\mathbb{E}_F(\mu^*)$</td>
<td>Optimal expected return</td>
<td>Equation 3</td>
</tr>
<tr>
<td>$\sigma^\alpha$</td>
<td>Canonical meta-algorithm exclusively selecting algorithm $\alpha$</td>
<td>Section 3.2</td>
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<tr>
<td>$\mathcal{P}_{\alpha}(T)$</td>
<td>Absolute pseudo-regret</td>
<td>Definition 1</td>
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<tr>
<td>$\mathcal{O}(f(x))$</td>
<td>Set of functions that get asymptotically dominated by $\kappa f(x)$</td>
<td>Section 4</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Constant number</td>
<td>Theorem 3</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>Stochastic $K$-armed bandit algorithm</td>
<td>Section 4.2</td>
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<tr>
<td>$\beta$</td>
<td>Epoch index</td>
<td>Section 4.2</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Parameter of the UCB algorithm</td>
<td>Pseudo-code 2</td>
</tr>
<tr>
<td>$\mathcal{P}_{\alpha}(T)$</td>
<td>Short-sighted pseudo-regret</td>
<td>Definition 2</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Gap between the best arm and another arm</td>
<td>Theorem 2</td>
</tr>
<tr>
<td>$\hat{\beta}$</td>
<td>Index of the second best algorithm</td>
<td>Theorem 2</td>
</tr>
<tr>
<td>$\Delta^2_{\beta}$</td>
<td>Gap of the second best arm at epoch $\beta$</td>
<td>Theorem 2</td>
</tr>
<tr>
<td>$\lfloor x \rfloor$</td>
<td>Rounding of $x$ at the closest integer below</td>
<td>Theorem 2</td>
</tr>
<tr>
<td>$\sigma_{\text{ESBAS}}$</td>
<td>The ESBAS meta-algorithm</td>
<td>Theorem 2</td>
</tr>
<tr>
<td>$\Theta(f(x))$</td>
<td>Set of functions asymptotically dominating $\kappa f(x)$ and dominated by $\kappa' f(x)$</td>
<td>Table 1</td>
</tr>
<tr>
<td>$\sigma^*$</td>
<td>Best meta-algorithm among the canonical ones</td>
<td>Theorem 3</td>
</tr>
<tr>
<td>Symbol</td>
<td>Designation</td>
<td>First use</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>-----------</td>
</tr>
<tr>
<td>(p, p_s, p_u)</td>
<td>Player, system player, and (simulated) user player</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\eta)</td>
<td>Option to agree or disagree on</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\nu^p)</td>
<td>Cost of booking/selecting option (\nu) for player (p)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\mathcal{U}[a, b])</td>
<td>Uniform distribution between (a) and (b)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(s_f)</td>
<td>Final state reached in a trajectory</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(R^\nu(s_f))</td>
<td>Immediate reward received by the system player at the end of the dialogue</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{RefPROP}(\eta))</td>
<td>Dialogue act consisting in proposing option (\eta)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{ASKREPEAT})</td>
<td>Dialogue act consisting in asking the other player to repeat what he said</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{ACCEPT}(\eta))</td>
<td>Dialogue act consisting in accepting proposition (\eta)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{ENDDIAL})</td>
<td>Dialogue act consisting in ending the dialogue</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(SER^u)</td>
<td>Sentence error rate of system (p_s) listening to user (p_u)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{score}<em>{\alpha</em>{\text{asr}}})</td>
<td>Speech recognition score</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\mathcal{N}(x, \nu^2))</td>
<td>Normal distribution of centre (x) and variance (\nu^2)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{RefINSIST})</td>
<td>(\text{RefPROP}(\eta)), with (\eta) being the last proposed option</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{RefNEWPROP})</td>
<td>(\text{RefPROP}(\eta)), with (\eta) being the best option that has not been proposed yet</td>
<td>Section 5.1</td>
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<td>(\epsilon_\beta)</td>
<td>(\epsilon)-greedy exploration in function of epoch (\beta)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\Phi^\alpha)</td>
<td>Set of features of algorithm (\alpha)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\phi_0)</td>
<td>Constant feature: always equal to 1</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\phi_{\alpha_{\text{asr}}})</td>
<td>ASR feature: equal to the last recognition score</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\phi_{\text{dif}})</td>
<td>Cost feature: equal to the difference of cost of proposed and targeted options</td>
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</tr>
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<td>(\phi_t)</td>
<td>RL-time feature</td>
<td>Section 5.1</td>
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<tr>
<td>(\phi_{\text{noise}})</td>
<td>Noise feature</td>
<td>Section 5.1</td>
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<tr>
<td>(\text{simple})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t})</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{fast})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}})</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{simple-2})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{dif}}, \phi_t^2, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t^2})</td>
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</tr>
<tr>
<td>(\text{fast-2})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t^2, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t^2})</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(n-1)-(\text{simple})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{noise}}, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{noise}}, \phi_{\text{asr}}, \phi_t})</td>
<td>Section 5.1</td>
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<tr>
<td>(n-1)-(\text{fast})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{noise}}, \phi_{\text{asr}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{noise}}, \phi_{\text{asr}}, \phi_t})</td>
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<tr>
<td>(n-1)-(\text{simple-2})</td>
<td>(\text{FQI with } \Phi = {\phi_0, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{noise}}, \phi_{\alpha_{\text{asr}}}, \phi_{\text{dif}}, \phi_t, \phi_{\text{noise}}, \phi_{\text{asr}}, \phi_t, \phi_{\text{dif}}, \phi_t^2, \phi_{\text{noise}}})</td>
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<tr>
<td>(n-1)-(\text{fast-2})</td>
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<td>Section 5.1</td>
</tr>
<tr>
<td>(\text{constant-}\mu)</td>
<td>Non-learning algorithm with average performance (\mu)</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\zeta)</td>
<td>Number of noisy features added to the feature set</td>
<td>Section 5.1</td>
</tr>
<tr>
<td>(\mathbb{P}(x</td>
<td>y))</td>
<td>Probability that (X = x) conditionally to (Y = y)</td>
</tr>
</tbody>
</table>
Table 3: ESBAS pseudo-regret after 12 epochs (i.e. 40,920 trajectories) compared with the best and the worst algorithms in the portfolio, in function of the algorithms in the portfolio (described in the first column). The '+' character is used to separate the algorithms. all-4 means all the four learning algorithms described in Section 5: simple + fast + simple-2 + fast-2. all-4-n-1 means the same four algorithms with one additional feature of noise. Finally, all-2-simple means simple + simple-2 and all-2-n-1-simple means n-1-simple + n-1-simple-2. On the second column, the redder the colour, the worse ESBAS is achieving in comparison with the best algorithm. Inversely, the greener the colour of the number, the better ESBAS is achieving in comparison with the best algorithm. If the number is neither red nor green, it means that the difference between the portfolio and the best algorithm is insignificant and that they are performing as good. This is already an achievement for ESBAS to be as good as the best. On the third column, the bluer the cell, the weaker is the worst algorithm in the portfolio. One can notice that positive regrets are always triggered by a very weak worst algorithm in the portfolio. In these cases, ESBAS did not allow to outperform the best algorithm in the portfolio, but it can still be credited with the fact it dismissed efficiently the very weak algorithms in the portfolio.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>w. best</th>
<th>w. worst</th>
</tr>
</thead>
<tbody>
<tr>
<td>simple-2 + fast-2</td>
<td>35</td>
<td>-181</td>
</tr>
<tr>
<td>simple + n-1-simple-2</td>
<td>-73</td>
<td>-131</td>
</tr>
<tr>
<td>simple + n-1-simple</td>
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<td>-2</td>
</tr>
<tr>
<td>simple-2 + n-1-simple-2</td>
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<td>-38</td>
</tr>
<tr>
<td>all-4 + constant-1.10</td>
<td>21</td>
<td>-2032</td>
</tr>
<tr>
<td>all-4 + constant-1.11</td>
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<td>-1414</td>
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<tr>
<td>all-4 + constant-1.13</td>
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<tr>
<td>all-4</td>
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<td>-275</td>
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<td>all-2-simple + constant-1.08</td>
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<td>-2734</td>
</tr>
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<td>all-2-simple + constant-1.11</td>
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<td>-2013</td>
</tr>
<tr>
<td>all-2-simple + constant-1.13</td>
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<td>all-2-simple</td>
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<td>fast + simple-2</td>
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<td>simple-2 + constant-1.01</td>
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<td>simple-2 + constant-1.11</td>
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<tr>
<td>simple-2 + constant-1.11</td>
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<td>simple + constant-1.10</td>
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<td>8*n-1-simple-2</td>
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<td>simple-2 + constant-1.13 (no reset)</td>
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<tr>
<td>simple-2 + constant-1.14 (no reset)</td>
<td>-125</td>
<td>-319</td>
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Appendix B. Not worse than the worst

