A Complete Characterization of Unitary Quantum Space

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**Our motivation**: How powerful are quantum computers with a small number of qubits?

- **Our results**: Give two natural problems *characterize* the power of quantum computation with *any* bound on the number of qubits
  1. **Precise Succinct Hamiltonian** problem
  2. **Well-conditioned Matrix Inversion** problem

- These characterizations have many applications
  - **QMA** proof systems and Hamiltonian complexity
  - The power of preparing **PEPS** states vs ground states of **Local Hamiltonians**
  - Classical **Logspace** complexity
Quantum space complexity

- **BQSPACE**[\(k(n)\)] is the class of promise problems \(L=(L_{\text{yes}}, L_{\text{no}})\) that can be decided by a bounded error quantum algorithm acting on \(k(n)\) qubits.
  - i.e., Exists uniformly generated family of quantum circuits \(\{Q_x\}_{x\in\{0,1\}^*}\) each acting on \(O(k(|x|))\) qubits:
    - “If answer is yes, the circuit \(Q_x\) accepts with high probability”
      \[x \in L_{\text{yes}} \Rightarrow \langle 0^k | Q_x \rangle_1 \langle 1 |_{\text{out}} Q_x | 0^k \rangle \geq 2/3\]
    - “If answer is no, the circuit \(Q_x\) accepts with low probability”
      \[x \in L_{\text{no}} \Rightarrow \langle 0^k | Q_x \rangle_1 \langle 1 |_{\text{out}} Q_x | 0^k \rangle \leq 1/3\]
- Our results show two natural complete problems for **BQSPACE**[\(k(n)\)]
  - For any \(k(n)\) so that \(\log(n) \leq k(n) \leq \text{poly}(n)\)
  - Our reductions use classical \(k(n)\) space and \(\text{poly}(n)\) time
- **Subtlety:** This is “unitary quantum space”
  - No intermediate measurements
  - Not known if “deferring” intermediate measurements can be done space efficiently
Quantum Merlin-Arthur

- Problems whose solutions can be verified quantumly given a quantum state as witness
- **QMA**(c,s) is the class of promise problems \(L=(L_{yes}, L_{no})\) so that:
  \[
  x \in L_{yes} \Rightarrow \exists |\psi\rangle \Pr[V(x, |\psi\rangle) = 1] \geq c
  \]
  \[
  x \in L_{no} \Rightarrow \forall |\psi\rangle \Pr[V(x, |\psi\rangle) = 1] \leq s
  \]
- **QMA = QMA**\((2/3, 1/3) = \bigcup_{c>0} QMA(c, c-1/poly)\)
- **k-Local Hamiltonian** problem is **QMA**-complete (when \(k \geq 2\))[Kitaev ’00]
  - Input: \(H = \sum_{i=1}^{M} H_i\), each term \(H_i\) is \(k\)-local
  - Promise either:
    - Minimum eigenvalue \(\lambda_{\min}(H) > b\) or \(\lambda_{\min}(H) < a\)
    - Where \(b-a \geq 1/poly(n)\)
  - Which is the case?
- Generalizations of **QMA**:
  1. **PreciseQMA** = \(\bigcup_{c>0} QMA(c, c-1/exp)\)
  2. **k-bounded QMA**\(_m\)(c,s)  
    - Arthur’s verification circuit acts on \(k\) qubits
    - Merlin sends an \(m\) qubit witness

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Characterization 1:
Precise Succinct Hamiltonian problem
The **Precise Succinct** Hamiltonian Problem

- **Definition:** “**Succinct Encoding**”
  - We say a classical Turing machine $M$ is a **Succinct Encoding** for $2^{k(n)} \times 2^{k(n)}$ matrix $A$ if:
    - On input $i \in \{0,1\}^{k(n)}$, $M$ outputs non-zero elements in $i$-th row of $A$
    - Using at most $\text{poly}(n)$ time and $k(n)$ space

- **$k(n)$-Precise Succinct Hamiltonian problem**
  - Input: Size $n$ Succinct Encoding of $2^{k(n)} \times 2^{k(n)}$ Hermitian PSD matrix $A$
  - Promised either:
    - Minimum eigenvalue $\lambda_{\min}(A) > b$ or $\lambda_{\min}(A) < a$
    - Where $b-a > 2^{-O(k(n))}$
  - Which is the case?

