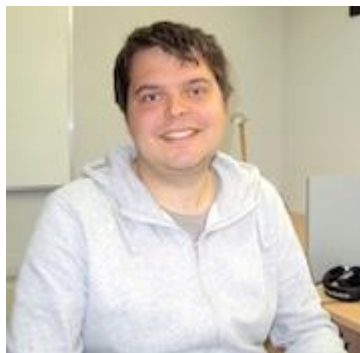


# Anyons and Matrix Product Operator Algebras

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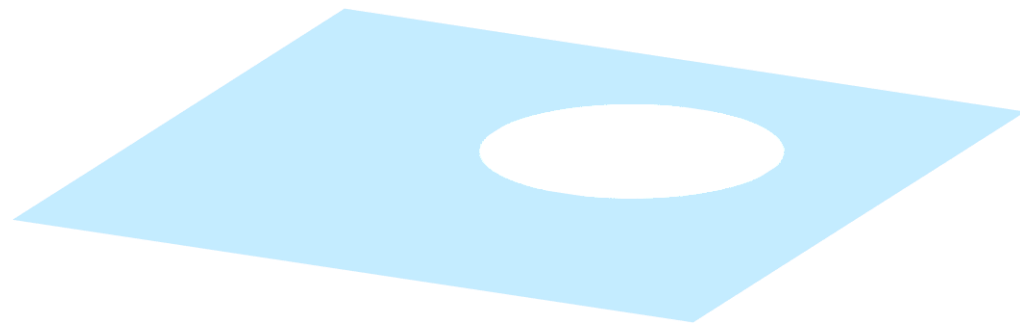
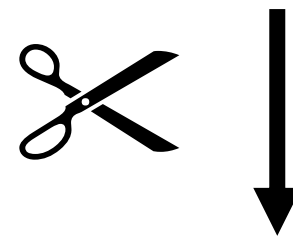
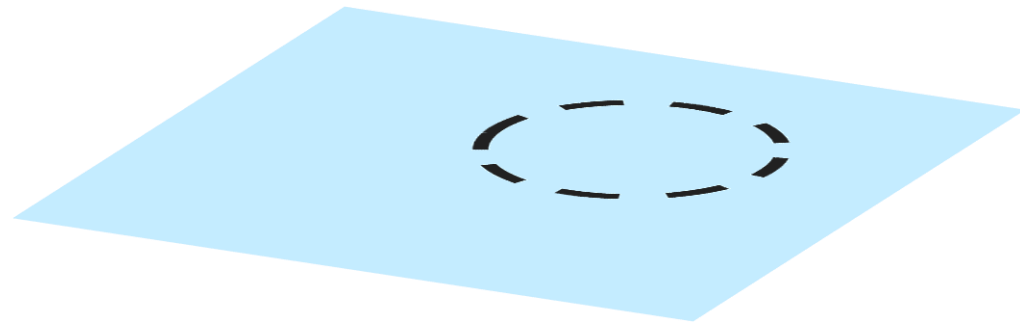
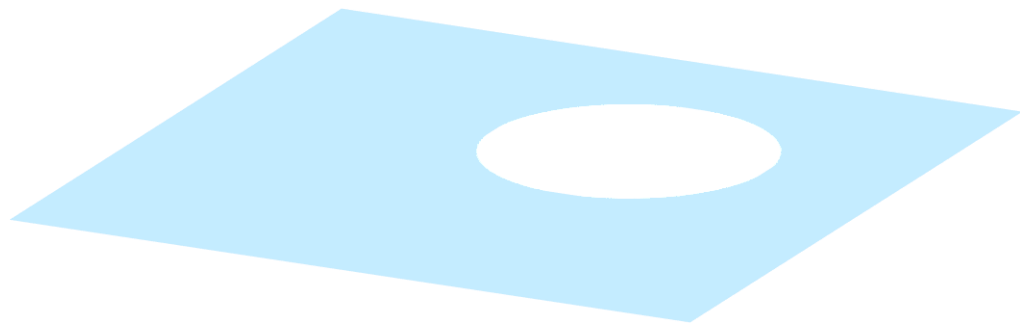
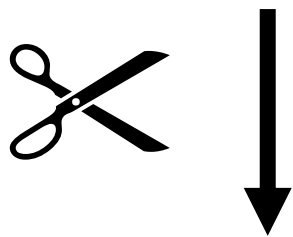
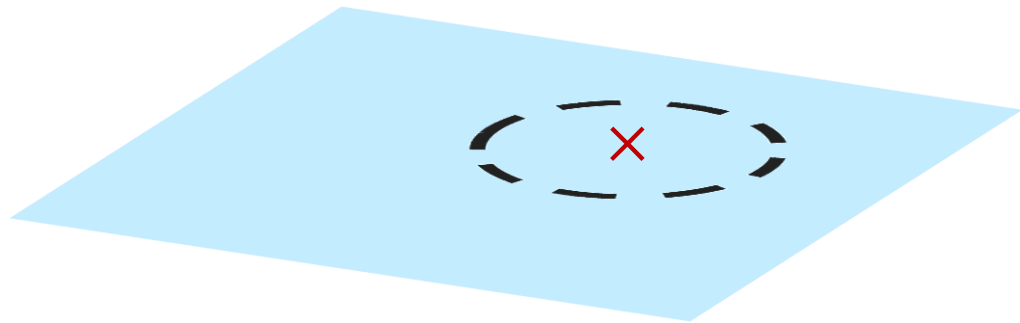
What states of matter are there?

$$[\psi] := \{\psi' \mid \psi' = U\psi, \exists U \text{ a local unitary}\}$$

# Anyons / Superselection sectors

(Modular Tensor Categories)

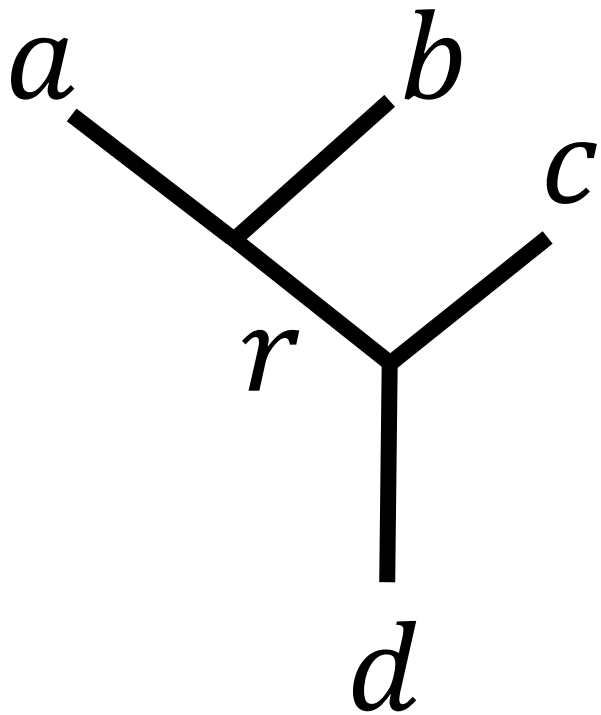
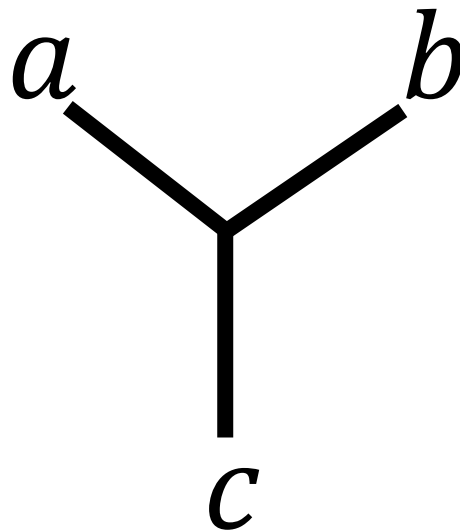
$$[\rho] := \{\rho' \mid \rho' = U\rho U^{-1}, \exists U\}$$



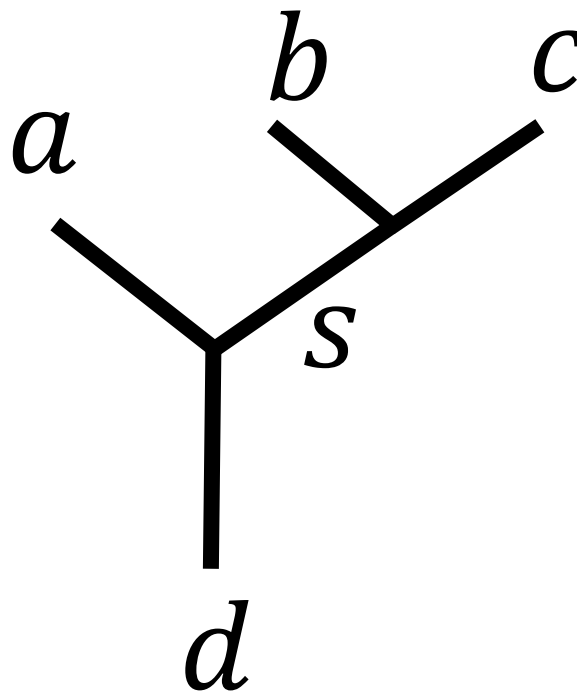
$\neq$

$$a \times b = c + d + \dots$$

$$= \sum N_{ab}^c c$$



$$= F_{d;rs}^{abc}$$



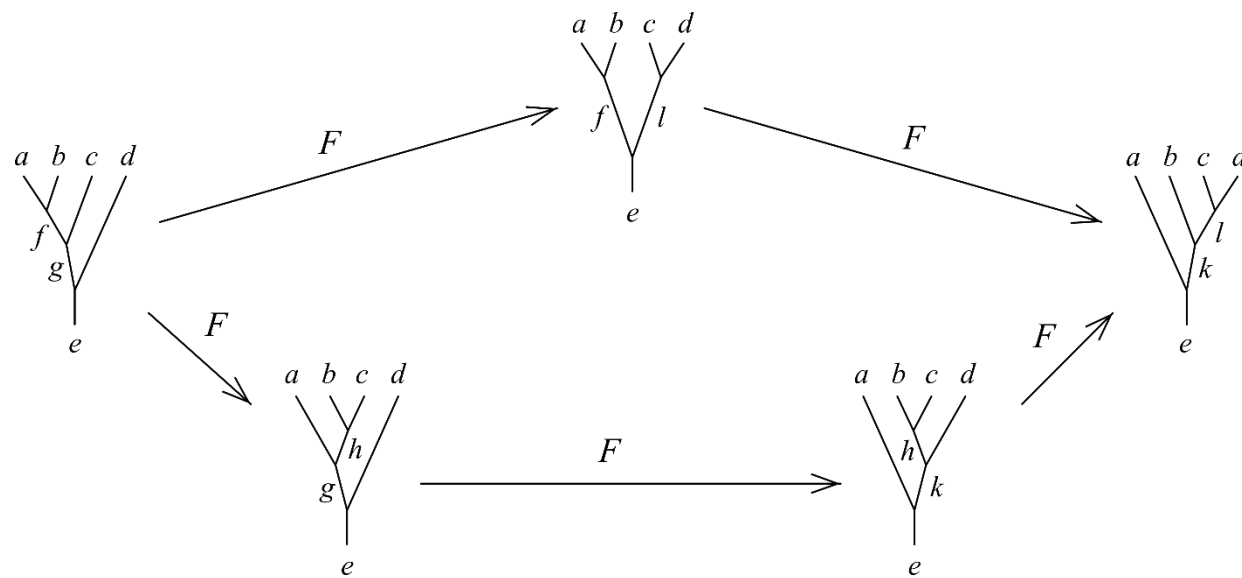
$$R^{ab} = \text{Diagram of a crossing with lines } a \text{ and } b$$

The diagram shows two lines, labeled  $a$  and  $b$ , crossing each other. Line  $a$  starts at the top left and ends at the bottom right. Line  $b$  starts at the top right and ends at the bottom left.

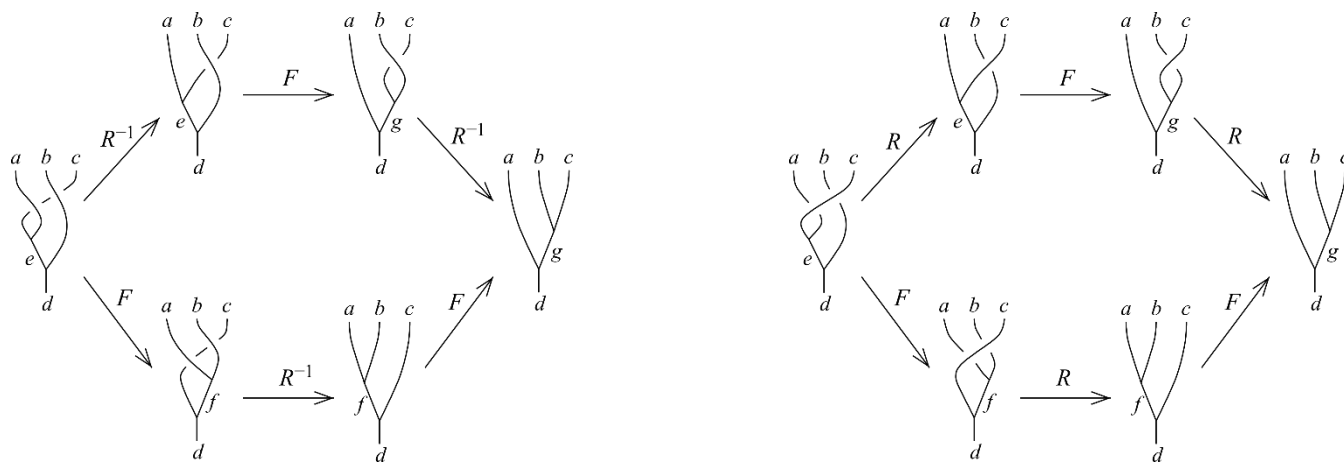
$$\text{Diagram of a loop with lines } a, b, \text{ and } c = R_c^{ab} \text{ Diagram of a Y-junction with lines } a, b, \text{ and } c$$

The left diagram shows a loop structure where lines  $a$  and  $b$  enter from the top, meet at a vertex, form a loop, and then line  $c$  exits from the bottom. The right diagram shows a Y-junction where lines  $a$  and  $b$  enter from the top and meet at a vertex, with line  $c$  exiting from the bottom.

