Threshold theorem for quantum supremacy

arXiv:1610.03632

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Outline

• Motivations

• Hardness proof by postselection

• Threshold theorem for quantum supremacy

• Applications: 3D topological cluster computation & 2D surface code

• Summary
How can we best achieve quantum supremacy with the relatively small systems that may be experimentally accessible fairly soon, systems with of order 100 qubits?

and Talk by S. Boixo et al
Intermediate models for non-universal quantum computation
Intermediate models for non-universal quantum computation

**Boson Sampling**  
*Aaronson-Arkhipov ‘13*

<table>
<thead>
<tr>
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<th>Universal linear optics</th>
<th>Linear optical quantum computation</th>
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**Experimental demonstrations**  
Intermediate models for non-universal quantum computation

**Boson Sampling**
Aarson-Arkhipov ‘13

**Experimental demonstrations**

**Linear optical quantum computation**

**IQP**
(commuting circuits)
Bremner-Jozsa-Shepherd ‘11

**Ising type interaction**

KF-Morimae ‘13
Bremner-Montanaro-Shepherd ‘15
Gao-Wang-Duan ‘15
Farhi-Harrow ‘16
Intermediate models for non-universal quantum computation

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**Experimental demonstrations**


- Ising type interaction
  - KF-Morimae ‘13
  - Bremner-Montanaro-Shepherd ‘15
  - Gao-Wang-Duan ‘15
  - Farhi-Harrow ‘16

- Bremner-Jozsa-Shepherd ‘11

- $|0\rangle \rightarrow \frac{1}{2^n} U$
Intermediate models for non-universal quantum computation

Boson Sampling  
*Aaronson-Arkhipov ‘13*

**Experimental demonstrations**

**IQP (commuting circuits)**  
*Bremner-Jozsa-Shepherd ‘11*

Ising type interaction

**DQC1 (one-clean qubit model)**  
*Knill-Laflamme ‘98*
*Morimae-KF-Fitzsimons ‘14*
*KF et al., ‘16*

The purpose of this study:
→ universal but (very) noisy quantum circuits
Noisy quantum circuits approaching fault-tolerance threshold
Noisy quantum circuits approaching fault-tolerance threshold

**IBM:**
Chow et al., Nat. Comm. 5 4015 (2015)
C´orcoles et al., Nat. Comm. 6, 6979 (2015)
Gambetta et al., npj quant. info. 3, 2 (2017)
Noisy quantum circuits approaching fault-tolerance threshold

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Delft (QuTech):
Riste et al., Nat. Comm. 6 6983 (2015)

UCSB(Martinis)+Google:
Barends et al., Nature 508, 500 (2014)

[fidelities]
single-qubit gate: 99.92%
two-qubit gate: 99.4%
measurement: 99%
Noisy quantum circuits above standard noise threshold

**Threshold theorem:** if the noise strength is smaller than a certain constant threshold value, quantum computation can be performed with an arbitrary accuracy.
Noisy quantum circuits above standard noise threshold

*Threshold theorem*: if the noise strength is smaller than a certain constant threshold value, quantum computation can be performed with an arbitrary accuracy.

- Phenomenological noise: 2.9-3.3%
- Circuit-based noise: 0.75%

---

Topological fault-tolerance in cluster state quantum computation

R Raussendorf, J Harrington and K Goyal
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- Phenomenological noise: 14.6%

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Topological fault-tolerance in cluster state quantum computation

R Raussendorf, J Harrington and K Goyal

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• Motivations

• Hardness proof by postselection

• Threshold theorem for quantum supremacy

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• Summary
Postselected computation
= solving a decision problem by using conditional probability distribution.

while(y==1)

\[ p(x|y = 1) = \frac{p(x, y)}{p(y = 1)} \]

yes: \[ p(x = 1|y = 1) \geq \frac{2}{3} \]

no: \[ p(x = 0|y = 1) \geq \frac{2}{3} \]

Hardness proof via \text{postBQP} = \text{PP}

A (fictitious) ability to \textit{postselect} a possibly exponentially rare events allows us to distinguish quantum and classical tasks!

| classical+ postselection | quantum+ postselection |

Hardness proof via postBQP = PP

A (fictitious) ability to postselect a possibly exponentially rare events allows us to distinguish quantum and classical tasks!