- Compared to the **Local Hamiltonian** problem...
  - Input is SuccinctlyEncoded instead of Local
  - Precision needed to determine the promise is $\frac{1}{2^k}$ instead of $\frac{1}{\text{poly}(n)}$

- **Our Result:** $k(n)$-P.S Hamiltonian problem is complete for $\text{BQSPACE}[k(n)]$
Upper bound (1/2):

\[ k(n) - \text{Precise Succinct Hamiltonian} \in \text{k(n)-bounded QMA}_{k(n)}(c,c-2^{-k(n)}) \]

- **Recall:** \( k(n) \)-precise succinct Hamiltonian problem
  - Given succinct encoding of \( 2^{k(n)} \times 2^{k(n)} \) Hermitian PSD matrix \( A \), is \( \lambda_{\min}(A) \leq a \) or \( \lambda_{\min}(A) \geq b \) where \( b-a \geq 2^{-O(k(n))} \)?

- Merlin send eigenstate \( |\psi\rangle \) with minimum eigenvalue
  - Arthur runs phase estimation with one ancilla qubit on \( e^{-iA} \) and \( |\psi\rangle \)

\[
|0\rangle \quad \begin{array}{c}
\text{H} \\
\text{H}
\end{array} \\
\frac{1 + e^{-i\lambda t}}{2} |0\rangle + \frac{1 - e^{-i\lambda t}}{2} |1\rangle

|\psi\rangle \quad e^{-iAt} \quad |\psi\rangle
\]

- Measure ancilla and accept iff “0”
- Easy to see that we get “0” outcome with probability that’s slightly \( (2^{-O(k)}) \) higher if \( \lambda_{\min}(A) < a \) than if \( \lambda_{\min}(A) > b \)
- But this is exactly what’s needed to establish the claimed bound!

- **Remaining question:** how do we implement \( e^{-iA} \)?
  - We need to implement this operator with precision \( 2^{-k} \), since otherwise the error in simulation overwhelms the gap!
  - Luckily, we can invoke recent “precise Hamiltonian simulation” results of [Childs et. al’14]
    - Implement \( e^{-iA} \) to within precision \( \epsilon \) in space that scales with \( \log(1/\epsilon) \) and time \( \text{polylog}(1/\epsilon) \)
    - See also Guang Hao Low’s talk on Thursday!

- Using these results, can implement Arthur’s circuit in \( \text{poly}(n) \) time and \( O(k(n)) \) space

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Upper bound (2/2):

\( k(n) \)-bounded QMA\(^{k(n)}(c,c\,-2^{-k(n)}) \subseteq BQSPACE[k(n)] \)

1. Error amplify the PreciseQMA protocol
   - \textit{Goal}: Obtain a protocol with error inverse exponential in the witness length, \( k(n) \)
   - We want to do this while simultaneously preserving verifier space \( O(k(n)) \)
   - We develop new “space-preserving” QMA amplification procedures
     - By combining ideas from “in-place” amplification [Marriott & Watrous ‘04] with phase estimation

2. “Guess the witness”!
   - Consider this amplified verification protocol run on a maximally mixed state on \( k(n) \) qubits
   - Not hard to see that this new “no witness” protocol has a “precise” gap of \( O(2^{-k(n)}) \)

3. Amplify again!
   - Use our “space-efficient” QMA error amplification technique again!
   - Obtain bounded error, at a cost of exponential time
   - But the space remains \( O(k(n)) \), establishing the BQSPACE\([k(n)]\) upper bound

   - Space-efficient amplification also used to prove hardness!
     - \( k(n) \)-P.S Hamiltonian is BQSPACE\([k(n)]\)-hard
     - Follows from first using our space-bounded amplification, and then Kitaev’s clock-construction to build sparse Hamiltonian from the amplified circuit

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Application: **PreciseQMA=PSPACE**

- **Question**: How does the power of QMA scale with the completeness-soundness gap?
- **Recall**: PreciseQMA=\(\bigcup_{c>0}^{}\)QMA\((c, c-2^{-\text{poly}(n)})\)
- Both upper and lower bounds follow from our completeness result, together with BQPSPACE=PSPACE [Watrous’03]
- **Corollary**: “precise k-Local Hamiltonian problem” is PSPACE-complete
- **Extension**: “Perfect Completeness case”: QMA\((1, 1-2^{-\text{poly}(n)})=\)PSPACE
  - **Corollary**: checking if a local Hamiltonian has zero ground state energy is PSPACE-complete
Where is this power coming from?