# Pentagon



# Hexagons





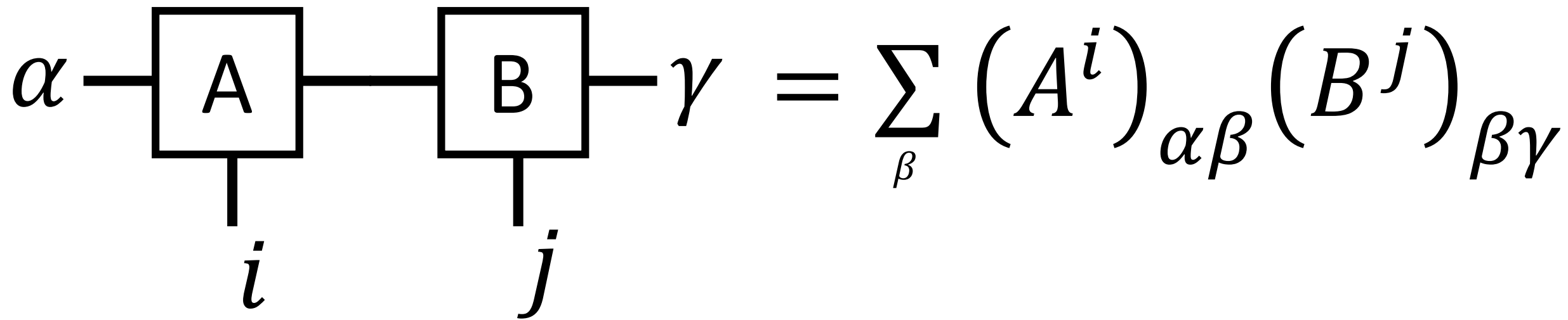
$$S_{ab} = a \text{ (two overlapping circles) } b$$

$$a \text{ (loop with vertical line) } = e^{i2\pi h_a} \text{ (vertical line) } a$$

# Tensor Networks

$$\alpha \text{ --- } \boxed{A} \text{ --- } \beta = (A^i)_{\alpha\beta} \in \mathbb{C}$$

$i$

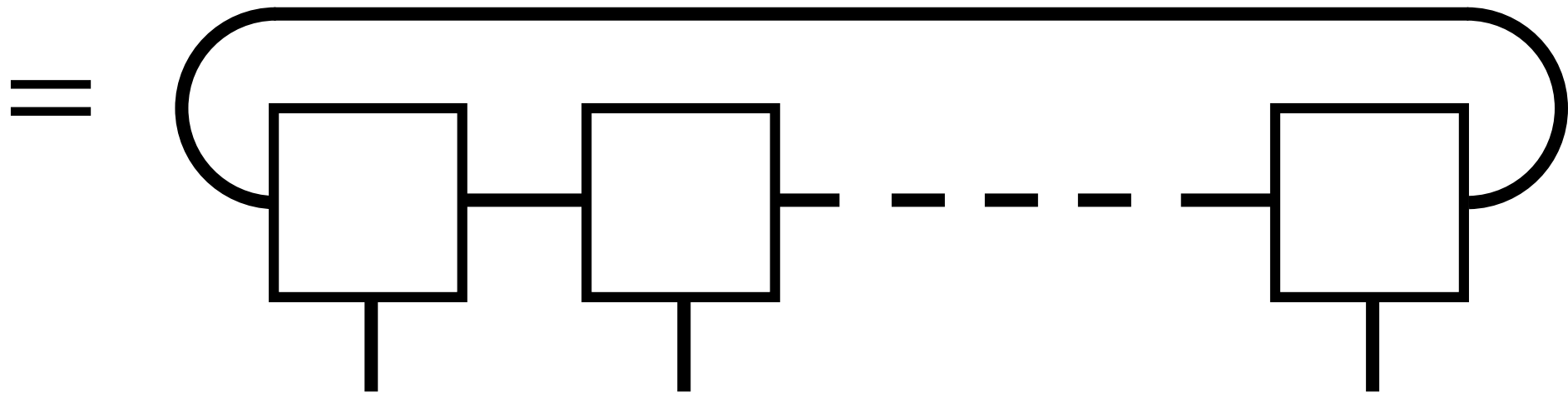


The diagram shows a sequence of two operations, A and B, connected by a horizontal line. Operation A is represented by a square box with the letter 'A' inside. A horizontal line enters A from the left, labeled with the Greek letter alpha ( $\alpha$ ). A horizontal line exits A to the right, entering the left side of operation B. Operation B is represented by a square box with the letter 'B' inside. A horizontal line exits B to the right, labeled with the Greek letter gamma ( $\gamma$ ). Below the box for A, a vertical line points down to the label  $i$ . Below the box for B, a vertical line points down to the label  $j$ . To the right of the diagram is an equals sign followed by a summation over the index  $\beta$ . The summation is  $\sum_{\beta} (A^i)_{\alpha\beta} (B^j)_{\beta\gamma}$ .

$$\alpha \text{---} \boxed{A} \text{---} \boxed{B} \text{---} \gamma = \sum_{\beta} (A^i)_{\alpha\beta} (B^j)_{\beta\gamma}$$

$i$                        $j$

$$|MPS_N(A)\rangle = \sum_{i_1 \dots i_N} \text{Tr}[A^{i_1} \dots A^{i_N}] |i_1 \dots i_N\rangle$$



# Normal form

$$A^i \sim \begin{pmatrix} A_0^i & M_{01}^i & \dots \\ 0 & A_1^i & \\ \vdots & & \ddots \end{pmatrix}$$

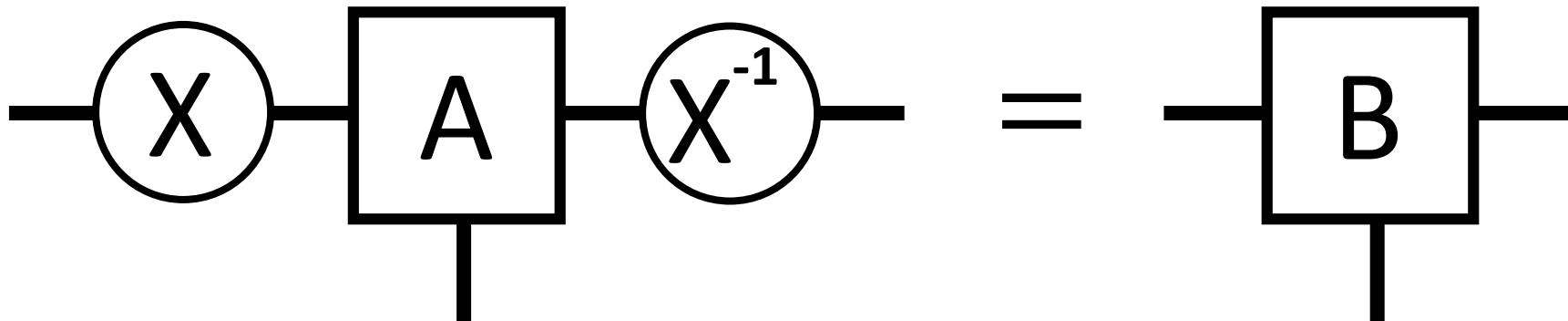
s.t.  $\{A_k^i\}_i$  generates  
an irred. algebra

$$|MPS_N(A)\rangle = \sum_k |MPS_N(A_k)\rangle$$

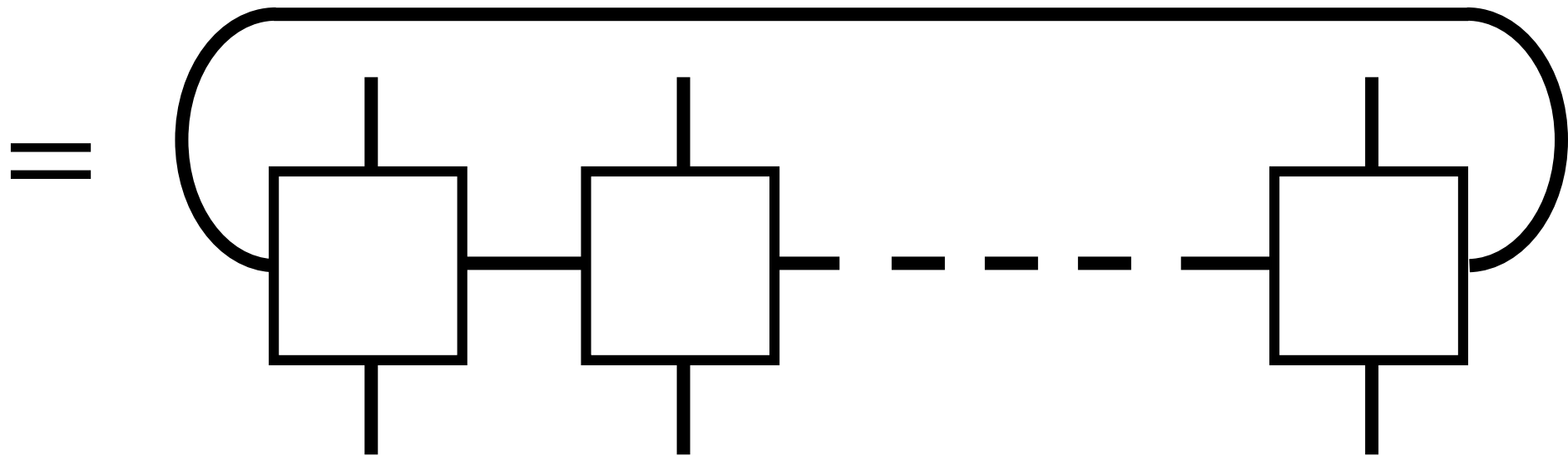
# Fundamental Thm. of MPS

$$|MPS_N(A)\rangle = |MPS_N(B)\rangle \quad \forall N$$

$\Rightarrow \exists X$  s.t.

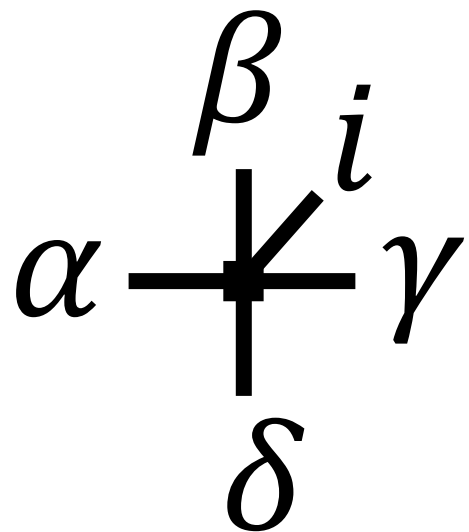


$$MPO_N(A) = \sum_{\substack{i_1 \dots i_N \\ j_1 \dots j_N}} \text{Tr} \left[ A^{i_1 j_1} \dots A^{i_N j_N} \right] |i_1 \dots i_N\rangle \langle j_1 \dots j_N|$$

