Asymptotic entanglement manipulation under PPT operations: new SDP bounds and irreversibility

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Joint work with Runyao Duan (UTS:QSI)
Background

• Entanglement
  • Entangled state: $\rho \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$
  • Non-entangled (separable) state: $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i$
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- Two fundamental processes in entanglement manipulations
  - **Entanglement distillation** (Bennett, DiVincenzo, Smolin, Wootters, 1996; Rains, 1999, 2001): To extract standard \( 2 \otimes 2 \) maximally entangled states (EPR pairs) from a given state \( \rho \) by LOCC

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- **Entanglement dilution**: To prepare a given state $\rho$ with the standard EPR pairs by LOCC
Distillable entanglement and entanglement cost

- **Distillable entanglement**: The optimal (maximal) number of EPR pairs we can extract from $\rho$ in an asymptotic setting,

$$E_D(\rho_{AB}) := \sup \{ r : \lim_{n \to \infty} \inf_{\Lambda \in \text{LOCC}} \| \Lambda(\rho_{AB}^{\otimes n}) - \Phi(2^{rn}) \|_1 = 0 \}.$$ 

Note that $\Phi(2^{rn})$ is local unitarily equivalent to $\Phi(2)^{rn}$.
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It is equal to the regularized entanglement of formation (Hayden, Horodecki, Terhal 2001).
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- It is natural to ask whether $E_C \neq E_D$.
Asymptotic entanglement manipulations and irreversibility

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\[ E_D(|\psi\rangle\langle\psi|) = E_C(|\psi\rangle\langle\psi|) = S(\text{Tr}_B |\psi\rangle\langle\psi|). \]
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- Enlarge the set of operations?
- One candidate is the set of **PPT operations** (quantum operations completely preserving positivity of partial transpose). Note that \( \text{LOCC} \not\subseteq \text{SEP} \not\subseteq \text{PPT} \).
Entanglement manipulations under PPT operations

- PPT distillable entanglement (Rains 1999, 2001)

\[ E_{D,PPT}(\rho_{AB}) := \sup \{ r : \lim_{n \to \infty} \inf_{\Lambda \in \text{PPT}} \| \Lambda(\rho_{AB}^\otimes n) - \Phi(2^rn) \|_1 = 0 \}. \]
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Clearly,

\[ E_D \leq E_{D,PPT} \leq E_{C,PPT} \leq E_C \]
The class of antisymmetric states is an example of reversibility under PPT operations (Audenaert, Plenio, Eisert 2003).
Reversibility under PPT operations?

- The class of *antisymmetric states* is an example of reversibility under PPT operations (Audenaert, Plenio, Eisert 2003).
- Any state with a nonpositive partial transpose is distillable under PPT operations (Eggeling, Vollbrecht, Werner, Wolf 2001).

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**Background**

Improved SDP bound

Rains’ bound is not additive

Irreversibility

Conclusion
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\textbf{An old open problem} (Audenaert, Plenio, Eisert 2003):

\[ E_{D,PPT}(\rho) = E_{C,PPT}(\rho)? \]

(The 20\textsuperscript{th} problem listed at the website of Werner’s group.)
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- (Brandão and Plenio 2008) Entanglement can be reversibly interconverted under asymptotically non-entangling operations.
Main question and outline

This talk is about

- How to efficiently estimate the distillable entanglement $E_D$ and entanglement cost $E_C$?
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- Are asymptotic entanglement transformations reversible under PPT operations?

Diagram:

- Distillation: $\phi(2^nE_D)$
- Dilution: $\phi(2^nE_C)$
- $\rho_{AB}^\otimes n$ (PPT operations)
- $\Delta = 0$?
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We will show

- **Improved** upper bounds for $E_{D,PPT}$
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- **Improved** upper bounds for $E_{D,PPT}$
- **Efficiently computable** lower bound for $E_{C,PPT}$
- The **irreversibility** under PPT operations:

$$\exists \rho, \text{ s.t. } E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$
An Upper bound of $E_D$: Logarithmic negativity

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  - **Negativity** $N(\rho_{AB}) = (\| \rho_{AB}^{T_B} \|_1 - 1)/2$ (Zyczkowski, Horodecki, Sanpera and Lewenstein 1998)
  - (Rains, 2001; Vidal and Werner 2002):

    $$E_D(\rho_{AB}) \leq E_{D,PPT}(\rho_{AB}) \leq E_N(\rho_{AB}).$$

  - $E_N$ has many nice properties (see later).
A better SDP upper bound of $E_D$

- Primal SDP:
  \[
  E_W(\rho) = \max \log_2 \text{Tr}\rho R, \text{ s.t. } |R^{TB}| \leq 1, R \geq 0. \tag{1}
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  iii) Detecting genuine PPT distillable entanglement: $E_W(\rho) > 0$ iff $E_D(\rho) > 0$, i.e., $\rho$ is PPT distillable.