\[
\text{classical+ postselection} \quad \text{quantum+ postselection}
\]

Hardness proof via \( \text{postBQP} = \text{PP} \)

A (fictitious) ability to **postselect** a possibly exponentially rare events allows us to distinguish quantum and classical tasks!

\[
\begin{align*}
\text{classical+ postselection} & \quad \text{postBPP} \\
\text{quantum+ postselection} & \quad \text{PP=\text{postBQP}} \\
& \quad \text{[Aaronson05]} 
\end{align*}
\]

Hardness proof via postBQP = PP

A (fictitious) ability to postselect a possibly exponentially rare events allows us to distinguish quantum and classical tasks!

\[ \text{classical+ postselection} \neq \text{quantum+ postselection} \]

unless the PH collapses to the 3rd level.

Hardness proof via postBQP = PP

A (fictitious) ability to postselect a possibly exponentially rare events allows us to distinguish quantum and classical tasks!

classical+ postselection ≠ quantum+ postselection

postBPP

unless the PH collapses to the 3rd level.

If your system is potentially as powerful as postBQP under postselection, then its classical simulation is hard unless the PH collapses!

Difficulty of quantum supremacy with noisy sampling

- multiplicative error (or exponentially small additive error)

\[ \frac{1}{c} p_{\text{ideal}}(x) < p_{\text{samp}}(x) < cp_{\text{ideal}}(x) \quad (c > 1) \]

[Bremner-Jozsa-Shepherd, ’11]
Difficulty of quantum supremacy with noisy sampling

- multiplicative error (or exponentially small additive error)
  \[ \frac{1}{c} p^{\text{ideal}}(x) < p^{\text{samp}}(x) < cp^{\text{ideal}}(x) \quad (c > 1) \]
  [Bremner-Jozsa-Shepherd, ’11]

- constant additive error with $l_1$-norm
  \[ \| p^{\text{samp}}(x) - p^{\text{ideal}}(x) \|_1 = \sum_x |p^{\text{samp}}(x) - p^{\text{ideal}}(x)| < c \]
  [Aaronson-Arkhipov, ’11, Bremner-Montanaro-Shepherd ‘16]
Difficulty of quantum supremacy with noisy sampling

- multiplicative error (or exponentially small additive error)

\[
\frac{1}{c} p_{\text{ideal}}(x) < p_{\text{samp}}(x) < c p_{\text{ideal}}(x) \quad (c > 1)
\]

[Bremner-Jozsa-Shepherd, ’11]

- constant additive error with \(l_1\)-norm

\[
\|p_{\text{samp}}(x) - p_{\text{ideal}}(x)\|_1 = \sum_x |p_{\text{samp}}(x) - p_{\text{ideal}}(x)| < c
\]

[Aaronson-Arkhipov, ’11, Bremner-Montanaro-Shepherd ‘16]

Small amount of noise can easily break these conditions.
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Main idea: simulation of fault-tolerant quantum computation under postselection

\[ |0\rangle \otimes n \xrightarrow{C_w} \bar{p}(x, y) \xrightarrow{\text{fault-tolerant version with noisy circuits}} p(x, y, z) \]

\[ p(x, y, z) \]

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Main idea: simulation of fault-tolerant quantum computation under postselection

$\bar{p}(x, y)$

logical output

fault-tolerant version with noisy circuits

classical processing

$p(x, y, z)$
Main idea: simulation of fault-tolerant quantum computation under postselection
Main idea: simulation of fault-tolerant quantum computation under postselection

$$\ket{0} \otimes n \xrightarrow{C_w} \overline{p}(x, y)$$

fault-tolerant version with noisy circuits

logical output

$$p(x, y, z)$$

classical processing

error syndrome

$$p(x, y, |z = 0)$$

under the condition of null syndrome

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Threshold theorem for quantum supremacy

- Part 1: An exponentially small additive error is enough.

\[
|\bar{p}(x, y) - p(x, y | z = 0)| < e^{-\kappa}
\]

where the overhead is polynomial in \((n, \kappa)\). Then, classical simulation of \(p(x, y, z)\) with a multiplicative error \(1 < c < \sqrt{2}\) is hard.