Theorem 1 (Not worse than the worst) The absolute pseudo-regret is bounded by the worst algorithm absolute pseudo-regret in order of magnitude:

\[ \forall \sigma, \quad \overline{p}_\sigma^\sigma(T) \in O \left( \max_{\alpha \in P} \overline{p}_\sigma^\alpha(T) \right). \]  

(7)

Proof From Definition 1:

\[ \overline{p}_\sigma^\sigma(T) = T \mu_\sigma^\sigma - \mathbb{E}_\sigma \left[ \sum_{\tau=1}^T \mathbb{E}_\sigma^{\sigma(\tau)} \right], \]  

(12a)

\[ \overline{p}_\sigma^\sigma(T) = T \mu_\sigma^\sigma - \sum_{\alpha \in P} \mathbb{E}_\sigma \left[ \sum_{i=1}^{\lvert \text{sub}^\alpha(\mathcal{D}_T^\sigma) \rvert} \mathbb{E}_{\mathcal{D}_T^\sigma}^{\alpha} \right], \]  

(12b)

\[ \overline{p}_\sigma^\sigma(T) = \sum_{\alpha \in P} \mathbb{E}_\sigma \left[ \lvert \text{sub}^\alpha(\mathcal{D}_T^\sigma) \rvert \mathbb{E}_\sigma^{\sigma(\tau)} - \sum_{i=1}^{\lvert \text{sub}^\alpha(\mathcal{D}_T^\sigma) \rvert} \mathbb{E}_{\mathcal{D}_T^\sigma}^{\alpha} \right], \]  