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  \item Improved over logarithmic negativity: $E_W(\rho) \leq E_N(\rho)$ and the inequality is strict in general.
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- $E_N$ has all above properties except v)!
Relative entropy of entanglement and Rains bound

- Relative Von Neumann entropy $S(\rho||\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$
- Relative entropy of entanglement (Vedral, Plenio, Rippin, Knight 1997; Vedral, Plenio, Jacobs, Knight 1997) with respect to PPT states

$$E_{R,PPT}(\rho) = \min S(\rho||\sigma) \quad \text{s.t.} \quad \sigma, \sigma^T_B \geq 0, \text{Tr} \sigma = 1.$$
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- Asymptotic relative entropy of entanglement w.r.t. PPT states
  
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  \[ E_{R,PPT}^\infty(\rho) = \lim_{n \to \infty} \frac{1}{n} E_{R,PPT}(\rho^\otimes n). \]
- (Rains 2001) Rains’ bound is the best known upper bound on the PPT distillable entanglement, i.e., $E_{D,PPT}(\rho) \leq R(\rho)$.
- **Rains’ bound** (Rains 2001; Audenaert, De Moor, Vollbrecht, Werner’02)
  \[ R(\rho) = \min S(\rho||\sigma) \text{ s.t. } \sigma \geq 0, \text{Tr} |\sigma^T_B| \leq 1, \]
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- Evidence: Rains’ bound equals to \( E^\infty_{R,PPT} \) for Werner states (Audenaert, Eisert, Jane, Plenio, Virmani, De Moor 2001) and orthogonally invariant states (Audenaert, et al. 2002).
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**Theorem**

There exists a two-qubit state \( \rho \) such that

\[ R(\rho \otimes^2) < 2R(\rho). \]

Meanwhile,

\[ E_\infty^{R,PPT}(\rho) < R(\rho). \]
Rains’ bound is not additive: Proof ideas

i) Construct a $2 \otimes 2$ state $\rho$ so that we can explicitly find a PPT state $\sigma$ such that

$$R(\rho) = E_{R,\text{PPT}}(\rho) = S(\rho \| \sigma)$$

via a technique in (Miranowicz, Ishizaka’08, $R = E_{R,\text{PPT}}$ for any $2 \otimes 2$ state; see also Gour, Friedland’11 and Girard+’14.)
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ii) Finding a PPT state $\tau$ via an algorithm developed in (Girard, Zinchenko, Friedland, Gour’15). This gives an upper bound on $E_{R,PPT}(\rho^{\otimes 2})$, i.e.,

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iii) Compare $S(\rho^2\|\tau)$ and $2E_{R,PPT}(\rho)$, achieve the goal by showing

$$R(\rho^2) \leq E_{R,PPT}(\rho^2) \leq S(\rho^2\|\tau) < 2S(\rho\|\sigma) = 2R(\rho).$$

and $E_{R,PPT}^\infty(\rho) \leq E_{R,PPT}(\rho^2)/2 < R(\rho)$. 
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iv) An example of semi-analytical and semi-numerical proof.
Rains’ bound is not additive: Proof ideas (cont.)

We construct $ \rho_r $ and $ \sigma_r $ such that $ R(\rho_r) = E_{R,PPT}(\rho_r) = S(\rho_r||\sigma_r)$:

$$
\rho_r = \frac{1}{8} |00\rangle\langle 00| + x |01\rangle\langle 01| + \frac{7 - 8x}{8} |10\rangle\langle 10| + \frac{32r^2 - (6 + 32x)r + 10x + 1}{4\sqrt{2}} (|01\rangle\langle 10| + |10\rangle\langle 01|) 
$$

$$
\sigma_r = \frac{1}{4} |00\rangle\langle 00| + \frac{1}{8} |11\rangle\langle 11| + r |01\rangle\langle 01| + \left(\frac{5}{8} - r\right) |10\rangle\langle 10| + \frac{1}{4\sqrt{2}} (|01\rangle\langle 10| + |10\rangle\langle 01|). 
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with $ x $ and $ y $ are determined by $ r $. 

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Rains’ bound is not additive: Proof ideas (cont.)