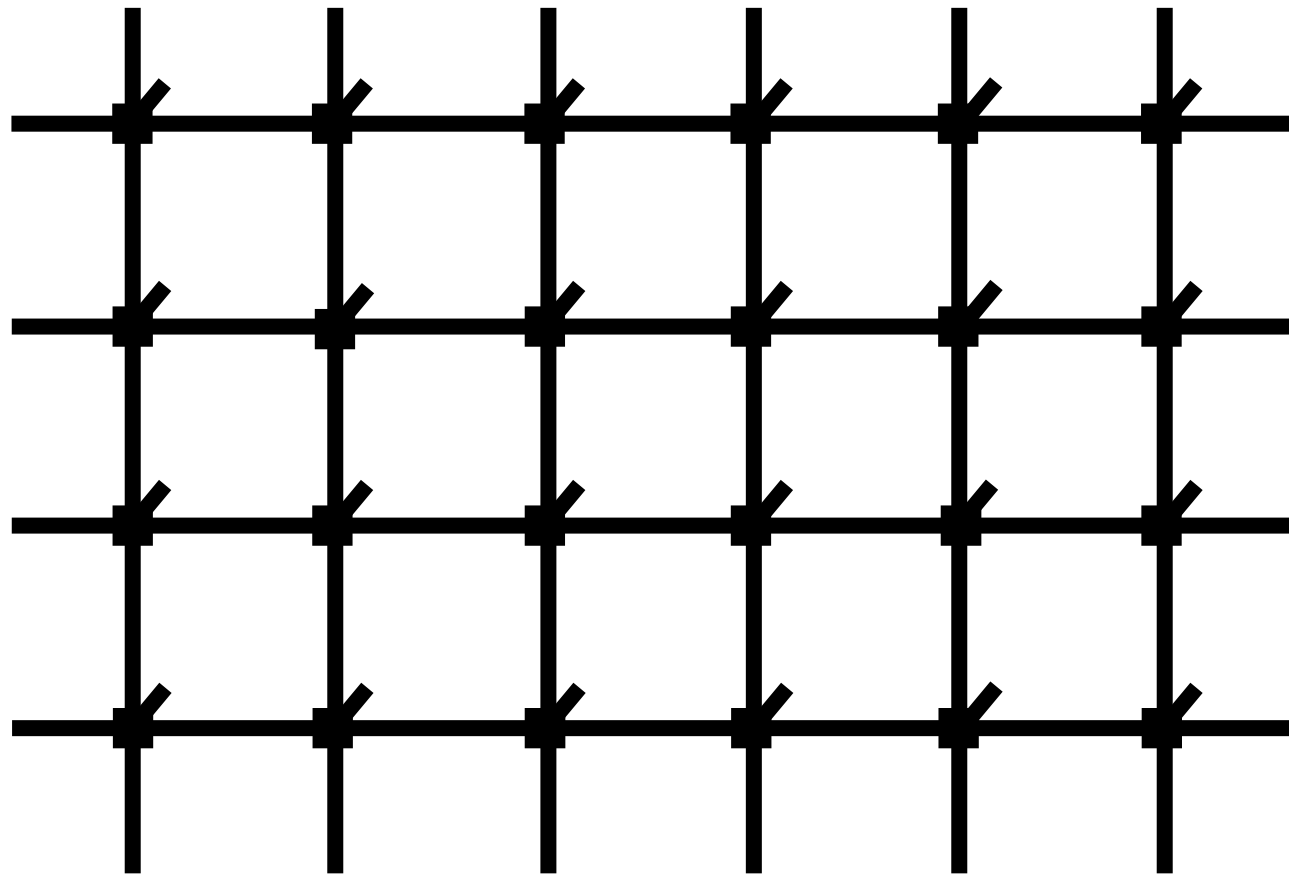


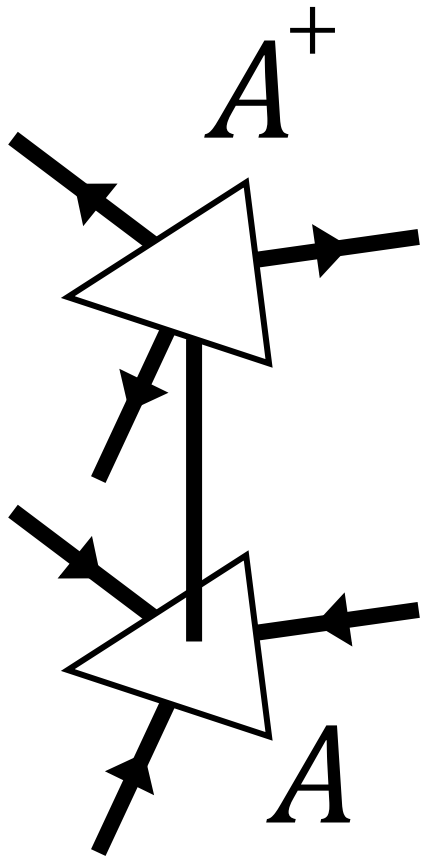


$$|PEPS(A)\rangle =$$

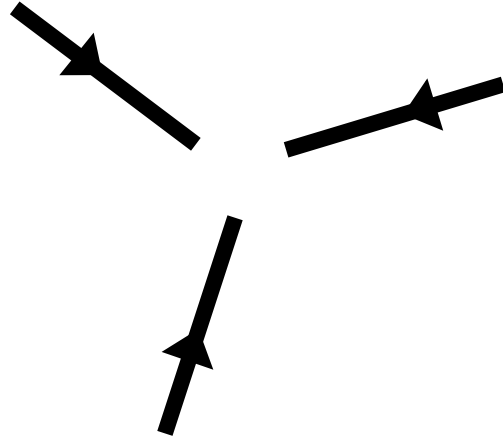


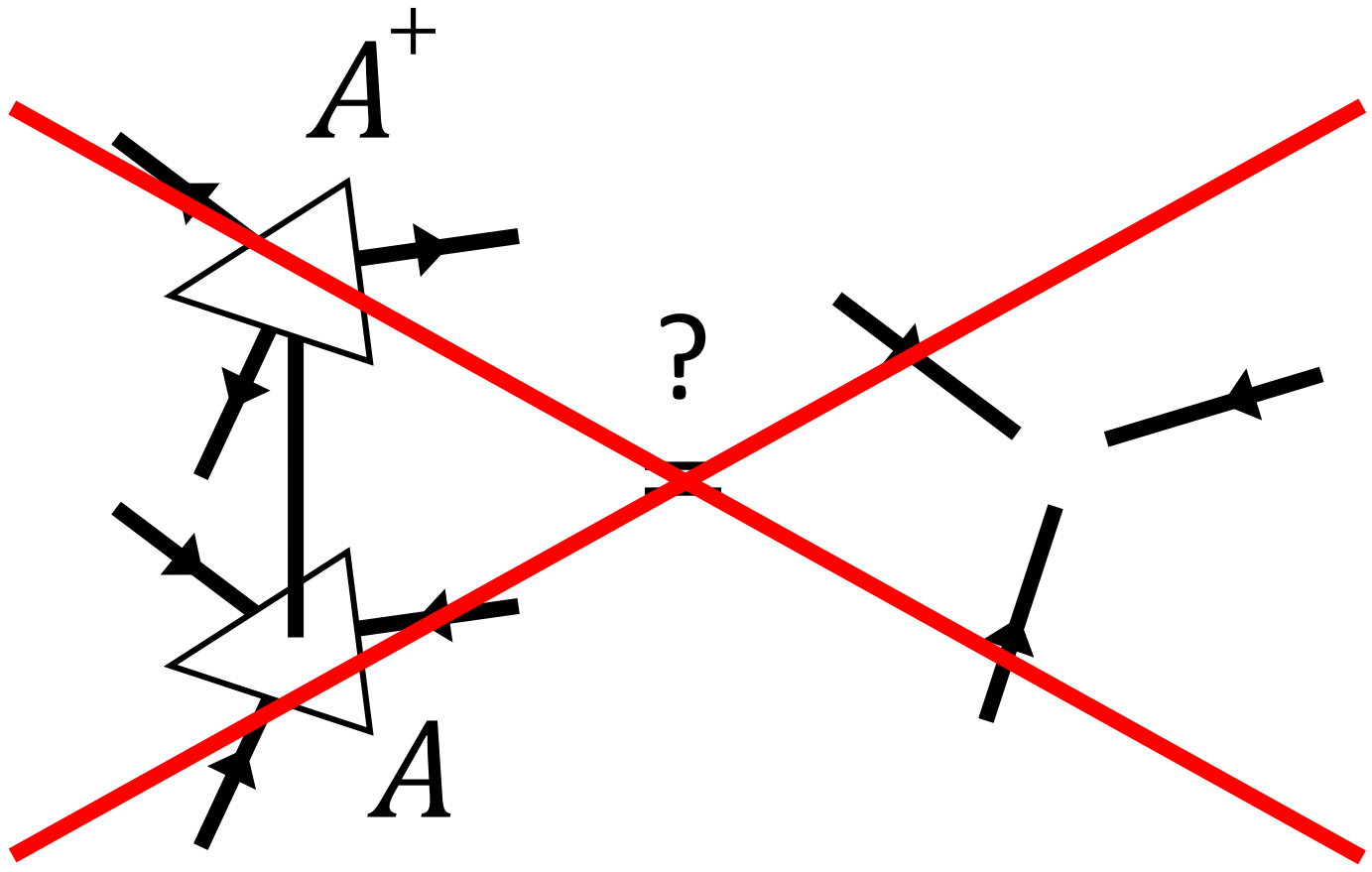
$$= (A^i)_{\alpha\beta\gamma\delta}$$

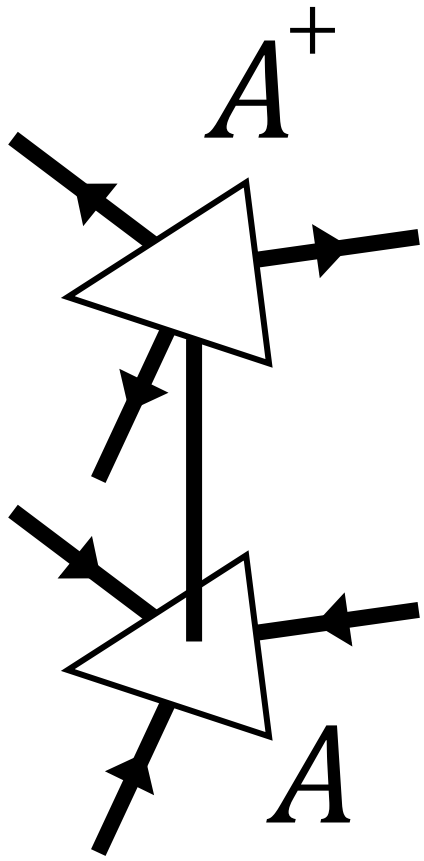




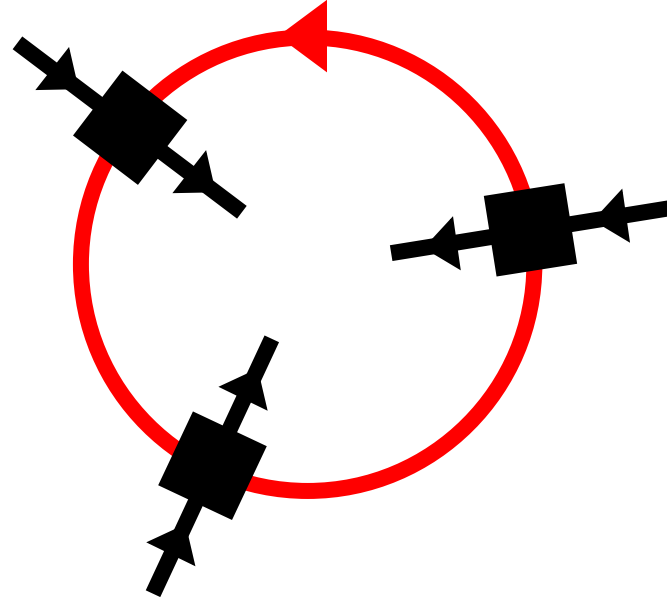
$?$   
 $=$

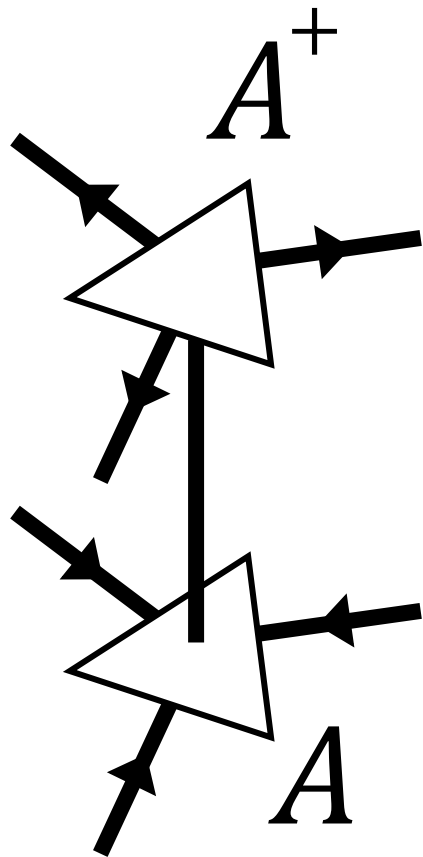




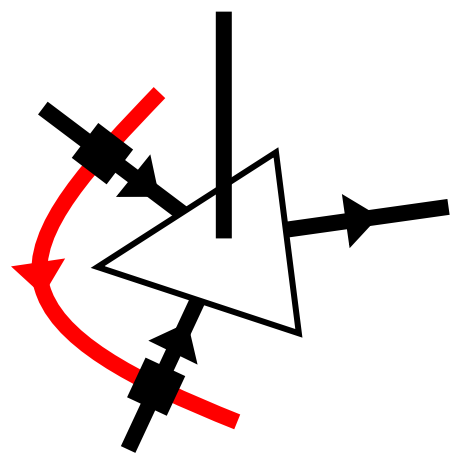
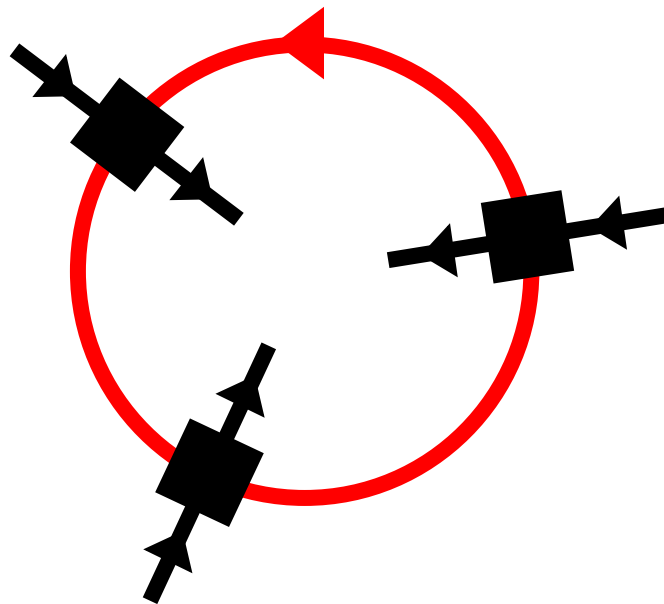


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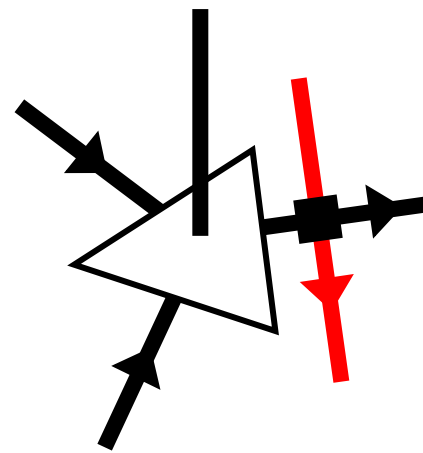




