Fault-tolerant error correction for non-abelian anyons

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QIP
January 18, 2017

1arXiv:1607.02159
Outline

1. Non-abelian anyons and quantum information
2. Error correction for abelian anyons
3. Error correction for non-abelian anyons
What are anyons\textsuperscript{1}?

- Localized gapped excitations living on a 2-dimensional surface

\textsuperscript{1}A. Kitaev, Annals Phys. \textbf{321}, 2-111 (2006)
What are anyons$^1$?

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- We can imagine bringing 2 excitation together (\(a\) and \(b\)), and ask what is their total charge \(c\).
- The possible outcomes are given by the fusion rules:

\[
 a \times b = \sum_c N_{ab}^c c
\]

Abelian vs non-abelian anyons

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- A Hilbert space is associated to each fusion/splitting process.

- Fusing two anyons \( a_1 \) and \( a_2 \) collapses the wavefunction into a definite super-selection sector, with probability given by Born’s rule:
  \[
P(c) = \langle \psi | \Pi_c^{a_1 a_2} | \psi \rangle.
  \] \hspace{1cm} (1)
Quantum computation with non-abelian anyons\textsuperscript{1}

Thermal processes can corrupt the information\textsuperscript{1}

- At $T > 0$, thermal excitations are present in finite density.
- Thermal excitations can diffuse at no energy cost.
- It really is a scalability issue: for large systems, such processes are bound to happen.

\textsuperscript{1}F. L. Pedrocchi \textit{et al.}, arXiv:1505.03712
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Fault-tolerant error correction for topologically ordered systems giving rise to abelian anyons have been studied extensively.\(^1\)

\(^1\)Dennis et al., J. Math. Phys. 43, 4452 (2002)
Anyons and topological order

Anyons appear as excitations in topologically ordered systems\(^1\). The ground space is degenerate and quantum information can be encoded in such states.

- Logical operations consist of creating a pair of excitations, performing non-trivial loop, and fuse the excitations back to the vacuum.
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- A decoding algorithm is used to find a correction procedure.
- The correction operations are performed.
Various families of decoding algorithms

- Perfect matching
- Mapping to statistical physics problems
- Clustering methods
- Cellular automaton
- Renormalization methods

\[1\] G. Duclos-Cianci et al., PRL 104, 050504 (2010)
Cellular Automata

- Classical device acting on a small neighborhood
- Apply predetermined local operations depending on the state of the sites in the neighborhood
- Can communicate with neighboring automata
- Can have a memory and instruction of a programs
Emerging structure of the noise\textsuperscript{1}

- Each actual error is characterized by a level \( n \).
- If fits in a box of size \( Q^n \times Q^n \times U^n \) and is separated by at least \( aQ^n \) sites (\( bU^n \) time steps) from other actual errors.
- The notion of \textit{actual error} is recursively defined over the level.

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The rate of appearance of a level-$n$ actual errors goes as $\epsilon_n \sim e^{-2^n}$
The idea behind Harrington’s algorithm

- Cellular automata periodically measure topological charges.

[Diagram showing cellular automata and topological charges]
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- If 2 excitations are close, they will be fused together.
- If an excitation is isolated, it is displaced to the colony center.
The need for renormalization

- An error chain extending over 2 or more colonies cannot get corrected using such simple local rules.
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- Colonies are periodically grouped into renormalized colonies.
- Renormalized transition rules are periodically applied.
Existence of a threshold

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- A level-$n$ actual errors gets corrected by the $n^{th}$ level transition rules.
- Actual errors stay well-separated from each other in time.

The properties above combined with the fact that $\epsilon_n \sim e^{-2n}$ leads to the existence of a threshold.
Complications for non-abelian anyons: probabilistic evolution

The fusion channel of 2 or more anyons is in general not deterministic:

\[ = \alpha + \beta \]

We introduce the notion of a \textit{trajectory domain} of an error. It roughly corresponds to the set of sites having a probability of becoming charged because of a given error.
Complications for non-abelian anyons: renormalized charge

The total charge present in a colony becomes path-dependent and subject to rapid fluctuations.

The notion of *renormalized charge* needs to be carefully defined, and must include the interactions of the errors with the transition rules.
Complications for non-abelian anyons: interactions between renormalization levels

The hierarchic classification of errors does not capture the 'topological interaction' between anyons caused by different actual errors.

We introduce the notion of *causally-linked clusters* of errors, sets of actual errors which can potentially interact with each others through the application of transition rules.
Key properties for non-cyclic anyons

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- Renormalized transition rules are always successful after being applied a constant number of times.

Non-cyclic anyons are anyons such that for any sequence of labels \(\{x_0, x_1, \ldots, x_n\}\) such that \(x_0 = x_n\) (and not the vacuum), then

\[
\prod_{i=0}^{n} N_{x_i \bar{x}_i}^{x_i + 1} = 0.
\]
A threshold for non-cyclic anyons

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Threshold theorem

If $A$ is non-cyclic, there exists a critical value $p_c > 0$ such that if $p + q < p_c$, for any number of time steps $T$ and any $\epsilon > 0$, there exists a linear system size $L = Q^n \in O(\log \frac{1}{\epsilon})$ such that with probability of at least $1 - \epsilon$, the encoded quantum state can in principle be recovered after $T$ time steps.
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**Threshold theorem**

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The theorem provides an upper bound on the numerical value of \( p_c < 2,7 \times 10^{-20} \times (3D + 1)^{-4} \).
We performed numerical simulations for Ising anyons. They suggest a threshold in the range of $10^{-4} \sim 10^{-3}$. 

![](image.png)
Future directions

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- How do we modify the algorithm to the case where we have computational anyons?
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- How do we modify the algorithm to the case where we have computational anyons?
- How about braiding in a fault-tolerant manner?
Thank you for your attention!