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Threshold theorem for quantum supremacy

- Part 1: An exponentially small additive error is enough.

\[ |\bar{p}(x, y) - p(x, y|z = 0)| < e^{-\kappa} \]

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- Part 2: The exponentially small additive error

\[ |\bar{p}(x, y) - p(x, y|z = 0)| < e^{-\kappa} \]

is achievable by quantum error correction under postselection.

\[ C_w \]

\[ \bar{p}(x, y) \]

\[ p(x, y, z) \]

error syndrome
Threshold theorem for quantum supremacy

- Part 1: An exponentially small additive error is enough.

$$\left| \tilde{p}(x, y) - p(x, y|z = 0) \right| < e^{-\kappa}$$

where the overhead is polynomial in \((n, \kappa)\). Then, classical simulation of 
\(p(x, y, z)\) with a multiplicative error \(1 < c < \sqrt{2}\) is hard.

- Part 2: The exponentially small additive error

$$\left| \tilde{p}(x, y) - p(x, y|z = 0) \right| < e^{-\kappa}$$

is achievable by quantum error correction under postselection.

\text{arXiv:1610.03632}
Part 1: an exponential small additive error is enough

\[ C_w \]

\[ \bar{p}(x, y) \]

\[ \bar{p}(x | y) \]

fault-tolerant version

\[ p(x, y, z) \]

\[ p(x, y, z) \]

\[ z \rightarrow \text{error syndrome} \]

Solve a PP-complete problem (MAJSAT) using \( \bar{p}(x | y) \) as in [Aaronson05]

\[ \rightarrow \text{probability for postselection: } \bar{p}(y = 0) > 2^{-6n-4} \]
Part 1: an exponential small additive error is enough

Solve a PP-complete problem (MAJSAT) using $\bar{p}(x|y)$ as in [Aaronson05]

$$\rightarrow \text{probability for postselection: } \bar{p}(y = 0) > 2^{-6n-4}$$

Therefore, if $|\bar{p}(x, y) - p(x, y|z = 0)| < e^{-\kappa}$ with $\kappa = \text{poly}(n)$

then we have

$$|\bar{p}(x|y = 0) - p(x|y = 0, z = 0)| < 1/2$$

$$\rightarrow p(x, y, z) \text{ can solve the PP-complete problem under postselection.}$$
Threshold theorem for quantum supremacy

- Part 1: An exponentially small additive error is enough.

\[ |\bar{p}(x, y) - p(x, y | z = 0)| < e^{-\kappa} \]

where the overhead is polynomial in \((n, \kappa)\). Then, classical simulation of \(p(x, y, z)\) with a multiplicative error \(1 < c < \sqrt{2}\) is hard.

- Part 2: The exponentially small additive error

\[ |\bar{p}(x, y) - p(x, y | z = 0)| < e^{-\kappa} \]

is achievable by quantum error correction under postselection.

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Part 2: error reduction under postselection (sketch)

$$P_{x,y} = |x, y\rangle\langle x, y|$$

$$Q_z = |z\rangle\langle z|$$

Initial state

Fault-tolerant circuit including classical processing

$$U_{\text{noisy}} = \prod_k (N_k U_k)$$

Error syndrome
Part 2: error reduction under postselection (sketch)

\[
\mathcal{U}^{\text{noisy}} = \prod_k (\mathcal{N}_k \mathcal{U}_k)
\]

error syndrome

\[
P_{x,y} = |x, y\rangle\langle x, y|
\]

\[
Q_z = |z\rangle\langle z|
\]

initial state

fault-tolerant circuit including classical processing

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Part 2: error reduction under postselection (sketch)

\[ \mathcal{N}_k = (1 - \epsilon_k)I + \mathcal{E}_k \]

initial state

fault-tolerant circuit including classical processing

stochastic noise

\[ U^{\text{noisy}} = \prod_{k} (\mathcal{N}_k U_k) \]

ideal operation

\[ P_{x,y} = |x, y\rangle \langle x, y| \]

error syndrome

\[ Q_z = |z\rangle \langle z| \]
Part 2: error reduction under postselection (sketch)

\[ \mathcal{N}_k = (1 - \epsilon_k)I + \mathcal{E}_k \]