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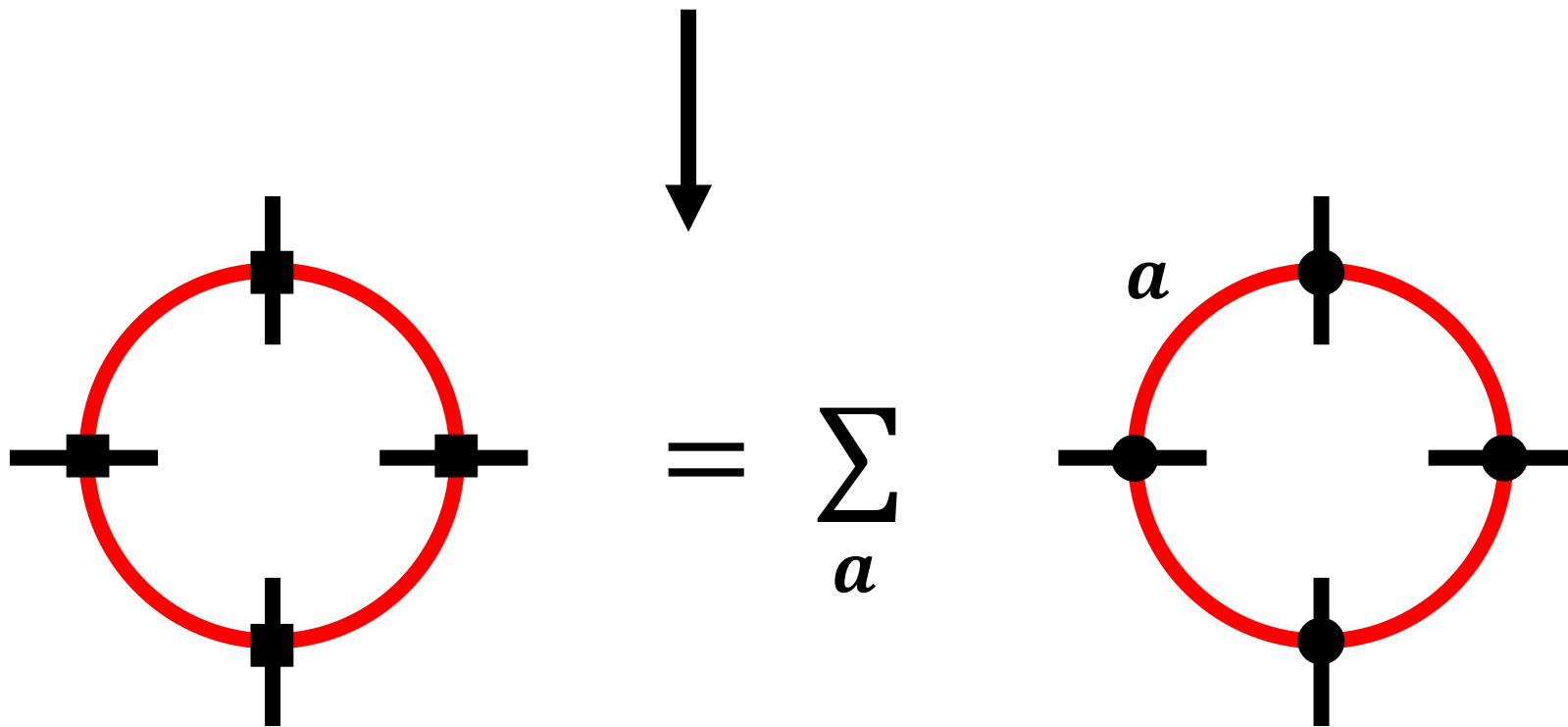
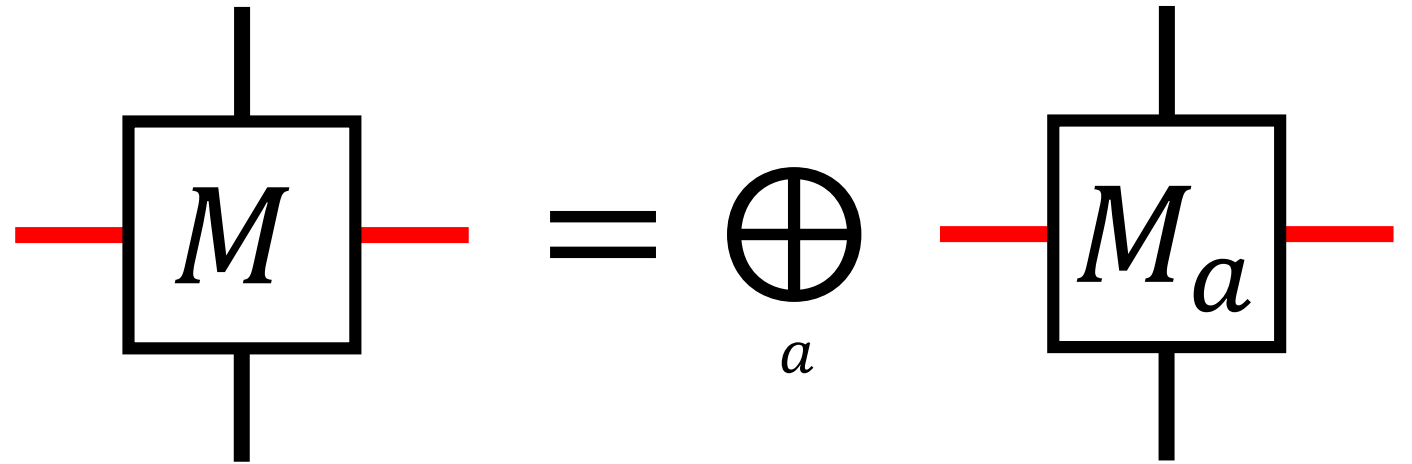


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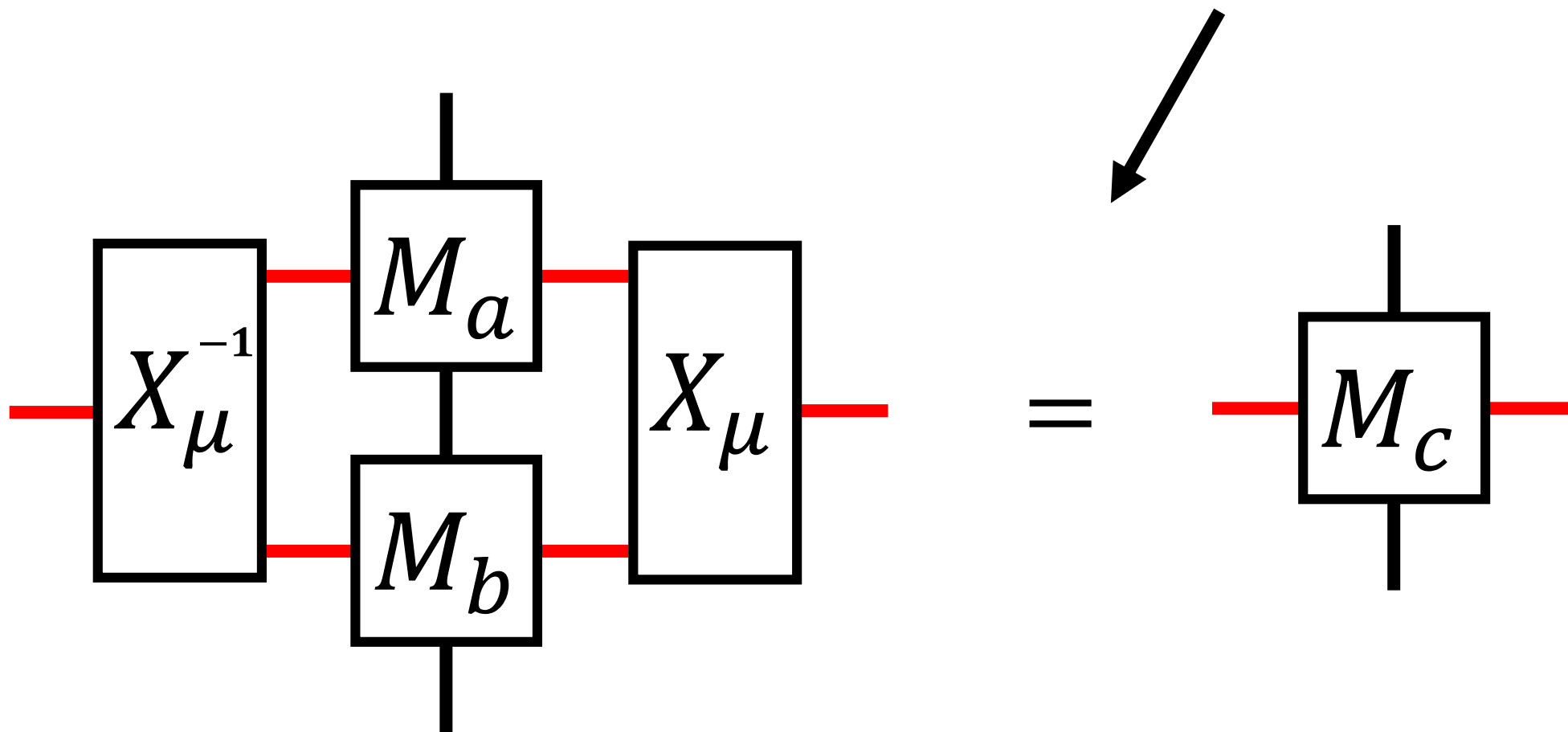


# Matrix Product Operator Algebras

Normal form:

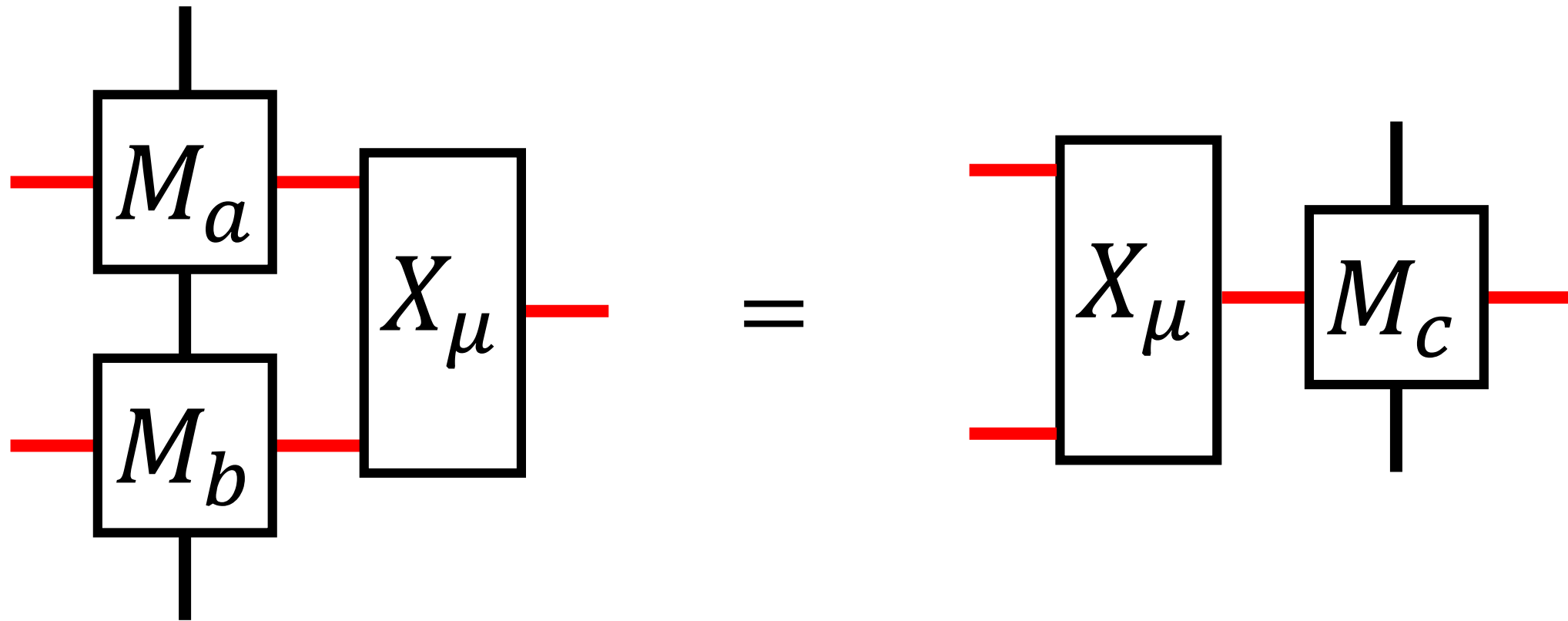


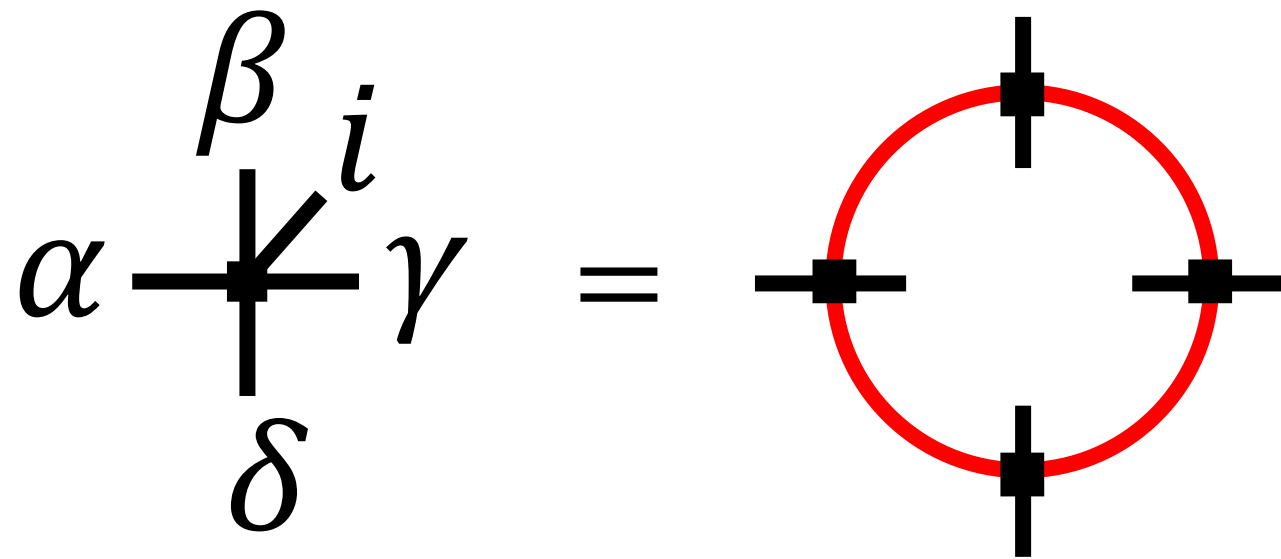
$$M(L)^2 = M(L) \longrightarrow M_a M_b = \sum_c N_{ab}^c M_c$$