(12c)

where \( \text{sub}^\alpha(\mathcal{D}) \) is the subset of \( \mathcal{D} \) with all the trajectories generated with algorithm \( \alpha \), where \( \tau_i^\alpha \) is the index of the \( i \)-th trajectory generated with algorithm \( \alpha \), and where \( \lvert S \rvert \) is the cardinality of finite set \( S \). By convention, let us state that \( \mathbb{E}_{\mathcal{D}_T^\sigma}^{\alpha} = \mathbb{E}_\sigma \) if \( \lvert \text{sub}^\alpha(\mathcal{D}_T^\sigma) \rvert < i \). Then:

\[ \overline{p}_\sigma^\sigma(T) = \sum_{\alpha \in P} \sum_{i=1}^T \mathbb{E}_\sigma \left[ \mu_\sigma^\sigma - \mu_\sigma^\alpha \right]. \]  

(13)

To conclude, let us prove by mathematical induction the following inequality:

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] \geq \mathbb{E}_\sigma^{\alpha} \left[ \mu_\sigma^\alpha \right]. \]

is true by vacuity for \( i = 0 \); both left and right terms equal \( \mu_\sigma^\alpha \). Now let us assume the property true for \( i \) and prove it for \( i + 1 \):

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] = \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha \right] \bigg|_{\mathcal{D}_T^\sigma} \bigcup_{\tau=1}^{\tau_{i+1}-1} \mathcal{D}_T^\alpha \bigg] \],

(14a)

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] = \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha \right] \bigg|_{\mathcal{D}_T^\sigma} \bigcup_{\tau=1}^{\tau_{i+1}-1} \mathcal{D}_T^\alpha \bigg] \],

(14b)

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] = \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha \right] \bigg|_{\mathcal{D}_T^\sigma} \bigcup_{\tau=1}^{\tau_{i+1}-1} \mathcal{D}_T^\alpha \bigg] \],

(14c)

If \( \lvert \text{sub}^\alpha(\mathcal{D}_T^\sigma) \rvert \geq i + 1 \), by applying mathematical induction assumption, then by applying Assumption 2 and finally by applying Assumption 1 recursively, we infer that:

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] \geq \mathbb{E}_\sigma^{\alpha} \left[ \mu_\sigma^\alpha \right], \]

(15a)

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] \geq \mathbb{E}_\sigma^{\alpha} \left[ \mu_\sigma^\alpha \right], \]

(15b)

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] \geq \mathbb{E}_\sigma^{\alpha} \left[ \mu_\sigma^\alpha \right], \]

(15c)

\[ \mathbb{E}_\sigma \left[ \mu_\sigma^\alpha_{\mathcal{D}_T^\sigma} \right] \geq \mathbb{E}_\sigma^{\alpha} \left[ \mu_\sigma^\alpha \right]. \]

(15d)
If \(|\text{sub}^{\alpha}(D_T)| < i + 1\), the same inequality is straightforwardly obtained, since, by convention \(\mathbb{E}\mu_{D_{r+1}}^k = \mathbb{E}\mu_\infty^\ast\), and since, by definition \(\forall D \in \mathcal{G}^+, \forall \alpha \in \mathcal{P}, \mathbb{E}\mu_\infty^\ast \geq \mathbb{E}\mu_D^\ast\).

The mathematical induction proof is complete. This result leads to the following inequalities:

\[
\begin{align*}
\mathcal{P}^{\sigma}_{\text{abs}}(T) &\leq \sum_{\alpha \in \mathcal{P}} \sum_{i=1}^{T} \mathbb{E}_{\sigma^\alpha} \left[ \mathbb{E}\mu_\infty^\ast - \mathbb{E}\mu_{D_{r+1}}^\ast \right], \tag{16a} \\
\mathcal{P}^{\sigma}_{\text{abs}}(T) &\leq \sum_{\alpha \in \mathcal{P}} \mathcal{P}^{\sigma}_{\text{abs}}(T), \tag{16b} \\
\mathcal{P}^{\sigma}_{\text{abs}}(T) &\leq K \max_{\alpha \in \mathcal{P}} \mathcal{P}^{\sigma}_{\text{abs}}(T), \tag{16c}
\end{align*}
\]

which leads directly to the result:

\[
\forall \sigma, \quad \mathcal{P}^{\sigma}_{\text{abs}}(T) \in \mathcal{O} \left( \max_{\alpha \in \mathcal{P}} \mathcal{P}^{\sigma}_{\text{abs}}(T) \right), \tag{17}
\]

This proof may seem to the reader rather complex for such an intuitive and loose result but algorithm selection \(\sigma\) and the algorithms it selects may act tricky. For instance selecting algorithm \(\alpha\) only when the collected trajectory sets contains misleading examples (i.e. with worse expected return than with an empty trajectory set) implies that the following unintuitive inequality is always true: \(\mathbb{E}\mu_{D_{r+1}}^\ast \leq \mathbb{E}\mu_{D_{r-1}}^\ast\). In order to control all the possible outcomes, one needs to translate the selections of algorithm \(\alpha\) into \(\sigma^\alpha\)’s view.
Appendix C. ESBAS short-sighted pseudo-regret upper order of magnitude

**Theorem 2 (ESBAS short-sighted pseudo-regret)** If the stochastic multi-armed bandit \( \Xi \) guarantees a regret of order of magnitude \( \mathcal{O}(\log(T)/\Delta) \), then:

\[
\rho_{\text{ss}}^{\text{ESBAS}}(T) \in \mathcal{O} \left( \sum_{\beta=0}^{\lfloor \log(T) \rfloor} \frac{\beta}{\Delta_{\beta}} \right).
\]  