• Could $\text{QMA} = \text{PreciseQMA} = \text{PSPACE}$?
  • Unlikely since $\text{QMA} = \text{PreciseQMA} \implies \text{PSPACE} = \text{PP}$
    • Using $\text{QMA} \subseteq \text{PP}$

• How powerful is $\text{PreciseMA}$, the classical analogue of $\text{PreciseQMA}$?
  • *Crude upper bound:* $\text{PreciseMA} \subseteq \text{NP}^{\text{PP}} \subseteq \text{PSPACE}$
    • And believed to be strictly less powerful, unless the “Counting Hierarchy” collapses

• So the power of $\text{PreciseQMA}$ seems to come from both the quantum witness and the small gap, together!

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Understanding “Precise” complexity classes

• We can answer questions in the “precise” regime that we have no idea how to answer in the “bounded-error” regime

• Example 1: How powerful is \textbf{QMA(2)}?
  • PreciseQMA=\textbf{PSPACE} (our result)
  • PreciseQMA(2)=\textbf{NEXP} [Blier & Tapp’07, Pereszlényi’12]
  • So, PreciseQMA(2) \neq PreciseQMA, unless NEXP=\textbf{PSPACE}

• Example 2: How powerful are quantum vs classical witnesses?
  • PreciseQCMA \subseteq \textbf{NP}^{\text{PP}}
  • So, PreciseQMA \neq PreciseQCMA, unless \textbf{PSPACE} \subseteq \textbf{NP}^{\text{PP}}

• Example 3: How powerful is \textbf{QMA} with perfect completeness?
  • PreciseQMA=PreciseQMA_1=\textbf{PSPACE}

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Characterization 2: 
Well-Conditioned Matrix Inversion
The Classical Complexity of **Matrix Inversion**

• The **Matrix Inversion** problem
  • Input: nonsingular $n \times n$ matrix $A$ with integer entries, promised either:
    • $A^{-1}[0,0]>2/3$ or
    • $A^{-1}[0,0]<1/3$
  • Which is the case?

• This problem can be solved in classical $O(\log^2(n))$ space [Csanky’76]

• Not believed to be solvable classically in $O(\log(n))$ space
  • If it is, then $L=NL$ (*Logspace* equivalent of $P=NP$)
Can we do better quantumly?

• “Well-Conditioned Matrix Inversion” can be solved in *non-unitary BQSPACE*[log(n)]! [Ta-Shma’12] building on [HHL’08]
  • i.e., same problem with poly(n) upper bound on the condition number, κ, so that κ^{-1}∥A∥1
  • *Appears* to attain quadratic speedup in space usage over classical algorithms

• *Begs the question*: how important is this “well-conditioned” restriction?
  • Can we also solve the *general Matrix Inversion* problem in quantum space O(log(n))?
Our results on **Matrix Inversion**

- **Well-conditioned Matrix Inversion** is complete for *unitary* $\text{BQSPACE}[\log(n)]!$
  1. We give a new quantum algorithm for **Well-conditioned Matrix Inversion** avoiding intermediate measurements
     - Combines techniques from [HHL’08] with amplitude amplification
  2. We also prove $\text{BQSPACE}[\log(n)]$ hardness—suggesting that “well-conditioned” constraint is *necessary* for quantum Logspace algorithms
Can generalize from $\log(n)$ to $k(n)$ qubits...

• **Result 3:** $k(n)$-**Well-conditioned Matrix Inversion** is complete for $\text{BQSPACE}[k(n)]$
  • Input: Succinct Encoding of $2^k \times 2^k$ PSD matrix $A$
    • Upper bound $\kappa < 2^{O(k(n))}$ on the condition number so that $\kappa^{-1} |A| \ll I$
    • Promised either $|A^{-1}[0,0]| \geq 2/3$ or $\leq 1/3$
    • Decide which is the case?

• Additionally, by varying the dimension and the bound on the condition number, can use **Matrix Inversion** problem to **characterize** the power of quantum computation with simultaneously bounded time and space!
Open questions

• Can we use our PreciseQMA=PSPACE characterization to give a PSPACE upper bound for other complexity classes?
  • For example, QMA(2)?

• How powerful is PreciseQIP?

• Natural complete problems for non-unitary quantum space?
Thanks!