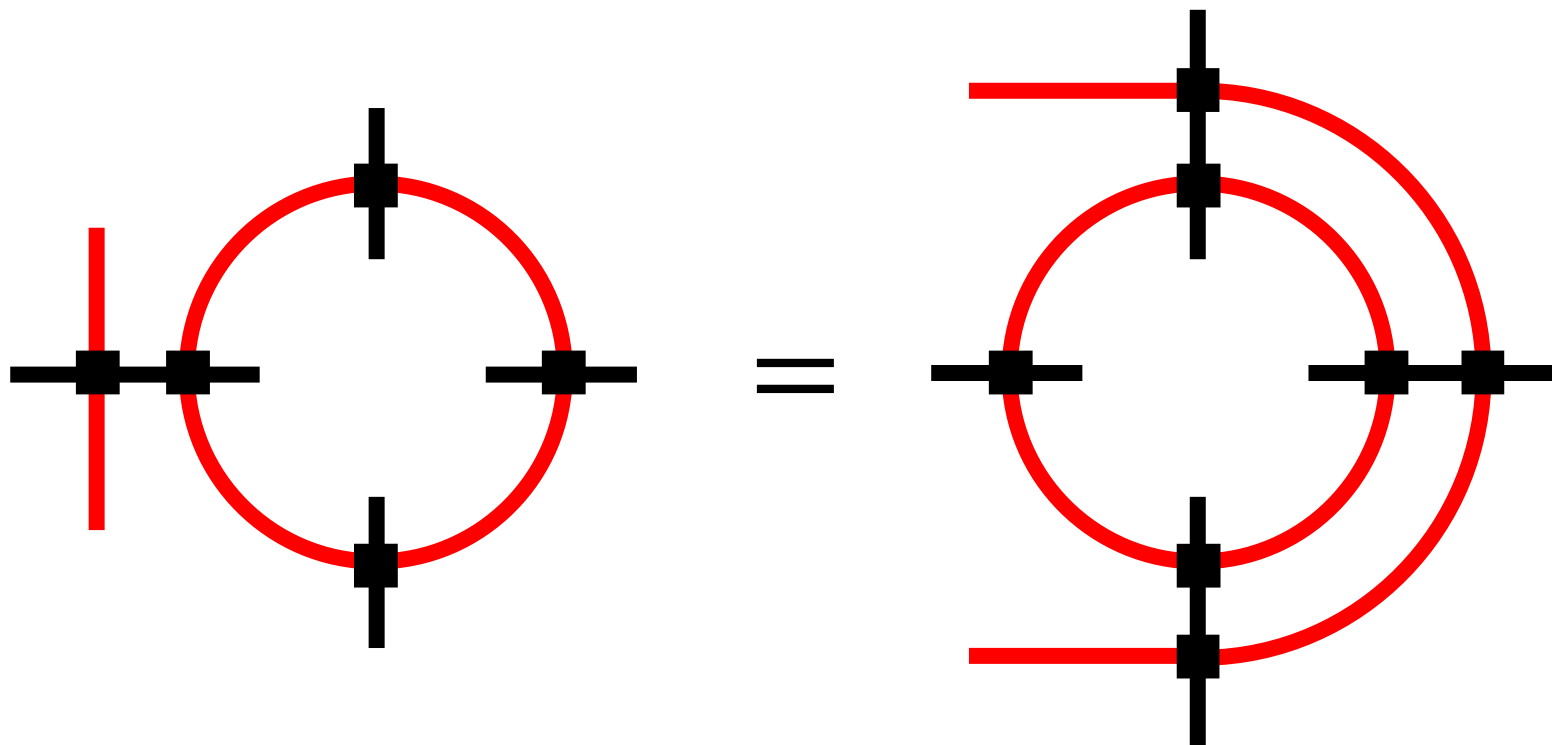


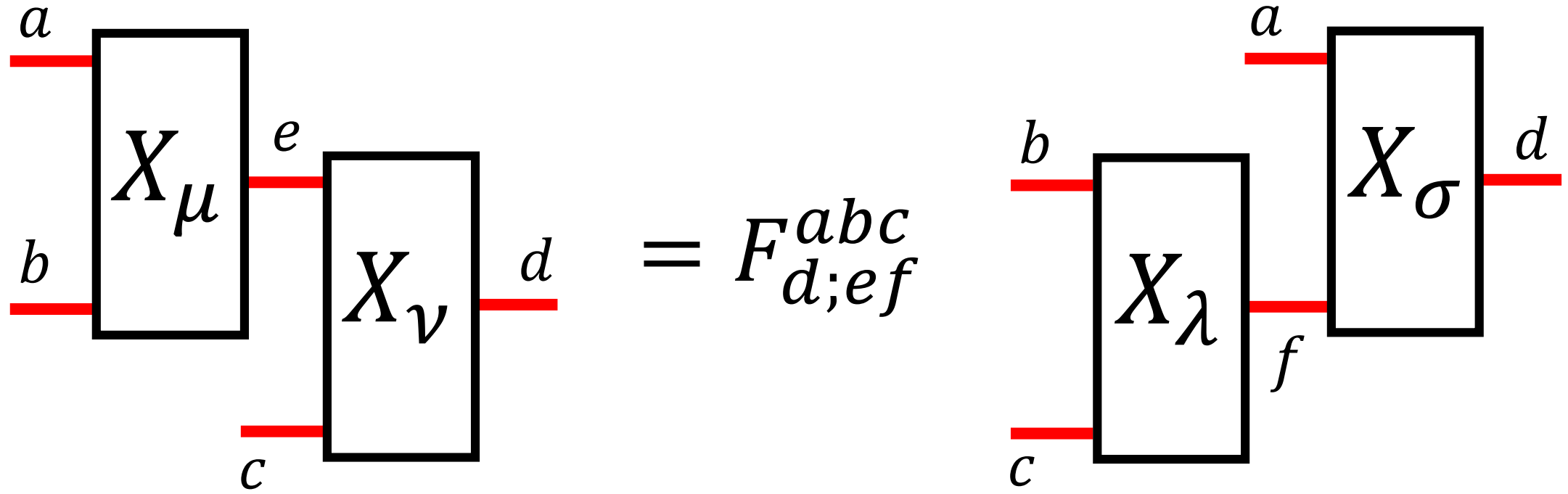
# The Zipper condition





Zipper  $\Rightarrow$



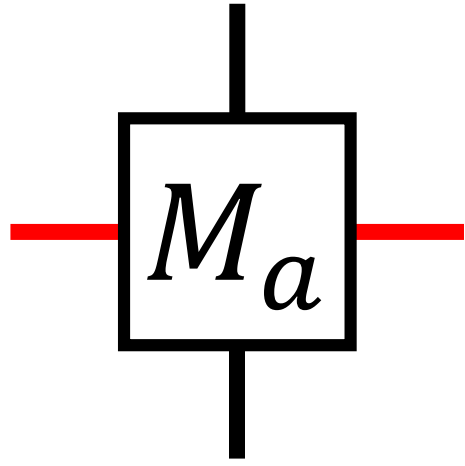


$$\sum_{de\mu\nu} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\mu \\ X_\nu \\ \bullet \\ X_\nu^+ \\ X_\mu^+ \end{array} \begin{array}{c} e \\ d \\ e \end{array} \begin{array}{c} a \\ b \\ c \end{array} = \sum_{df\sigma\lambda} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\lambda \\ X_\sigma \\ \bullet \\ X_\sigma^+ \\ X_\lambda^+ \end{array} \begin{array}{c} f \\ d \\ f \end{array} \begin{array}{c} a \\ b \\ c \end{array}$$

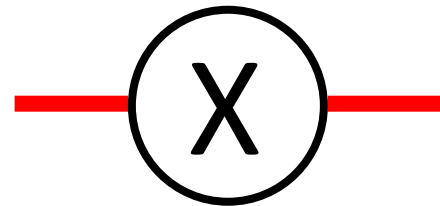
$$\begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\mu \\ X_\nu \\ \bullet \end{array} \begin{array}{c} e \\ d \end{array} = \sum_{d'f\sigma\lambda} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\lambda \\ X_\sigma \\ \bullet \\ X_\sigma^+ \\ X_\lambda^+ \\ X_\mu \\ X_\nu \end{array} \begin{array}{c} f \\ d' \\ f \\ e \\ d \end{array}$$

$$\begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\mu \\ X_\nu \end{array} \begin{array}{c} e \\ d \end{array} \otimes d = \sum_{f\sigma\lambda} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\lambda \\ X_\sigma \\ \otimes \\ X_\sigma^+ \\ X_\lambda^+ \\ X_\mu \\ X_\nu \end{array} \begin{array}{c} f \\ d \\ f \\ e \\ d \end{array}$$