**Proof** By simplification of notation, \( \mathbb{E}_{\mu_{2^\beta-1}} = \mathbb{E}_{\mu_{2^\beta}} \). From Definition 2:

\[
\rho_{\text{ss}}^{\text{ESBAS}}(T) = \mathbb{E}_{\sigma_{\text{ESBAS}}} \left[ \sum_{\tau=2^0}^{\lfloor \log(T) \rfloor} \left( \max_{\alpha \in \mathcal{P}} \mathbb{E}_{\mu_{2^\beta}}^{\alpha} - \mathbb{E}_{\mu_{\beta}}^{\text{ESBAS}(\tau)} \right) \right],
\]  

where \( \beta_{\tau} \) is the epoch of meta-time \( \tau \). A bound on short-sighted pseudo-regret \( \rho_{\text{ss}}^{\text{ESBAS}}(\beta) \) for each epoch \( \beta \) can then be obtained by the stochastic bandit \( \Xi \) regret bounds in \( \mathcal{O}(\log(2^\beta)/\Delta) \):

\[
\rho_{\text{ss}}^{\text{ESBAS}}(\beta) = \mathbb{E}_{\sigma_{\text{ESBAS}}} \left[ \sum_{\tau=2^0}^{2^{\beta+1}-1} \left( \max_{\alpha \in \mathcal{P}} \mathbb{E}_{\mu_{2^\beta}}^{\alpha} - \mathbb{E}_{\mu_{\beta}}^{\text{ESBAS}(\tau)} \right) \right],
\]  

where \( \Delta_{\beta} = \sum_{\alpha \in \mathcal{P}} \frac{1}{\Delta_{\beta}} \)

and where

\[
\Delta_{\beta} = \begin{cases} +\infty & \text{if } \mathbb{E}_{\mu_{\beta}}^{\alpha} = \max_{\alpha' \in \mathcal{P}} \mathbb{E}_{\mu_{\beta}}^{\alpha'}, \\ \max_{\alpha' \in \mathcal{P}} \mathbb{E}_{\mu_{\beta}}^{\alpha'} - \mathbb{E}_{\mu_{\beta}}^{\alpha} & \text{otherwise}. \end{cases}
\]

Since we are interested in the order of magnitude, we can once again only consider the upper bound of \( \frac{1}{\Delta_{\beta}} \):

\[
\frac{1}{\Delta_{\beta}} \in \bigcup_{\alpha \in \mathcal{P}} \mathcal{O} \left( \frac{1}{\Delta_{\beta}} \right),
\]  

and

\[
\frac{1}{\Delta_{\beta}} \in \mathcal{O} \left( \max_{\alpha \in \mathcal{P}} \frac{1}{\Delta_{\beta}} \right),
\]  

where

\[
\rho_{\text{ss}}^{\text{ESBAS}}(T) \leq \sum_{\beta=0}^{\lfloor \log(2^\beta) \rfloor} \rho_{\text{ss}}^{\text{ESBAS}}(\beta),
\]

and

\[
\rho_{\text{ss}}^{\text{ESBAS}}(T) \leq \sum_{\beta=0}^{\lfloor \log(T) \rfloor} \rho_{\text{ss}}^{\text{ESBAS}}(\beta),
\]  

where

\[
\rho_{\text{ss}}^{\text{ESBAS}}(T) \in \mathcal{O} \left( \sum_{\beta=0}^{\lfloor \log(T) \rfloor} \frac{\beta}{\Delta_{\beta}} \right).
\]
\[ \exists \kappa_2 > 0, \quad \frac{1}{\Delta_\beta} \leq \frac{\kappa_2}{\Delta_\beta^{1}}, \quad (20c) \]

where the second best algorithm at epoch \( \beta \) such that \( \Delta_\beta^{1} > 0 \) is noted \( \alpha_\beta^{1} \). Injected in Equation 18d, it becomes:

\[ \hat{p}_{ss}^{\text{ESBAS}}(T) \leq \kappa_1 \kappa_2 \sum_{\beta=0}^{\log_2(T)} \frac{\beta}{\Delta_\beta^{1}}, \quad (21) \]

which proves the result.

\[ \phi \]

C.1 Corollaries of Theorem 2

**Corollary 1** If \( \Delta^{1}_\beta \in \Theta(1) \), then \( \hat{p}_{ss}^{\text{ESBAS}}(T) \in \mathcal{O}\left(\frac{\log^2(T)}{\Delta^{1}_\infty}\right) \), where \( \Delta^{1}_\infty = \mu^{*}_\infty - \mu^{\dagger}_\infty > 0 \).