We construct $\rho_r$ and $\sigma_r$ such that $R(\rho_r) = E_{R,PPT}(\rho_r) = S(\rho_r||\sigma_r)$:

$$\rho_r = \frac{1}{8}|00\rangle\langle 00| + x|01\rangle\langle 01| + \frac{7 - 8x}{8}|10\rangle\langle 10| + \frac{32r^2 - (6 + 32x)r + 10x + 1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|)$$

$$\sigma_r = \frac{1}{4}|00\rangle\langle 00| + \frac{1}{8}|11\rangle\langle 11| + r|01\rangle\langle 01| + \left(\frac{5}{8} - r\right)|10\rangle\langle 10| + \frac{1}{4\sqrt{2}}(|01\rangle\langle 10| + |10\rangle\langle 01|).$$

with $x$ and $y$ are determined by $r$. When $0.45 \leq r \leq 0.548$, we show the gap between $2R(\rho_r)$ and $E_R^+(\rho_r^\otimes 2) = S(\rho_r^\otimes 2||\tau_r)$:

![Graph showing the comparison between $2R(\rho_r)$ and $E_R^+(\rho_r^\otimes 2)$](image-url)
Application & New problem

- Regularization of Rains’ bound: \( R^\infty(\rho) = \inf_{k \geq 1} \frac{R(\rho^\otimes k)}{k} \).
- A better upper bound on distillable entanglement:

\[
E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq R(\rho),
\]

and the second inequality could be strict.
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- Remark: Hayashi introduced \( R^\infty \) in his book in 2006.

New problem and an old open problem
- \( R^\infty(\rho) = E_{R,PPT}^\infty(\rho) \)?
- Note that

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E_{D,PPT}(\rho) \leq R^\infty(\rho) \leq E_{R,PPT}^\infty(\rho) \leq E_{C,PPT}(\rho).
\]
- Dream: if \( R^\infty(\rho) < E_{R,PPT}^\infty(\rho) \), then we will have \( E_{D,PPT}(\rho) < E_{C,PPT}(\rho) \)!
Application & New problem

- Regularization of Rains’ bound: $R^\infty(\rho) = \inf_{k \geq 1} \frac{R(\rho \otimes^k)}{k}$.
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- Dream: if $R^\infty(\rho) < E_{R,PPT}^\infty(\rho)$, then we will have $E_{D,PPT}(\rho) < E_{C,PPT}(\rho)$!
- How to evaluate $R^\infty$ and $E_{R,PPT}^\infty$?

Xin Wang & Runyao Duan (UTS:QSI) | Asymptotic entanglement manipulation under PPT operations: new bounds & irreversibility
Irreversibility under PPT operations

**Theorem (Key result)**

There exists entangled state $\rho$ such that $R^\infty(\rho) < E^\infty_{R,PPT}(\rho)$. Thus, the asymptotic entanglement manipulation under PPT operations is irreversible:

$$\exists \rho, \text{ s.t. } E_{D,PPT}(\rho) < E_{C,PPT}(\rho).$$
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The lower bound of $E_{R,PPT}^\infty$

Our key contribution is an efficiently computable lower bound on the regularized relative entropy of entanglement w.r.t. PPT states.

**A lower bound for $E_{R,PPT}^\infty$**

Let $P$ be the projection over the support of state $\rho$. Then

$$E_{R,PPT}^\infty(\rho) \geq E_\eta(\rho) = -\log_2 \eta(P),$$

where

$$\eta(P) = \min t, \text{ s.t. } -t \mathbb{1} \leq Y_{TB} \leq t \mathbb{1}, -Y \leq P_{TB}^T \leq Y.$$
Lower bound of $E_{R,PPT}^\infty$: Sketch of the proof

- Relate the problem to an SDP:

$$\min_{\sigma \in \text{PPT}} S(\rho \| \sigma) \geq \min_{\rho_0 \in D(\rho), \sigma_0 \in \text{PPT}} S(\rho_0 \| \sigma_0) \geq \min_{\sigma_0 \in \text{PPT}} -\log \text{Tr } P\sigma_0.$$ 

Also see min-relative entropy (Datta 2009):

$$S(\rho \| \sigma) \geq D_{\text{min}}(\rho \| \sigma) = -\log \text{Tr } P\sigma$$
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- Utilizing the weak duality of SDP and did a further relaxation

$$E_{R,PPT}(\rho) \geq \min_{\sigma_0 \in \text{PPT}} -\log \text{Tr} P \sigma_0 \geq \max -\log t \text{ s.t. } Y^{T_B} \leq t \mathbb{1}, P^{T_B} \leq Y \ (\text{not additive} \; \odot)$$
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$$E_{R,PPT}(\rho) \geq \min_{\sigma_0 \in \text{PPT}} -\log \text{Tr} P\sigma_0 \geq \max -\log t \text{ s.t. } Y^{T_B} \leq t1, P^{T_B} \leq Y \quad \text{(not additive 😞)}$$

$$\geq \max -\log t \text{ s.t. } -t1 \leq Y^{T_B} \leq t1, -Y \leq P^{T_B} \leq Y = E_\eta.$$
Lower bound of $E_{R,PPT}^\infty$: Sketch of the proof