# Fusion category $\mathcal{C}$



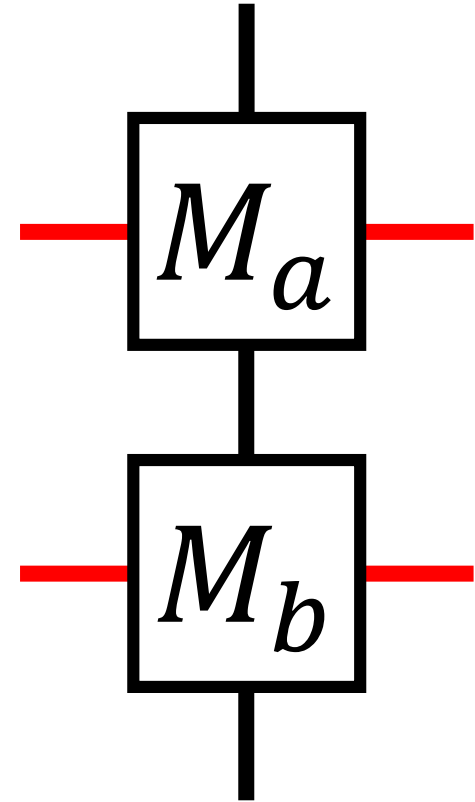
Objects



morphisms

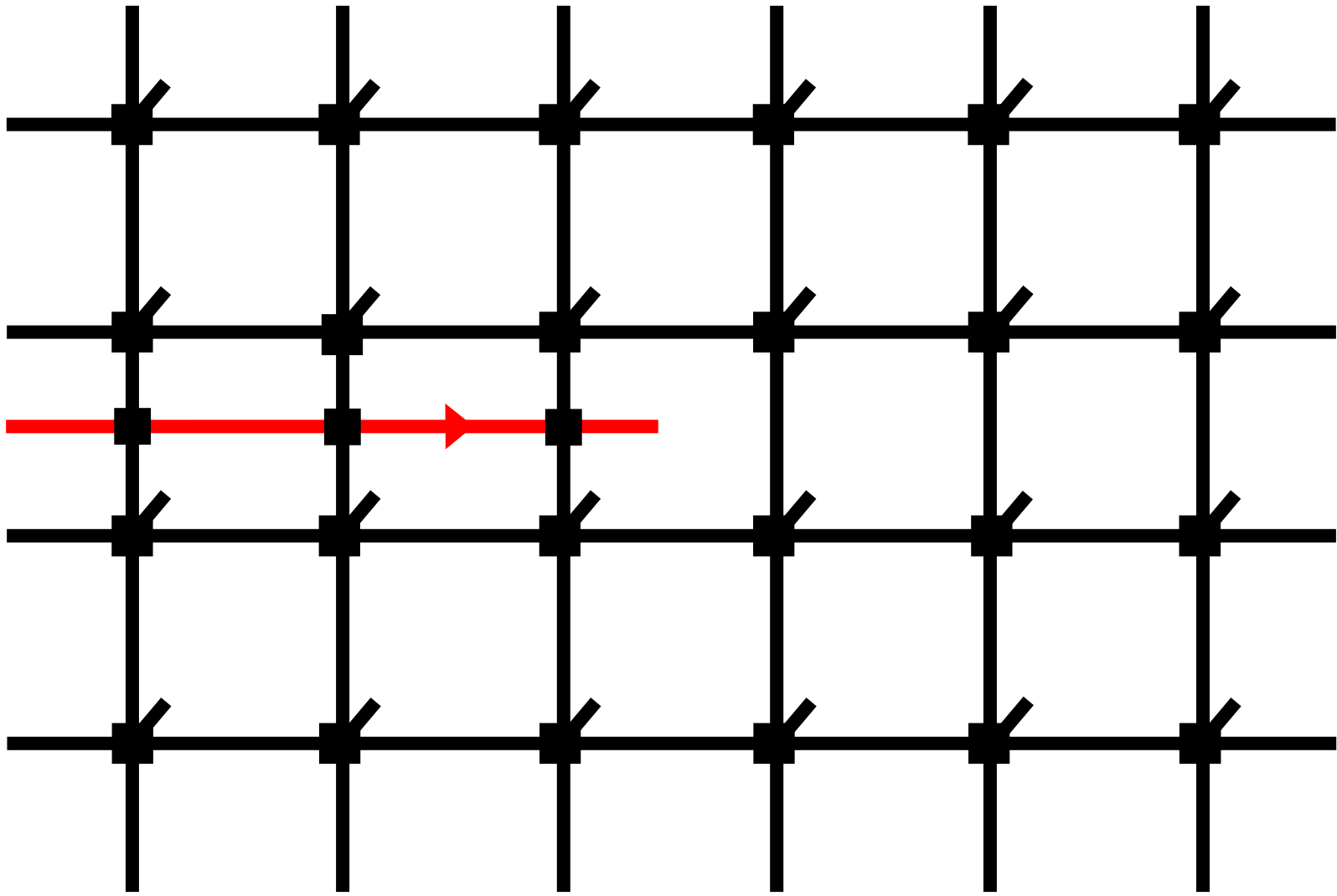
Tensor product

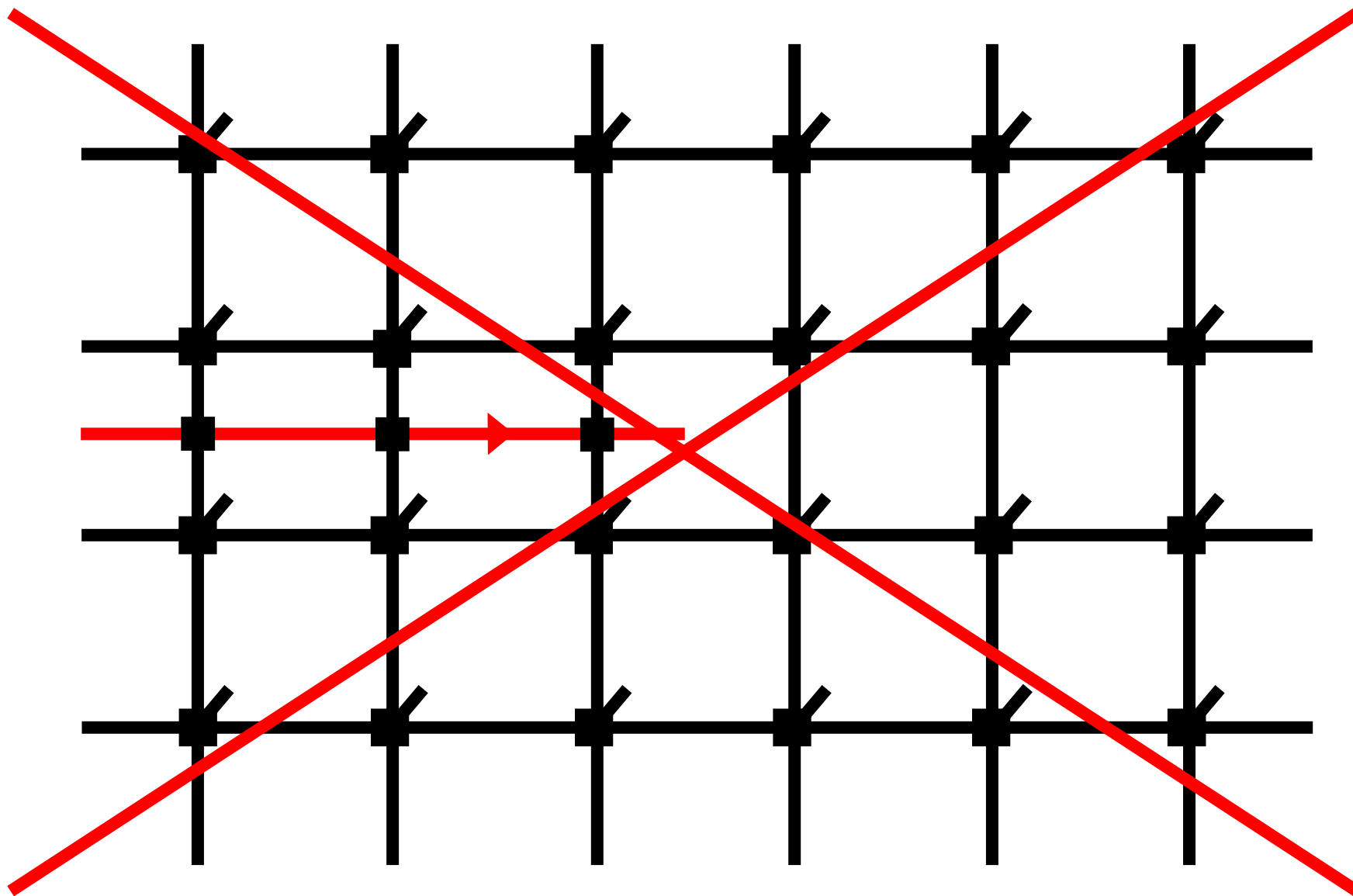
$$a \otimes b$$



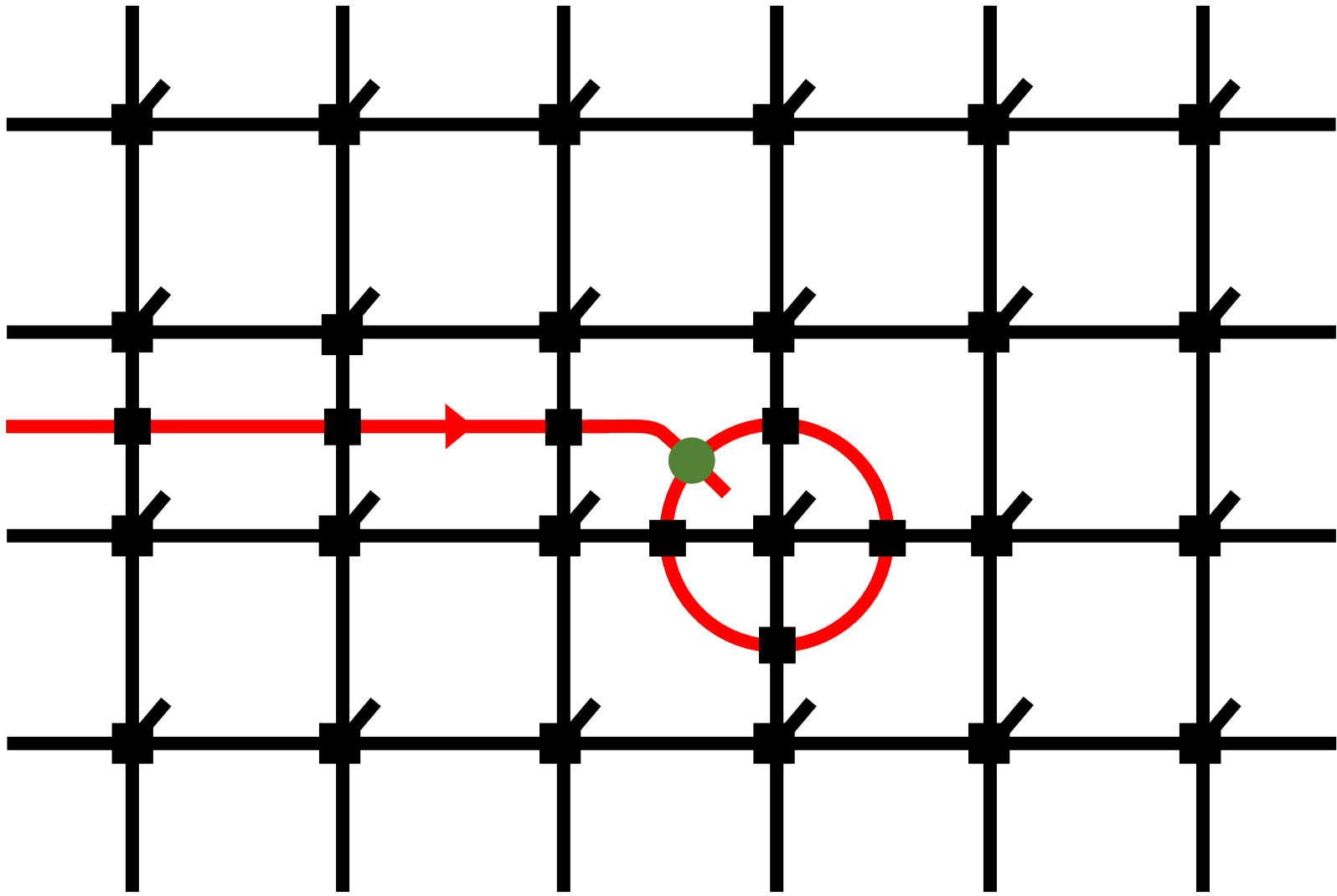


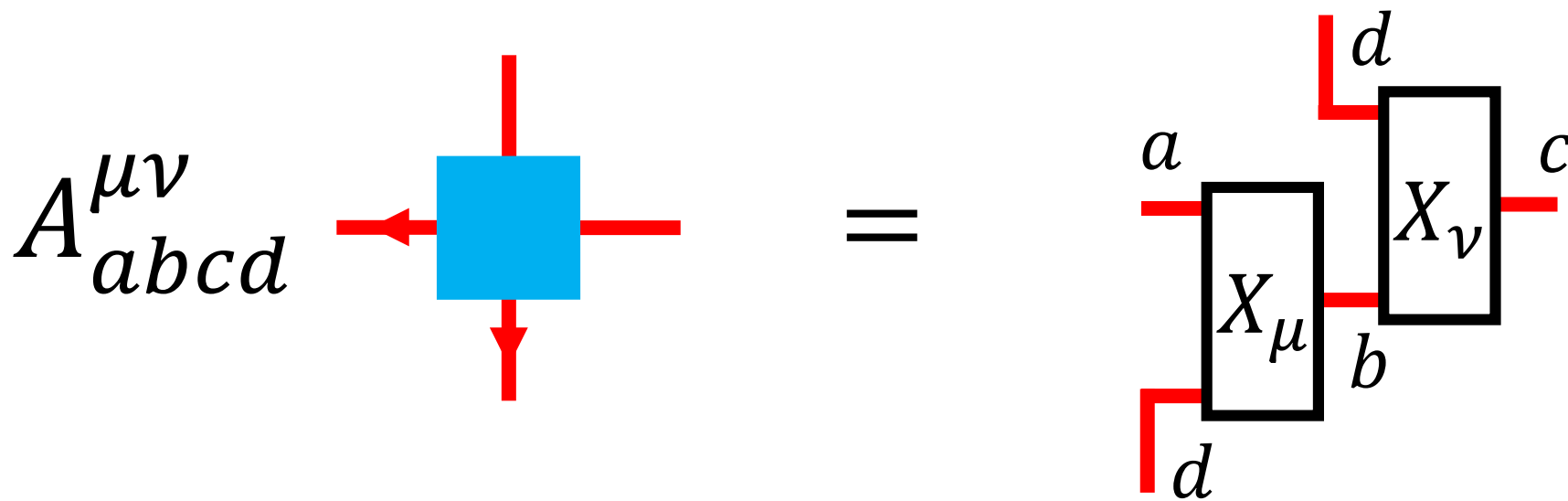
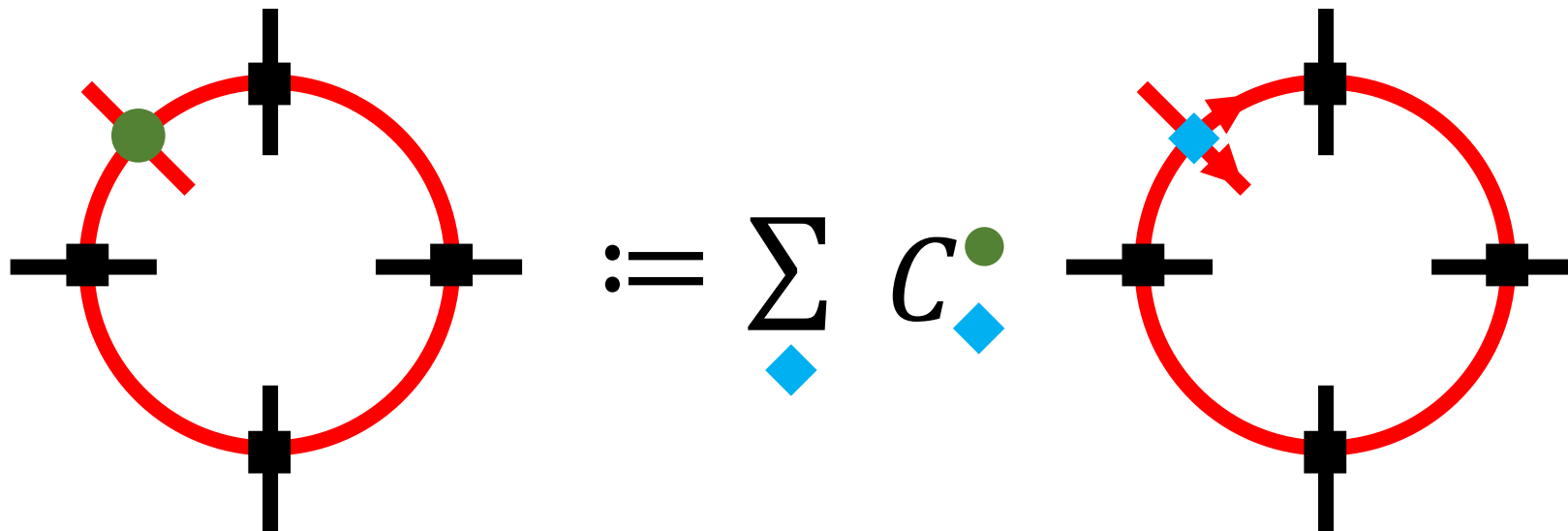
How to construct  
an anyon?

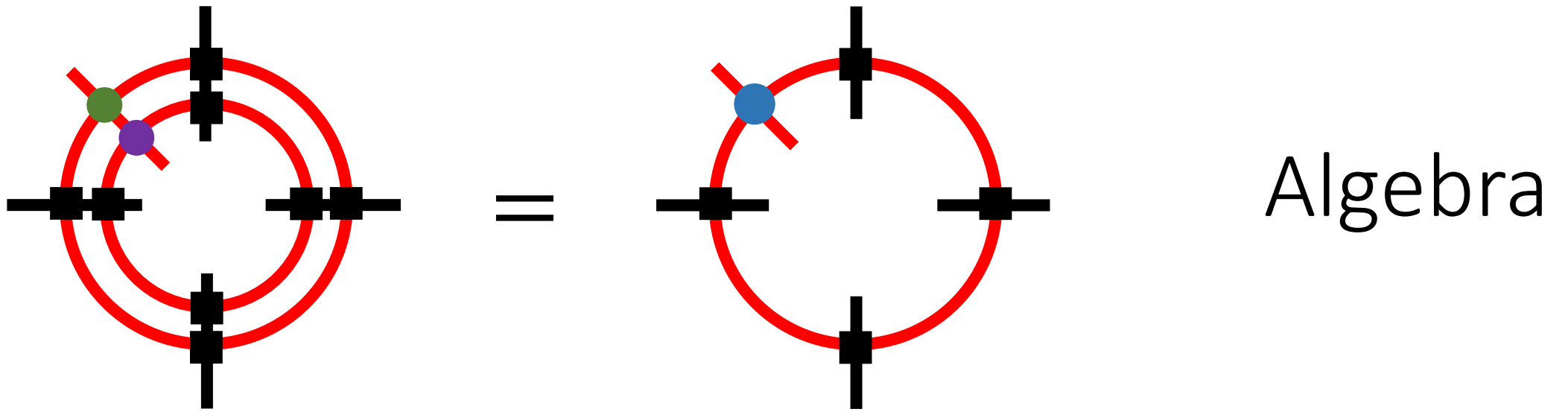
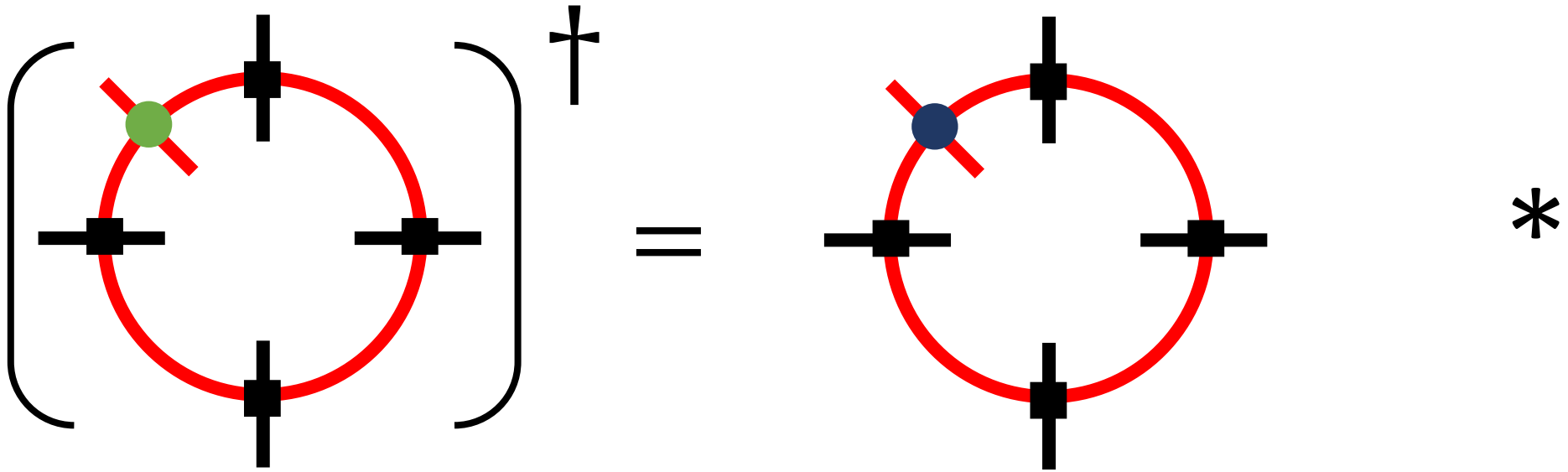