**Proof** \( \Delta^{1}_\beta \in \Omega(1) \) means that only one algorithm \( \alpha^{*} \) converges to the optimal asymptotic performance \( \mu^{*}_\infty \) and that \( \exists \Delta^{1}_\infty = \mu^{*}_\infty - \mu^{\dagger}_\infty > 0 \) such that \( \forall \beta_2 > 0, \exists \beta_1 \in \mathbb{N} \), such that \( \forall \beta \geq \beta_1, \Delta^{1}_\beta > \Delta^{1}_\infty - \epsilon \). In this case, the following bound can be deduced from equation 21:

\[ \hat{p}_{ss}^{\text{ESBAS}}(T) \leq \kappa_4 + \sum_{\beta=\beta_1}^{\log_2(T)} \frac{\kappa_1 \kappa_2}{\Delta^{1}_\infty - \epsilon}, \quad (22a) \]

\[ \hat{p}_{ss}^{\text{ESBAS}}(T) \leq \kappa_4 + \frac{\kappa_1 \kappa_2 \log^2(T)}{2(\Delta^{1}_\infty - \epsilon)}, \quad (22b) \]

where \( \kappa_4 \) is a constant equal to the short-sighted pseudo-regret before epoch \( \beta_1 \):

\[ \kappa_4 = \hat{p}_{ss}^{\text{ESBAS}}\left(2^{\beta_1-1}\right) \quad (23) \]

Equation 22b directly leads to the corollary.

\[ \phi \]

**Corollary 2** If \( \Delta^{1}_\beta \in \Theta\left(\beta^{-m^{1}}\right) \), then \( \hat{p}_{ss}^{\text{ESBAS}}(T) \in \mathcal{O}\left(\log^{m^{1}+2}(T)\right) \).

**Proof** If \( \Delta^{1}_\beta \) decreases slower than polynomially in epochs, which implies decreasing polylogarithmically in meta-time, \( \text{i.e.} \ \exists \kappa_5 > 0, \exists m^{1} > 0, \exists \beta_2 \in \mathbb{N} \), such that \( \forall \beta \geq \beta_2, \Delta^{1}_\beta > \kappa_5 \beta^{-m^{1}} \), then, from Equation 21:

\[ \hat{p}_{ss}^{\text{ESBAS}}(T) \leq \kappa_6 + \sum_{\beta=\beta_2}^{\log(T)} \frac{\kappa_1 \kappa_2}{\kappa_5 \beta^{-m^{1}}}, \quad (24a) \]

\[ \hat{p}_{ss}^{\text{ESBAS}}(T) \leq \kappa_6 + \sum_{\beta=\beta_2}^{\log(T)} \frac{\kappa_1 \kappa_2}{\kappa_5 \beta^{m^{1}+1}}, \quad (24b) \]

\[ \hat{p}_{ss}^{\text{ESBAS}}(T) \leq \frac{\kappa_1 \kappa_2}{\kappa_5} \log^{m^{1}+2}(T), \quad (24c) \]

where \( \kappa_6 \) is a constant equal to the short-sighted pseudo-regret before epoch \( \beta_2 \):

\[ \kappa_6 = \hat{p}_{ss}^{\text{ESBAS}}\left(2^{\beta_2-1}\right) \quad (25) \]

Equation 24c directly leads to the corollary.

\[ \phi \]
Corollary 3 If $\Delta_\beta \in \Theta \left(T^{-c^1}\right)$, then $\overline{\rho}_{ss}^{\text{ESBAS}}(T) \in O \left(T^{c^1} \log(T)\right)$.

Proof If $\Delta_\beta^1$ decreases slower than a fractional power of meta-time $T$, then $\exists \kappa_7 > 0, 0 < c^1 < 1, \exists \beta_3 \in \mathbb{N}$, such that $\forall \beta \geq \beta_3, \Delta_\beta^1 > \kappa_7 T^{-c^1}$, and therefore, from Equation 21:

$$\overline{\rho}_{ss}^{\text{ESBAS}}(T) \leq \kappa_8 + \sum_{\beta = \beta_3}^{\lfloor \log(T) \rfloor} \frac{\kappa_1 \kappa_2}{\kappa_7 T^{-c^1} \beta},$$

$$\overline{\rho}_{ss}^{\text{ESBAS}}(T) \leq \kappa_8 + \sum_{\beta = \beta_3}^{\lfloor \log(T) \rfloor} \frac{\kappa_1 \kappa_2}{\kappa_7 \left(2^\beta\right)^{-c^1} \beta},$$

$$\overline{\rho}_{ss}^{\text{ESBAS}}(T) \leq \kappa_8 + \sum_{\beta = \beta_3}^{\lfloor \log(T) \rfloor} \frac{\kappa_1 \kappa_2}{\kappa_7} \left(2^{c^1}\right) \beta,$$

where $\kappa_8$ is a constant equal to the short-sighted pseudo-regret before epoch $\beta_3$:

$$\kappa_8 = \overline{\rho}_{ss}^{\text{ESBAS}}\left(2^{\beta_3-1}\right).$$

The sum in Equation 26c is solved as follows:

$$\sum_{i=1_0}^{n} ix^i = x \sum_{i=1_0}^{n} ix^{i-1},$$

$$\sum_{i=1_0}^{n} ix^i = x \sum_{i=1_0}^{n} \frac{d(x^i)}{dx},$$

$$\sum_{i=1_0}^{n} ix^i = x \frac{d \left(\sum_{i=1_0}^{n} x^i\right)}{dx},$$

$$\sum_{i=1_0}^{n} ix^i = x \left(1 - \frac{x}{x-1}\right),$$

$$\sum_{i=1_0}^{n} ix^i = \frac{x}{(x-1)^2} \left((x-1)nx^n - x^n - (x-1)i_0x^{i_0-1} + x^{i_0}\right).$$