Stochastic noise

Ideal operation

CP map

Fault-tolerant circuit including classical processing

Initial state

Error syndrome

Stochastic noise

\[ U^{\text{noisy}} = \prod_k (\mathcal{N}_k U_k) \]

\[ p(x, y, z) = \text{Tr}[P_{x,y} Q_z U^{\text{noisy}}(\rho_{\text{ini}})] \]

\[ P_{x,y} = |x, y\rangle\langle x, y| \]

\[ Q_z = |z\rangle\langle z| \]

ArXiv:1610.03632
Part2: error reduction under postselection (sketch)

Using $\mathcal{N}_k = (1 - \epsilon_k)\mathcal{I} + \mathcal{E}_k$, we decompose $\mathcal{U}^{\text{noisy}}$ into

$$\mathcal{U}^{\text{noisy}}(\rho_{\text{ini}}) = \rho_{\text{sparse}} + \rho_{\text{faulty}}$$

such that $\bar{p}(x, y) \propto \text{Tr}[P_{x,y}Q_{z=0}\rho_{\text{sparse}}]$.
Part2: error reduction under postselection (sketch)

Using $\mathcal{N}_k = (1 - \epsilon_k)\mathcal{I} + \mathcal{E}_k$, we decompose $\mathcal{U}^{\text{noisy}}$ into

$$\mathcal{U}^{\text{noisy}}(\rho_{\text{ini}}) = \rho_{\text{sparse}} + \rho_{\text{faulty}}$$

such that $\bar{p}(x, y) \propto \text{Tr}[P_{x, y}Q_{z=0}\rho_{\text{sparse}}]$.

Then we can show that

$$\|\bar{p}(x, y) - p(x, y|z = 0)\|_1 < 2\|\rho_{\text{faulty}}\|_1/q_{z=0}$$

where $q_{z=0} \equiv \text{Tr}[Q_{z=0}\mathcal{U}^{\text{noisy}}(\rho_{\text{ini}})]$. (prob. of null syndrome measurement)

$$< 2 \sum_{r \geq d} C(r) \left(\frac{\epsilon}{1 - \epsilon}\right)^r \quad (\epsilon \equiv \max_k \epsilon_k)$$

arXiv:1610.03632
Part 2: error reduction under postselection (sketch)

Using $N_k = (1 - \epsilon_k)I + \mathcal{E}_k$, we decompose $U^{\text{noisy}}$ into

\[
\text{noisy} \rightarrow \text{ini} = \text{sparse} \oplus \text{faulty}
\]

such that $\bar{p}(x, y) = \text{Tr}[P_{x,y}Q_zU^{\text{noisy}}(\rho_{\text{ini}})]$

Then we can show that

\[
\bar{p}(x, y) - p(x, y | z = 0) < 2 \frac{1}{q_z=0} \frac{\| \rho_{\text{faulty}} \|_1}{q_z=0}
\]

where $q_z=0 = \text{Tr}[Q_z=0U^{\text{noisy}}(\rho_{\text{ini}})]$. (prob. of null syndrome measurement)

\[
< 2 \sum_{r \geq d} C(r) \left( \frac{\epsilon}{1 - \epsilon} \right)^r
\]

There is a constant threshold $\epsilon_{\text{th}}$ below which the output $p(x, y, z) = \text{Tr}[P_{x,y}Q_zU^{\text{noisy}}(\rho_{\text{ini}})]$ from the noisy quantum circuits cannot be simulated efficiently on a classical computer unless the PH collapses to the 3rd level.
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• Motivations

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• Summary
Topological MBQC on a 3D cluster state

- MBQC on a graph state of degree $\log(n)$
  (corresponds to commuting circuits of depth $\log(n)$)

see also KF-Tamate ‘16
Topological MBQC on a 3D cluster state

- MBQC on a graph state of degree $\log(n)$ (corresponds to commuting circuits of depth $\log(n)$)
- Noise: independent single-qubit dephasing w prob. $\epsilon$ (phenomenological noise model)

see also KF-Tamate ‘16
Topological MBQC on a 3D cluster state

- MBQC on a graph state of degree $\log(n)$ (corresponds to commuting circuits of depth $\log(n)$)
- Faulty part comes from either clifford operations or magic state distillation
- Noise: independent single-qubit dephasing with prob. $\epsilon$
  (phenomenological noise model)
- Faulty part comes from either clifford operations or magic state distillation
  $\rho_{\text{faulty}} = \rho_{\text{cl}} + \rho_{\text{magic}}$