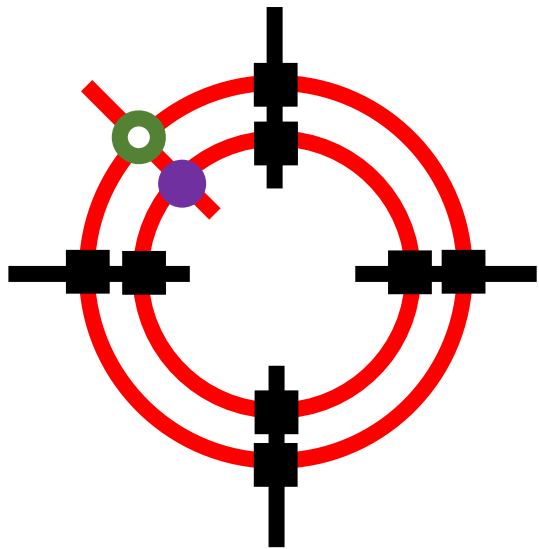




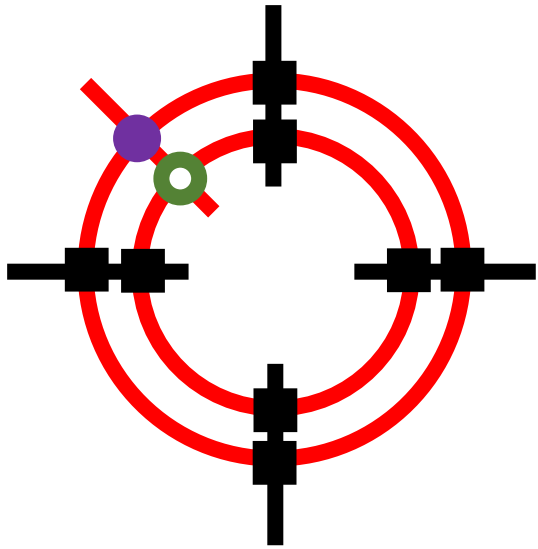




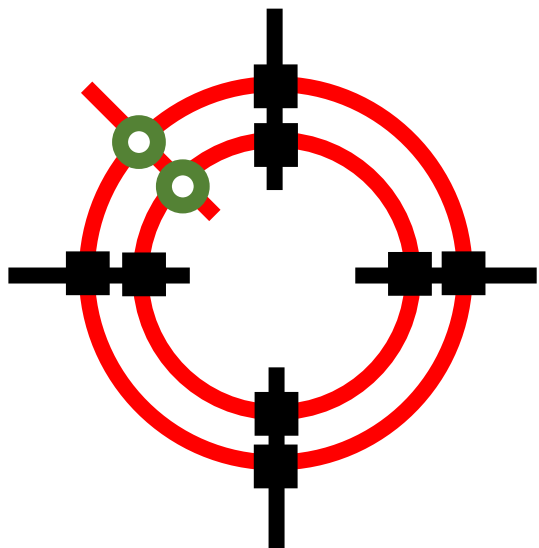




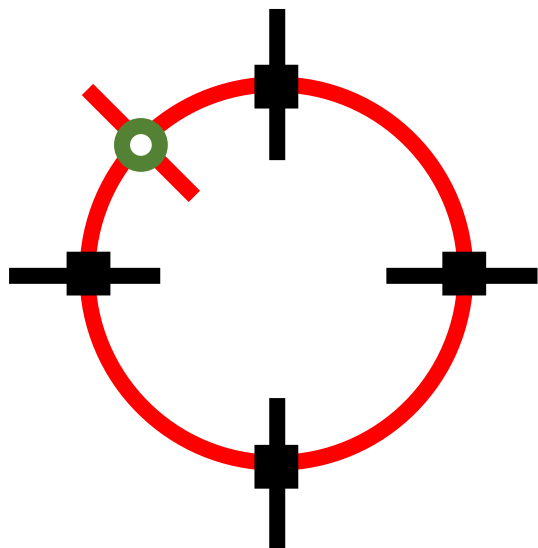
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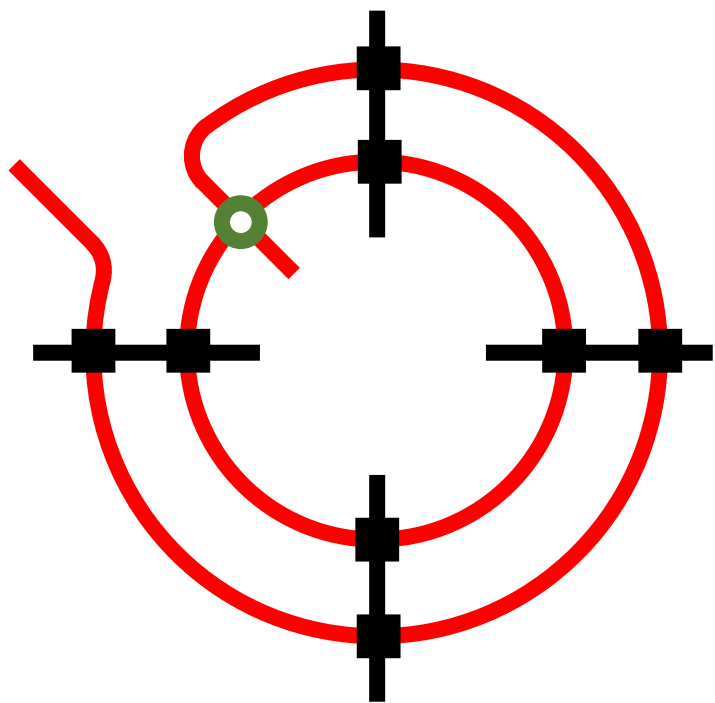
Central



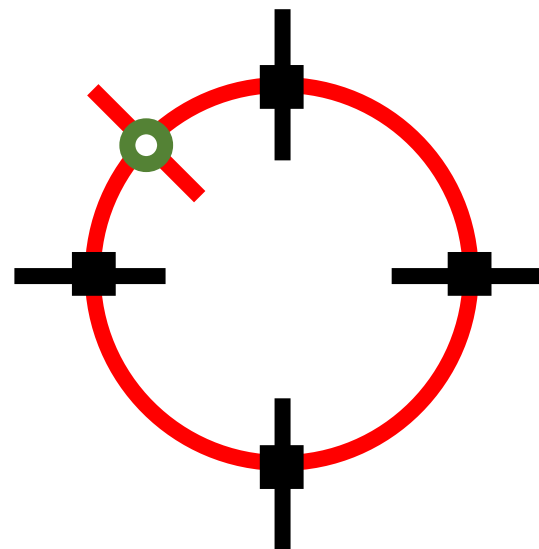
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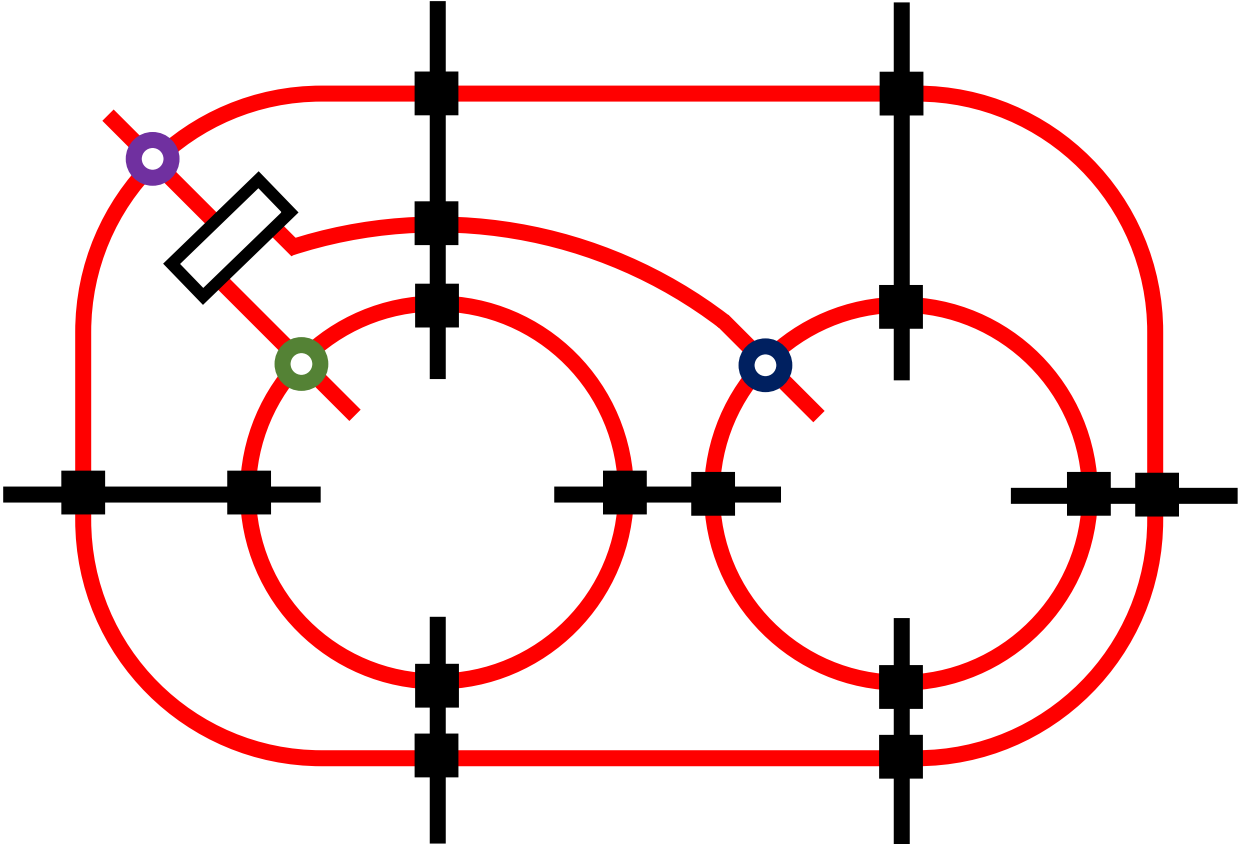