This result, injected in Equation 26c, induces that $\forall \epsilon_3 > 0, \exists T_1 \in \mathbb{N}, \forall T \geq T_1$:

$$\overline{\rho}_{ss}^{\text{ESBAS}}(T) \leq \kappa_8 + \frac{\kappa_1 \kappa_2 \left(1 + c^1\right) 2^{c^1}}{\kappa_7 \left(2^{c^1} - 1\right)} \log(T) 2^{c_1} \log(T),$$

$$\overline{\rho}_{ss}^{\text{ESBAS}}(T) \leq \kappa_8 + \frac{\kappa_1 \kappa_2 \left(1 + c^1\right) 2^{c_1}}{\kappa_7 \left(2^{c_1} - 1\right)} T^{c_1} \log(T),$$

which proves the corollary.
Appendix D. ESBAS absolute pseudo-regret bound

Theorem 3 (ESBAS absolute pseudo-regret upper bound) Under assumption 3 f the stochastic multi-armed bandit $Ξ$ guarantees that the best arm has been selected in the $T$ first episodes at least $T/K$ times, with high probability $δ_T ∈ O(1/T)$, then:

$$\exists \kappa > 0, \ \forall T \geq 9K^2, \ \overline{p}_{abs}^{ESBAS}(T) \leq (3K + 1)\overline{p}_{abs}^{\sigma}(\frac{T}{3K}) + \overline{p}_{abs}^{ESBAS}(T) + \kappa \log(T),$$  \hspace{1cm} (11)

where meta-algorithm $\sigma^*$ selects exclusively algorithm $\alpha^* = \arg\min_{\alpha \in \mathcal{P}} \overline{p}_\alpha^{abs}(T)$.

Proof The ESBAS absolute pseudo-regret is written with the following notation simplifications: $D_{\tau-1} = D_{\tau-1}^{ESBAS}$ and $k_\tau = \sigma^{ESBAS}(\tau)$:

$$\overline{p}_\sigma^{ESBAS}(T) = T\mathbb{E}\mu_\sigma^\infty - \mathbb{E}_{\sigma^{ESBAS}} \left[ \sum_{\tau=1}^{T} \mathbb{E}_{\mu_{D_{\tau-1}^{ESBAS}}}^{\sigma} \right],$$  \hspace{1cm} (30a)

$$\overline{p}_\sigma^{ESBAS}(T) = T\mathbb{E}\mu_\sigma^\infty - \mathbb{E}_{\sigma^{ESBAS}} \left[ \sum_{\tau=1}^{T} \mathbb{E}_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{k_\tau} \right].$$  \hspace{1cm} (30b)

Let $\sigma^*$ denote the algorithm selection selecting exclusively $\alpha^*$, and $\alpha^*$ be the algorithm minimising the algorithm absolute pseudo-regret:

$$\alpha^* = \arg\min_{\alpha \in \mathcal{P}} \overline{p}_\alpha^{abs}(T).$$  \hspace{1cm} (31)

Note that $\sigma^*$ is the optimal constant algorithm selection at horizon T, but it is not necessarily the optimal algorithm selection: there might exist, and there probably exists a non constant algorithm selection yielding a smaller pseudo-regret.

The ESBAS absolute pseudo-regret $\overline{p}_\alpha^{ESBAS}(T)$ can be decomposed into the pseudo-regret for not having followed the optimal constant algorithm selection $\sigma^*$ and the pseudo-regret for not having selected the algorithm with the highest return, i.e. between the pseudo-regret on the trajectory and the pseudo-regret on the immediate optimal return:

$$\overline{p}_\sigma^{ESBAS}(T) = T\mathbb{E}\mu_\sigma^\infty - \mathbb{E}_{\sigma^{ESBAS}} \left[ \sum_{\tau=1}^{T} \mathbb{E}_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{\sigma^*} \right]$$

$$+ \mathbb{E}_{\sigma^{ESBAS}} \left[ \sum_{\tau=1}^{T} \mathbb{E}_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{\sigma^*} \right] - \mathbb{E}_{\sigma^{ESBAS}} \left[ \sum_{\tau=1}^{T} \mathbb{E}_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{k_\tau} \right],$$  \hspace{1cm} (32)

where $E_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{\sigma^*}$ is the expected return of policy $\pi_{\tilde{D}_{\tau-1}^{ESBAS}}$, learnt by algorithm $\alpha^*$ on trajectory set $\tilde{D}_{\tau-1}^{ESBAS}$, which is the trajectory subset of $D_{\tau-1}$ obtained by removing all trajectories that were not generated with algorithm $\alpha^*$.

First line of Equation 32 can be rewritten as follows:

$$T\mathbb{E}\mu_\sigma^\infty - \mathbb{E}_{\sigma^{ESBAS}} \left[ \sum_{\tau=1}^{T} \mathbb{E}_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{\sigma^*} \right] = \sum_{\tau=1}^{T} \left( \mathbb{E}_{\mu_\sigma^\infty} - \mathbb{E}_{\sigma^{ESBAS}} \left[ \mathbb{E}_{\mu_{\tilde{D}_{\tau-1}^{ESBAS}}}^{\alpha^*} \right] \right).$$  \hspace{1cm} (33)

The key point in Equation 33 is to evaluate the size of $\mu_{\tilde{D}_{\tau-1}^{ESBAS}}$.

