Quantum Recommendation Systems.

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- Assumptions: $|b\rangle$ can be prepared polylog(n) time and A is polylog(n) sparse.
- Incomparable to classical linear system solver which returns vector $x \in \mathbb{R}^n$ as opposed to $|x\rangle$.

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- Sometimes a variant of the classical problem is solved: ℓ_1 vs ℓ_2-SVM.
- Incomparable with classical.

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- *Quantum recommendation systems*.
- An exponential speedup over classical with similar assumptions and guarantees.
- An end to end application with no assumptions on the data set.
- Solves the 'same' problem as a classical recommendation system.

The Recommendation Problem

- The preference matrix P.

	P_1	P_2	P_3	P_4	$\cdots$	$\cdots$	P_{n-1}	P_n
U_1	.1	.4	?	?	$\cdots$	$\cdots$	?	.9
U_2	.2	?	.6	?	$\cdots$	$\cdots$	.85	?
U_3	?	?	.8	.9	$\cdots$	$\cdots$	?	.2
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
U_m	?	.75	?	?	$\cdots$	$\cdots$	?	.2

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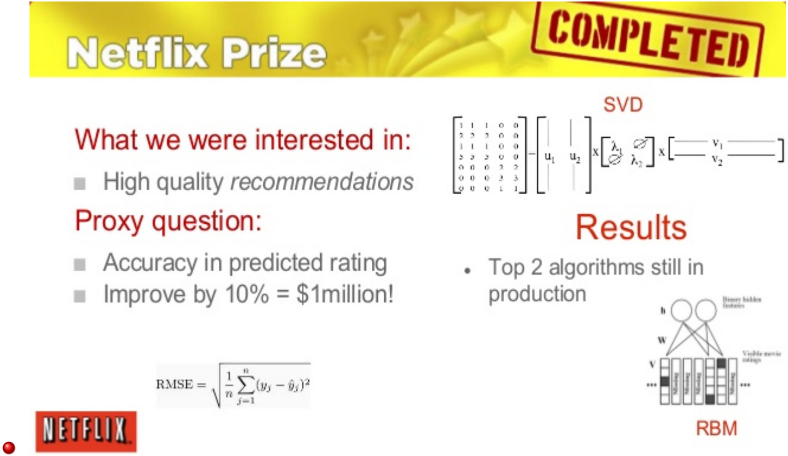
- P_{ij} is the value of item j for user i. Samples from P arrive in an online manner.
- The assumption that P has a good rank-k approximation for small k is widely used.

The Netflix problem



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Theorem

There is a quantum recommendation algorithm with running time $O(poly(k)polylog(mn))$.

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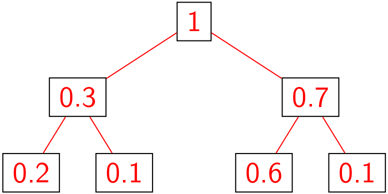
- The quantum algorithm has oracle access to binary tree data structure storing additional metadata.



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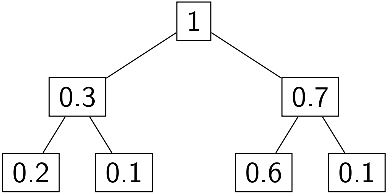

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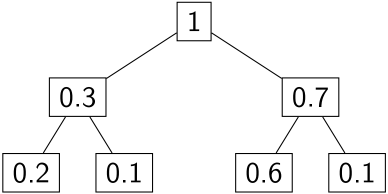

- We use the standard memory model used for algorithms like Grover search.
- Users arrive into system in an online manner and system provides recommendations in time $poly(k)polylog(mn)$.

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Theorem

There is an algorithm with running time $O(polylog(mn)/\epsilon)$ that transforms $\sum_i \alpha_i |v_i\rangle \rightarrow \sum_i \alpha_i |v_i\rangle |\overline{\sigma_i}\rangle$ where $\overline{\sigma_i} \in \sigma_i \pm \epsilon \|A\|_F$ with probability at least $1 - 1/poly(n)$.

Matrix Sampling

- Let T be a $0/1$ matrix such that $T_{ij} = 1$ if item j is 'good' recommendation for user i.

	P_1	P_2	P_3	P_4	$\cdots$	$\cdots$	P_{n-1}	P_n
U_1	0	0	?	?	$\cdots$	$\cdots$	?	1
U_2	0	?	0	?	$\cdots$	$\cdots$	1	?
U_3	?	?	1	1	$\cdots$	$\cdots$	?	0
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
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- Set the ?s to 0 and rescale to obtain a *subsample* matrix $\widehat{T}$.

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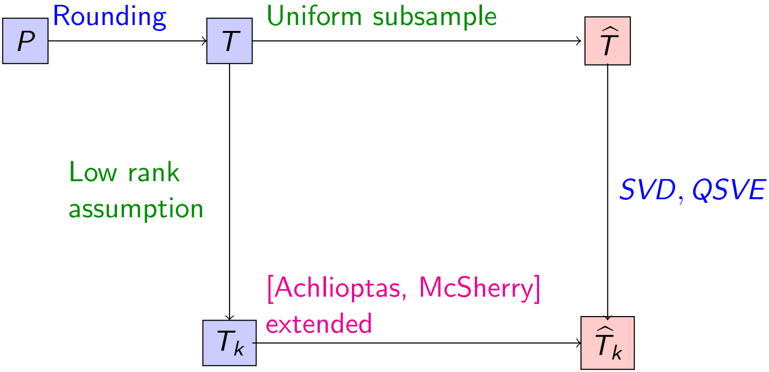

Figure: Matrix sampling based recommendation system.

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- Analysis: Sampling from matrix 'close to' $\widehat{T}_k$ yields good recommendations.

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Theorem (AM02)

If $\widehat{A}$ is obtained from a $0/1$ matrix A by subsampling with probability $p = 16n/\eta \|A\|_F^2$ then with probability at least $1 - exp(-19(\log n)^4)$, for all k,

$$||A - \widehat{A}_k||_F \leq ||A - A_k||_F + 3\sqrt{\eta}k^{1/4}||A||_F$$

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Analysis

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- We extend $AM02$ to this setting showing that:

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- For most typical users, samples from $(\widehat{T}_{\sigma,\kappa})_i$ are good recommendations with high probability.

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- The threshold $\sigma = \frac{\epsilon\sqrt{p}\|A\|_F}{\sqrt{2k}}$ and $\kappa = \frac{1}{3}$.
- Running time depends on the threshold and not the condition number.

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- Post-select on $t = 1$.
- The output $|A_{\geq \sigma,\kappa} x\rangle$ a projection the space of singular vectors with singular values $\geq \sigma$ and some in the range $[(1-\kappa)\sigma, \sigma)$.

Open Questions

- Find a classical algorithm matrix sampling based recommendation algorithm that runs in time $O(\text{poly}(k)\text{polylog}(mn))$.

  OR

  Prove a lower bound to rule out such an algorithm.
- Find more quantum machine learning algorithms.