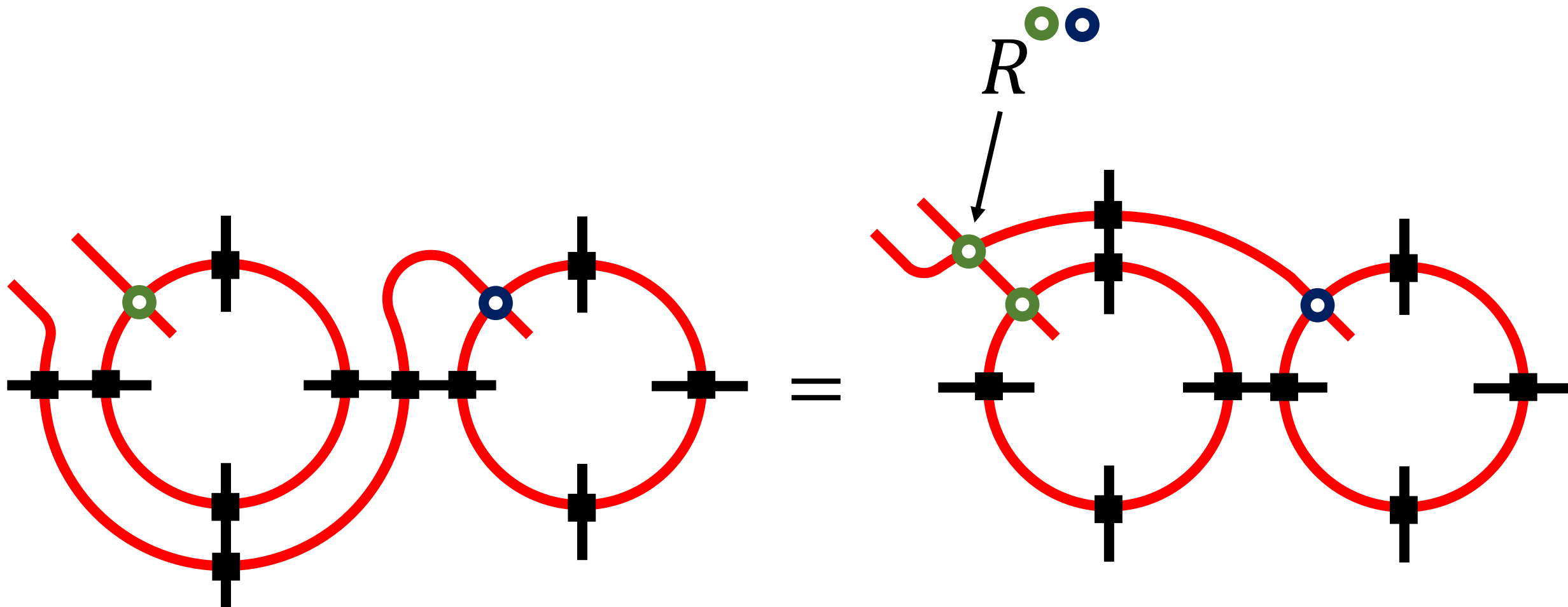
Idempotent

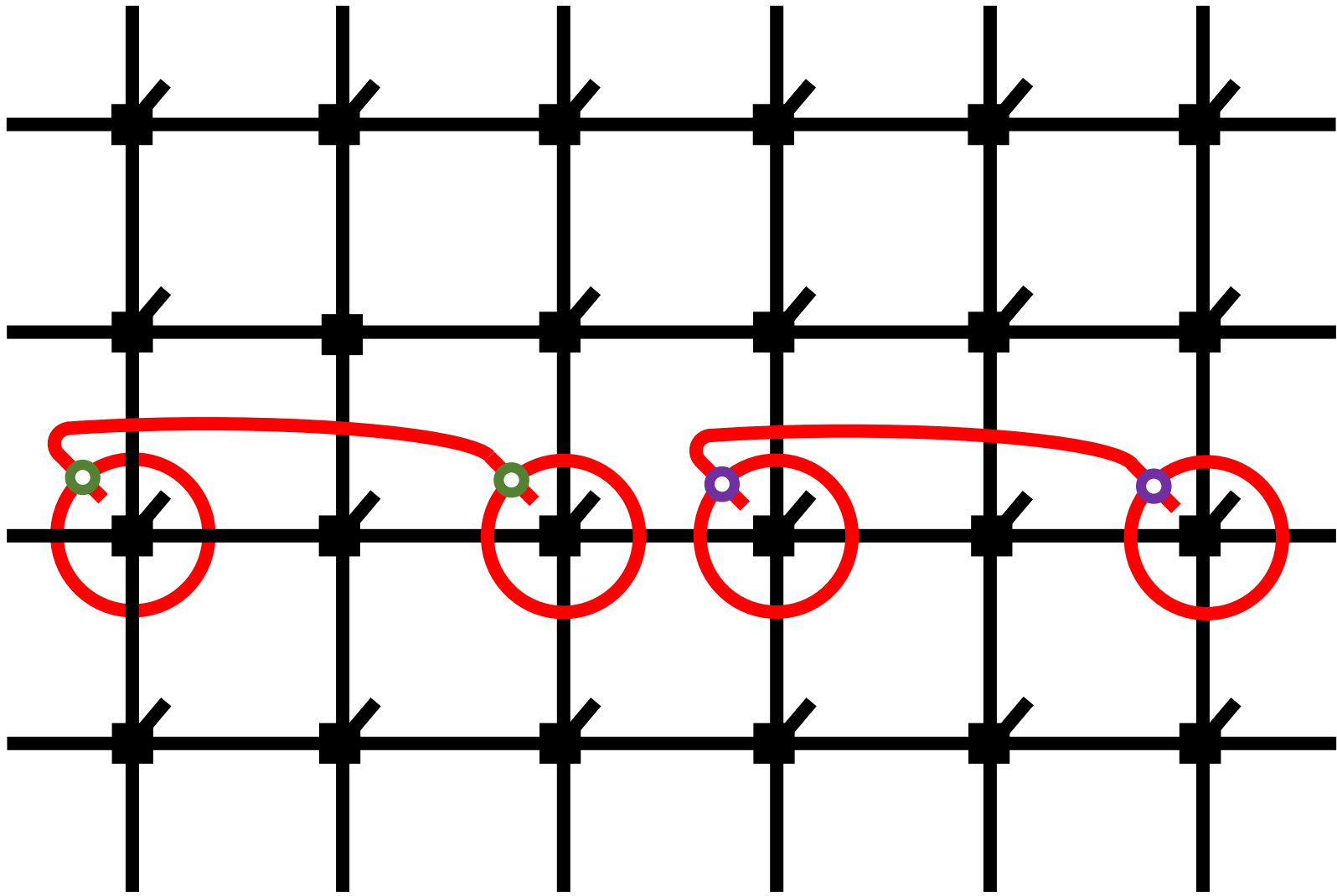


$$= e^{i2\pi h_i}$$

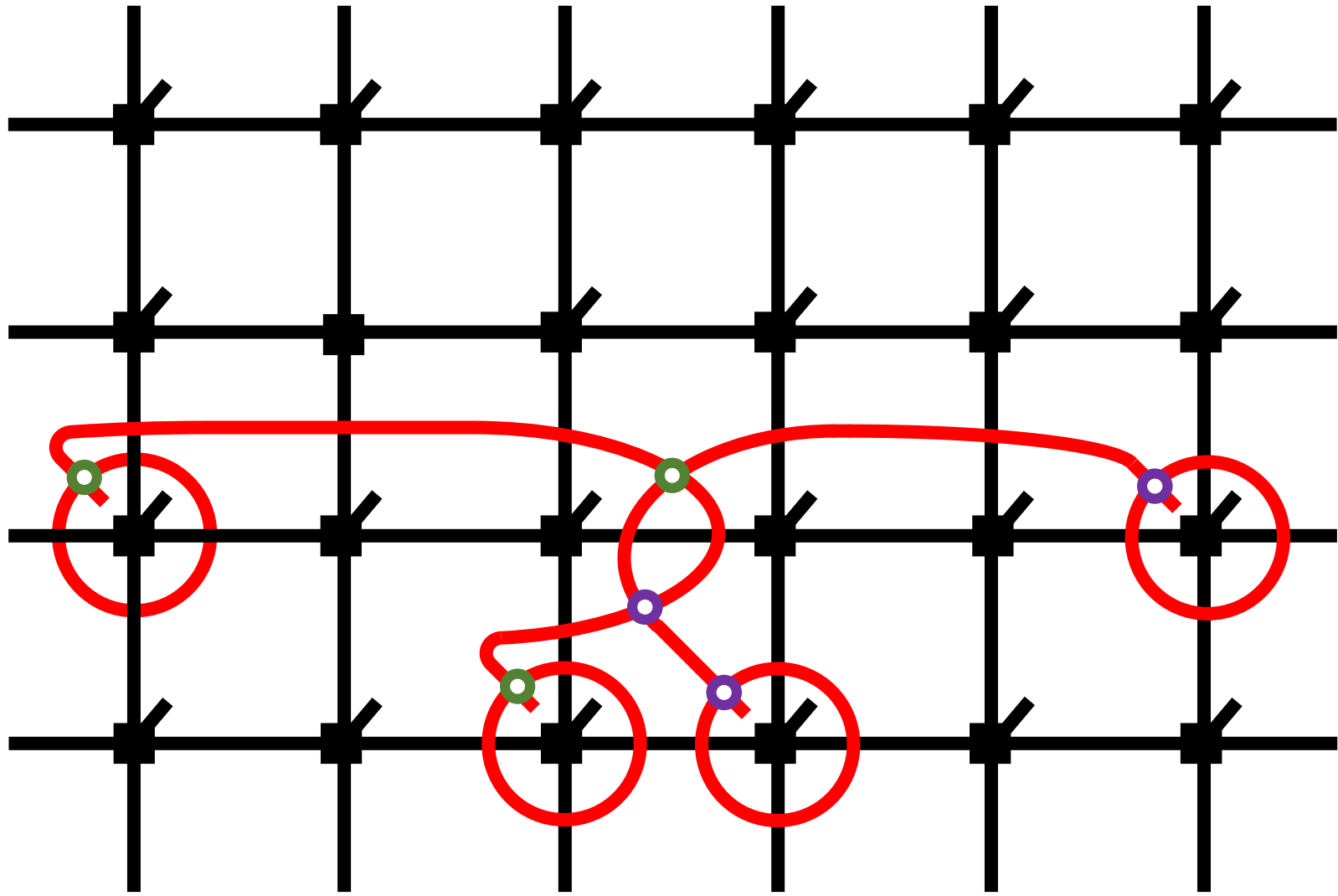


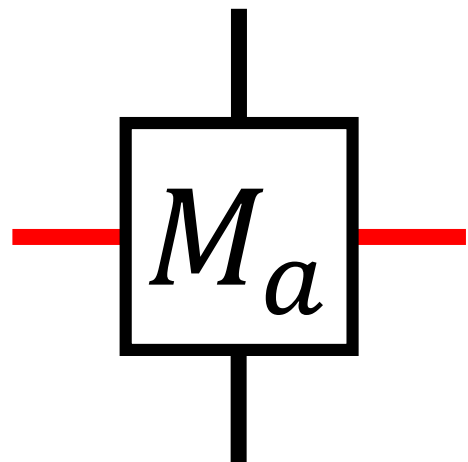




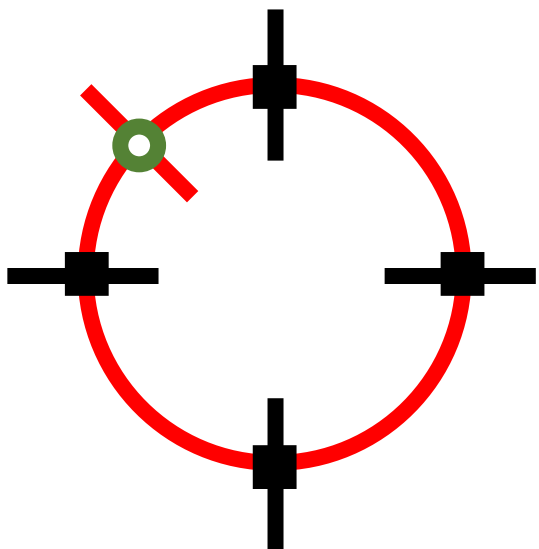






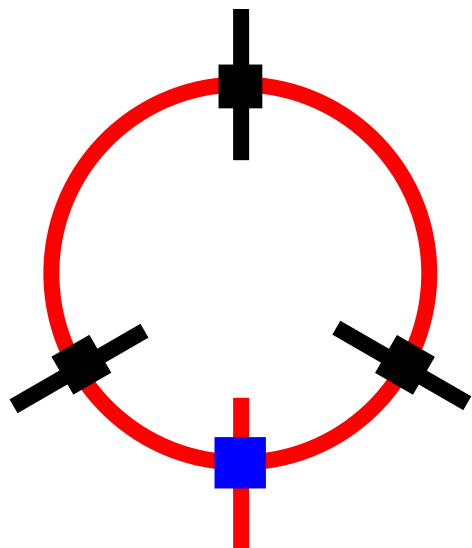


:  $\mathcal{C}$



:  $Z(\mathcal{C})$

# Toric Code



$$0 \text{ --- } \blacksquare \text{ --- } 0 = I$$

$$1 \text{ --- } \blacksquare \text{ --- } 1 = X$$

1	$0 \begin{array}{c} 0 \\   \\ \blacksquare \\   \\ 0 \end{array} 0 = 1$	$1 \begin{array}{c} 0 \\   \\ \blacksquare \\   \\ 0 \end{array} 1 = 1$
e	$0 \begin{array}{c} 0 \\   \\ \blacksquare \\   \\ 0 \end{array} 0 = 1$	$1 \begin{array}{c} 0 \\   \\ \blacksquare \\   \\ 0 \end{array} 1 = -1$
m	$0 \begin{array}{c} 1 \\   \\ \blacksquare \\   \\ 1 \end{array} 0 = 1$	$1 \begin{array}{c} 1 \\   \\ \blacksquare \\   \\ 1 \end{array} 1 = 1$
em	$0 \begin{array}{c} 1 \\   \\ \blacksquare \\   \\ 1 \end{array} 0 = 1$	$1 \begin{array}{c} 1 \\   \\ \blacksquare \\   \\ 1 \end{array} 1 = -1$

# Double Fibonacci

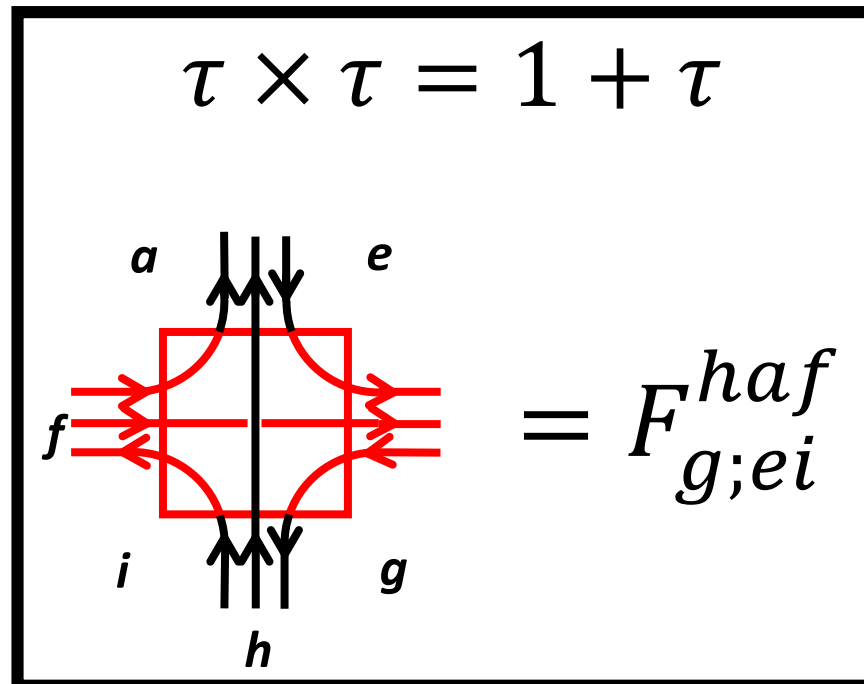
$$A_{11111}, A_{\tau\tau\tau1}, A_{1\tau1\tau}, A_{1\tau\tau\tau}, A_{\tau1\tau\tau}, A_{\tau\tau1\tau}, A_{\tau\tau\tau\tau}$$

$$(1, 1) = \frac{1}{\sqrt{5}} \left( \frac{1}{\phi} A_{11111} + \sqrt{\phi} A_{1\tau1\tau} \right)$$

$$(1, \bar{\tau}) = \frac{1}{\sqrt{5}} \left( \frac{1}{\phi} A_{\tau\tau\tau1} + \frac{1}{\sqrt{\phi}} e^{-\frac{4\pi i}{5}} A_{\tau1\tau\tau} + e^{\frac{3\pi i}{5}} A_{\tau\tau\tau\tau} \right)$$

$$(\tau, 1) = \frac{1}{\sqrt{5}} \left( \frac{1}{\phi} A_{\tau\tau\tau1} + \frac{1}{\sqrt{\phi}} e^{\frac{4\pi i}{5}} A_{\tau1\tau\tau} + e^{-\frac{3\pi i}{5}} A_{\tau\tau\tau\tau} \right)$$

$$(\tau, \bar{\tau}) = \frac{1}{\sqrt{5}} \left( \phi A_{11111} + A_{\tau\tau\tau1} - \sqrt{\phi} A_{1\tau1\tau} + \sqrt{\phi} A_{\tau1\tau\tau} + \frac{1}{\phi} A_{\tau\tau\tau\tau} \right)$$



$$h_{(1, \bar{\tau})} = -4/5$$

$$h_{(\tau, 1)} = 4/5$$

# Future Directions

- Fermions
- SET & transversal gates
- Topological Quantum Computation
- Domain Walls
- $\geq 3$  Dimensions

- Tensor networks with topological order are not generic
- By studying their entanglement structure we extract a fusion algebra
- From this we can construct a tube algebra to extract (all) information about the topological order in the system