On the one side, Assumption 3 of fairness states that one algorithm learns as fast as any another over any history. The asymptotically optimal algorithm(s) when $\tau \rightarrow \infty$ is(are) therefore the same one(s) whatever the the algorithm selection is. On the other side, let $1 - \delta_\tau$ denote the probability, that at time $\tau$, the following inequality is true:

$$|\mu_{\tilde{D}_{\tau-1}^{ESBAS}}| \geq \left\lceil \frac{\tau - 1}{3K} \right\rceil.$$

$$\hspace{1cm} (34)$$
With probability $\delta_\tau$, inequality 34 is not guaranteed and nothing can be inferred about $E\mu^*_{\text{ESBAS}}$, except it is bounded under by $R_{\text{min}}/(1-\gamma)$. Let $E^{\tau-1}_{3K}$ be the subset of $E^{\tau-1}$ such that $\forall D \in E^{\tau-1}_{3K}, |\text{sub}^*(D)| \geq \lfloor (\tau-1)/3K \rfloor$. Then, $\delta_\tau$ can be expressed as follows:

$$
\delta_\tau = \sum_{D \in E^{\tau-1} \setminus E^{\tau-1}_{3K}} P(D|\sigma),
$$

(35)

With these new notations:

$$
E\mu^*_\infty - E\sigma_{\text{ESBAS}} E\mu^*_{\text{sub}^*(D_{\tau-1})} \leq E\mu^*_\infty - \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) E\mu^*_{\text{sub}^*(D)} - \delta_\tau \frac{R_{\text{min}}}{1 - \gamma},
$$

(36a)

$$
E\mu^*_\infty - E\sigma_{\text{ESBAS}} E\mu^*_{\text{sub}^*(D_{\tau-1})} \leq (1 - \delta_\tau) E\mu^*_\infty - \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) E\mu^*_{\text{sub}^*(D)} + \delta_\tau \left( E\mu^*_\infty - \frac{R_{\text{min}}}{1 - \gamma} \right).
$$

(36b)

Let consider $E^*(\alpha, N)$ the set of all sets $D$ such that $|\text{sub}^\alpha(D)| = N$ and such that last trajectory in $D$ was generated by $\alpha$. Since ESBAS, with $\Xi$, a stochastic bandit with regret in $O(\log(T)/\Delta)$, guarantees that all algorithms will eventually be selected an infinity of times, we know that:

$$
\forall \alpha \in P, \forall N \in N, \sum_{D \in E^{\tau}(\alpha, N)} P(D|\sigma) = 1.
$$

(37)

By applying recursively Assumption 2, one demonstrates that:

$$
\sum_{D \in E^{\tau+}(\alpha, N)} P(D|\sigma) E\mu^\alpha_{\text{sub}^*}(D) \geq \sum_{D \in E^N} P(D|\sigma) E\mu^\alpha_D,
$$

(38a)

$$
\sum_{D \in E^{\tau+}(\alpha, N)} P(D|\sigma) E\mu^\alpha_{\text{sub}^*}(D) \geq E\sigma^\alpha \left[ E\mu^\alpha_{D^N} \right].
$$

(38b)

One also notices the following piece-wisely domination from applying recursively Assumption 1

$$
(1 - \delta_\tau) E\mu^*_\infty - \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) E\mu^*_{\text{sub}^*(D)} = \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) \left( E\mu^*_\infty - E\mu^*_{\text{sub}^*(D)} \right),
$$

(39a)

$$
(1 - \delta_\tau) E\mu^*_\infty - \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) E\mu^*_{\text{sub}^*(D)} \leq \sum_{D \in E^{\tau+}(\alpha, \lfloor \frac{\tau-1}{3K} \rfloor)} P(D|\sigma) \left( E\mu^*_\infty - E\mu^*_{\text{sub}^*(D)} \right),
$$

(39b)

$$
(1 - \delta_\tau) E\mu^*_\infty - \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) E\mu^*_{\text{sub}^*(D)} \leq \sum_{D \in E^{\tau+}(\alpha, \lfloor \frac{\tau-1}{3K} \rfloor)} P(D|\sigma) \left( E\mu^*_\infty - E\mu^*_{\text{sub}^*(D)} \right),
$$

(39c)

$$
(1 - \delta_\tau) E\mu^*_\infty - \sum_{D \in E^{\tau-1}_{3K}} P(D|\sigma) E\mu^*_{\text{sub}^*(D)} \leq E\mu^*_\infty - \sum_{D \in E^{\tau+}(\alpha, \lfloor \frac{\tau-1}{3K} \rfloor)} P(D|\sigma) E\mu^*_{\text{sub}^*(D)}.
$$

(39d)

Then, by applying results from Equations 38b and 39d into Equation 36b, one obtains:

$$
E\mu^*_\infty - E\sigma_{\text{ESBAS}} E\mu^*_{\text{sub}^*(D_{\tau-1})} \leq E\mu^*_\infty - E\sigma^* \left[ E\mu^*_{D^*_{\tau-1}} \right] + \delta_\tau \left( E\mu^*_\infty - \frac{R_{\text{min}}}{1 - \gamma} \right).
$$

(40)
Next, the terms in the first line of Equation 32 are bounded as follows:

\[ T\mathbb{E}\mu_{\infty}^\star - \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\text{sub}^\star (D_{\tau-1})}^\star \right] \leq T\mathbb{E}\mu_{\infty}^\star - \mathbb{E}_{\alpha^\star} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\text{D}_{\tau}^\star}^\star \right] + \sum_{\tau=1}^{T} \delta_{\tau} \left( \mathbb{E}\mu_{\infty}^\star - \frac{R_{\min}}{1-\gamma} \right), \]  
\[ (41a) \]

\[ T\mathbb{E}\mu_{\infty}^\star - \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\text{sub}^\star (D_{\tau-1})}^\star \right] \leq \frac{T}{3K} \mathbb{E}\mu_{\text{abs}}^\star \left[ \frac{T}{3K} (T \mathbb{E}\mu_{\infty}^\star - \frac{R_{\min}}{1-\gamma}) \right] \sum_{\tau=1}^{T} \delta_{\tau}. \]  
\[ (41b) \]