see also KF-Tamate ‘16
Topological MBQC on a 3D cluster state

- MBQC on a graph state of degree $\log(n)$
  (corresponds to commuting circuits of depth $\log(n)$)
- Faulty part comes from either clifford operations or magic state distillation
  \[ \rho_{\text{faulty}} = \rho_{\text{cl}} + \rho_{\text{magic}} \]
- Noise: independent single-qubit dephasing w prob. $\epsilon$
  (phenomenological noise model)
- Faulty part comes from either clifford operations or magic state distillation
  \[ \frac{12}{5} \text{poly}(n) \left( \frac{5\epsilon}{1 - \epsilon} \right)^d \rightarrow \epsilon_{\text{cl}} = 0.167 \]

See also KF-Tamate '16
Topological MBQC on a 3D cluster state

- MBQC on a graph state of degree $\log(n)$ (corresponds to commuting circuits of depth $\log(n)$)
- Noise: independent single-qubit dephasing with prob. $\epsilon$ (phenomenological noise model)
- Faulty part comes from either clifford operations or magic state distillation

$\rho_{\text{faulty}} = \rho_{\text{cl}} + \rho_{\text{magic}}$

$\frac{12}{5} \text{poly}(n) \left( \frac{5\epsilon}{1 - \epsilon} \right)^d \rightarrow \epsilon_{\text{cl}} = 0.167$

magic state distillation

$\epsilon_{\text{magic}} = 0.146$

(Clifford operations (counting # of self-avoiding walks: Dennis et al ‘02))

see also KF-Tamate ‘16
Threshold theorem: if the noise strength is smaller than a certain constant threshold value, quantum computation can be done with an arbitrary accuracy poly(logarithmic) overhead.

phenomenological noise 2.9-3.3%
circuit-based noise 0.75%

phenomenological noise 14.6%

Topological fault-tolerance in cluster state
quantum computation
R Raussendorf, J Harrington and K Goyal
Circuit-based noise model with 2D surface code

- 2D nearest-neighbor gates on a square grid

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Circuit-based noise model with 2D surface code

- 2D nearest-neighbor gates on a square grid

$$
e(\nu, \mu) \equiv \left( \frac{\nu}{1-\nu} \right)^{1/2} \left[ \left( \frac{\nu}{1-\nu} \right) + \left( \frac{2\mu}{1-\mu} \right) \right]^{1/2}$$

$\nu$: independent error rate
$\mu$: correlated error rate

$\nu = \frac{54p}{15}, \mu = \frac{6p}{5}$

arXiv:1610.03632
Circuit-based noise model with 2D surface code

- 2D nearest-neighbor gates on a square grid

$$e(\nu, \mu) \equiv \left( \frac{\nu}{1-\nu} \right)^{1/2} \left[ \left( \frac{\nu}{1-\nu} \right) + \left( \frac{2\mu}{1-\mu} \right) \right]^{1/2}$$

$\nu$: independent error rate

$\mu$: correlated error rate

$\nu = 54p/15, \mu = 6p/5$

- threshold value: $p=2.84\%$ (distillability of magic state)
Circuit-based noise model with 2D surface code

- 2D nearest-neighbor gates on a square grid

$$
\epsilon(\nu, \mu) \equiv \left( \frac{\nu}{1-\nu} \right)^{1/2} \left[ \left( \frac{\nu}{1-\nu} \right) + \left( \frac{2\mu}{1-\mu} \right) \right]^{1/2}
$$

- $\nu$: independent error rate
- $\mu$: correlated error rate

$\nu = \frac{54p}{15}, \mu = \frac{6p}{5}$

- threshold value: $p=2.84\%$ (distillability of magic state)
- higher than the standard threshold 0.75%

arXiv:1610.03632
Summary

• Sampling with noisy quantum circuits can exhibit quantum supremacy.

• The threshold for supremacy is much higher than that for universal fault-tolerant quantum computation.

• The threshold is determined purely by distillability of the magic state (in a phenomenological model it sharply separate classically simulatable and not-simulatable regions).

• Can we directly verify or identify quantum supremacy of the near-term noisy quantum devices in a pre-fault-tolerant region?

Thank you for your attention!