- Relax the problem to an SDP:
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  \min_{\sigma \in \text{PPT}} S(\rho \| \sigma) \geq \min_{\sigma_0 \in D(\rho), \sigma_0 \in \text{PPT}} S(\rho_0 \| \sigma_0) \\
  \geq \min_{\sigma_0 \in \text{PPT}} -\log \Tr P \sigma_0.
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- Utilizing the strong duality of SDP to obtain
  \[
  E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2), \quad \odot
  \]
Lower bound of $E_{\infty,\text{PPT}}^\infty$: Sketch of the proof

- Relax the problem to an SDP:

  $$\min_{\sigma \in \text{PPT}} S(\rho \| \sigma) \geq \min_{\rho_0 \in D(\rho), \sigma_0 \in \text{PPT}} S(\rho_0 \| \sigma_0) \geq \min_{\sigma_0 \in \text{PPT}} -\log \text{Tr} P\sigma_0.$$ 

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- Utilizing the strong duality of SDP to obtain

  $$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2), \; ☺$$

  thus we have

  $$E_{R,\text{PPT}}^\infty(\rho) \geq \lim_{n \to \infty} \frac{1}{n} E_\eta(\rho^\otimes n) = E_\eta(\rho).$$
Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace
  \[
  \text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle\}
  \]
Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace
  \[\text{span}\{ |01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle \}\]

- **Example 1:** We choose the rank-2 state. Let
  \[
  \rho = \frac{1}{2}( |v_1\rangle \langle v_1 | + |v_2\rangle \langle v_2 |) \quad \text{with}
  \]
  \[|v_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), |v_2\rangle = \frac{1}{\sqrt{2}}(|02\rangle - |20\rangle),\]
Explicit examples of irreversibility under PPT operations

- Consider the $3 \otimes 3$ anti-symmetric subspace
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  \[ E_{D,PPT}(\rho) = R^\infty(\rho) = \log_2\left(1 + \frac{1}{\sqrt{2}}\right) < 1 = E_{R,PPT}(\rho) = E_{C,PPT}(\rho). \]
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- Sufficient condition for the irreversibility: If

  \[
  E_\eta(\rho) > E_W(\rho) = \min_{X_{AB} \geq \rho} \log_2 \| X_{AB}^{TB} \|_1,
  \]

  then

  \[
  E_{D,PPT}(\rho) \leq E_W(\rho) < E_\eta(\rho) \leq E_{C,PPT}(\rho).
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Explicit examples of irreversibility under PPT operations

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- Sufficient condition for the irreversibility: If
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  E_{D,PPT}(\rho) \leq E_W(\rho) < E_\eta(\rho) \leq E_{C,PPT}(\rho),
  \]

- **Example 2:** The above example can be generalized to any rank-2 state \( \rho \) supporting on the $3 \otimes 3$ anti-symmetric subspace:
  \( E_{D,PPT}(\rho) \leq E_W(\rho) < 1 = E_\eta(\rho) = E_{C,PPT}(\rho) \).
Conclusion

Results:
- Better SDP upper bound on $E_D$
- Non-additivity of Rains’ bound
- SDP lower bound for $E_R^{\infty,\text{PPT}}$
- Irreversibility under PPT operations:

$$E_{D,\text{PPT}} \neq E_{C,\text{PPT}}.$$
Conclusion

Results:

- Better SDP upper bound on $E_D$
- Non-additivity of Rains’ bound
- SDP lower bound for $E_{R,PPT}^\infty$
- Irreversibility under PPT operations:

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Discussions:

- $E_{D,PPT}(\rho) = R^\infty(\rho)$?
**Conclusion**

**Results:**
- Better SDP upper bound on $E_D$
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**Discussions:**
- $E_{D,PPT}(\rho) = R^\infty(\rho)$?
- Note that $E_\eta$ is not tight for the $3 \otimes 3$ anti-symmetric state $\sigma_a$, how to improve $E_\eta$?
Conclusion

Results:

- Better SDP upper bound on $E_D$
- Non-additivity of Rains’ bound
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- Irreversibility under PPT operations:
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Discussions:

- $E_{D,PPT}(\rho) = R^\infty(\rho)$?
- Note that $E_\eta$ is not tight for the 3 $\otimes$ 3 anti-symmetric state $\sigma_a$, how to improve $E_\eta$?
- How to evaluate the distillable entanglement without using PPT operations?
arXiv: 1606.09421, 1605.00348, 1601.07940
Thank you for your attention!