Again, for \( T \geq 9K^2 \):

\[ T\mathbb{E}\mu_{\infty}^\star - \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\text{sub}^\star (D_{\tau-1})}^\star \right] \leq (3K + 1) \mathbb{E}\mu_{\text{abs}}^\star \left( \frac{T}{3K} \right) + \left( \mathbb{E}\mu_{\infty}^\star - \frac{R_{\min}}{1-\gamma} \right) \sum_{\tau=1}^{T} \delta_{\tau}. \]  
\[ (42) \]

Regarding the first term in the second line of Equation 32 from applying recursively Assumption 2:

\[ \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \mathbb{E}\mu_{\text{sub}^\star (D_{\tau})}^\star \right] \leq \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \mathbb{E}\mu_{D_{\tau}}^\star \right], \]  
\[ (43a) \]

\[ \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \mathbb{E}\mu_{\text{sub}^\star (D_{\tau})}^\star \right] \leq \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \max_{\alpha \in P} \mathbb{E}\mu_{D_{\tau}}^\star \right]. \]  
\[ (43b) \]

From this observation, one directly concludes the following inequality:

\[ \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\text{sub}^\star (D_{\tau})}^\star \right] - \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{D_{\tau}}^\star \right] \leq \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \max_{\alpha \in P} \mathbb{E}\mu_{D_{\tau}}^\star \right] - \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{D_{\tau}}^\star \right], \]  
\[ (44a) \]

\[ \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{\text{sub}^\star (D_{\tau})}^\star \right] - \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ \sum_{\tau=1}^{T} \mathbb{E}\mu_{D_{\tau}}^\star \right] \leq \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ T \mathbb{E}\mu_{\infty}^\star \right]. \]  
\[ (44b) \]

Injecting results from Equations 42 and 44b into Equation 32 provides the result:

\[ \mathbb{E}\mu_{\infty}^\star \left( T \right) \leq (3K + 1) \mathbb{E}\mu_{\text{abs}}^\star \left( \frac{T}{3K} \right) + \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ T \mathbb{E}\mu_{\infty}^\star \right] + \left( \mathbb{E}\mu_{\infty}^\star - \frac{R_{\min}}{1-\gamma} \right) \sum_{\tau=1}^{T} \delta_{\tau}. \]  
\[ (45) \]

We recall here that the stochastic bandit algorithm \( \Xi \) was assumed to guarantee to try the best algorithm \( \alpha^\star \) at least \( N/K \) times with high probability \( 1 - \delta_N \) and \( \delta_N \in \mathcal{O}(N^{-1}) \). Now, we show that at any time, the longest stochastic bandit run (i.e. the epoch that experienced the biggest number of pulls) lasts at least \( \tau = \frac{L}{\delta} \).

At epoch \( \beta_{\tau} \), the meta-time spent on epochs before \( \beta_{\tau} - 2 \) is equal to \( \sum_{\beta_{\tau} - 2}^{\beta_{\tau} - 1} 2 \beta_{\tau} = 2 \beta_{\tau} - 1 \); the meta-time spent on epoch \( \beta_{\tau} - 1 \) is equal to \( 2 \beta_{\tau} - 1 \); the meta-time spent on epoch \( \beta_{\tau} \) is either below \( 2 \beta_{\tau} - 1 \), in which case, the meta-time spent on epoch \( \beta_{\tau} - 1 \) is higher than \( \frac{L}{\delta} \), or the meta-time spent on epoch \( \beta_{\tau} \) is over \( 2 \beta_{\tau} - 1 \) and therefore higher than \( \frac{L}{\delta} \). Thus, ESBAS is guaranteed to try the best algorithm \( \alpha^\star \) at least \( \tau/3K \) times with high probability \( 1 - \delta_{\tau} \) and \( \delta_{\tau} \in \mathcal{O}(\tau^{-1}) \). As a result:

\[ \exists \kappa > 0, \quad \mathbb{E}\mu_{\text{abs}}^\star \left( T \right) \leq (3K + 1) \mathbb{E}\mu_{\text{abs}}^\star \left( \frac{T}{3K} \right) + \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ T \mathbb{E}\mu_{\text{sub}^\star (D_{\tau-1})}^\star \right] + \left( \mathbb{E}\mu_{\infty}^\star - \frac{R_{\min}}{1-\gamma} \right) \sum_{\tau=1}^{T} \kappa_{\tau}, \]  
\[ (46a) \]

\[ \exists \kappa > 0, \quad \mathbb{E}\mu_{\text{abs}}^\star \left( T \right) \leq (3K + 1) \mathbb{E}\mu_{\text{abs}}^\star \left( \frac{T}{3K} \right) + \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ T \mathbb{E}\mu_{\infty}^\star \right] + \mathbb{E}_{\alpha_{\text{ESBAS}}} \left[ T \mathbb{E}\mu_{\text{sub}^\star (D_{\tau-1})}^\star \right] + \kappa \log(T), \]  
\[ (46b) \]

which proves the theorem.