Semidefinite programming strong converse bounds for quantum channel capacities

Xin Wang

UTS: Centre for Quantum Software and Information

Joint work with Wei Xie, Runyao Duan (UTS:QSI)
Before

In last year’s QIP,

- Aram Harrow gave the tutorial of Quantum Shannon theory (also ask for non-trivial upper bounds for classical capacity),
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Let’s combine them!
Channel & capacity

- **Quantum Channel**: completely positive (CP) trace-preserving (TP) linear map $\mathcal{N}$. 

\[ \rho \rightarrow \text{Tr}_E (V \rho V^\dagger), \text{ with isometry } V : A \rightarrow B \otimes E \]

\[ \rho_{AN} (\rho) \rightarrow \rho_{AB} (V \rho V^\dagger) \]

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  - Complementary $\mathcal{N}^c : \rho \rightarrow \text{Tr}_B(V\rho V^\dagger)$
Introduction One-shot information theory Strong converse bounds Summary

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- Choi-Jamiołkowski representation of $\mathcal{N}$:

$$J_{\mathcal{N}} = \sum_{ij} |i\rangle\langle j|_{A'} \otimes \mathcal{N}(|i\rangle\langle j|_A) = (\operatorname{id}_{A'} \otimes \mathcal{N})|\Phi_{A' A}\rangle |\Phi_{A' A}\rangle,$$

with $|\Phi_{A' A}\rangle = \sum_k |k_{A'}\rangle |k_A\rangle$. 

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Classical communication via quantum channels

- Classical capacity (Holevo’73, 98; Schumacher & Westmoreland’97):

\[
C(\mathcal{N}) = \sup_{k \to \infty} \frac{1}{k} \chi(\mathcal{N}^\otimes k),
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with \( \chi(\mathcal{N}) = \max\{(p_i;\rho_i)\} H(\sum_i p_i \mathcal{N}(\rho_i)) - \sum_i p_i H(\mathcal{N}(\rho_i)). \)
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- Difficulties of evaluating \( C(\mathcal{N}) \)
  - \( \chi(\mathcal{N}) \): NP-hard (Beigi & Shor’07)
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  - \( \chi(\mathcal{N}) \): NP-hard (Beigi & Shor’07)
  - Worse: \( \chi(\mathcal{N}) \) is not additive (Hastings’09)
  - Classical capacity of amplitude damping channel is unknown.
Practical setting and assisted communication

- Resource is finite and we are in the early stage of quantum information processing.
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  - Entanglement-assisted capacity (Bennett, Shor, Smolin, Thapliyal 1999, 2002)
Main question and outline

- **Non-asymptotic** communication capability
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- All these results are given by SDPs.
  - An analytical tool in proof (Watrous’ Book)
  - There are efficient algorithms.
  - Implementations: CVX for MATLAB, toolbox QETLAB.
Non-asymptotic communication capability
Optimal success probability and capacity

- (Shannon, 1948) The fundamental problem of communication is that of reproducing at one point, either exactly or approximately, a message selected at another point.

\[ k \in \{1, \ldots, m\} \xrightarrow{A} E \xrightarrow{N} D \xrightarrow{B} \hat{k} \in \{1, \ldots, m\} \]

\[ M = D \circ N \circ E \]
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- Optimal success probability

\[ p_s(\mathcal{N}, m) := \sup_{\mathcal{E}, \mathcal{D}} \frac{1}{m} \sum_{k=1}^{m} p(k = \hat{k}) \]

\[ = \sup_{\mathcal{E}, \mathcal{D}} \frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|]. \]
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- Classical capacity \( C(\mathcal{N}) := \sup\{r : \lim_{n \to \infty} p_s(\mathcal{N}^\otimes n, 2^{rn}) = 1\} \).

- Question: how to solve or estimate \( p_s(\mathcal{N}, m) \)?
General codes

\[ p_s(\mathcal{N}, m) = \sup_{\mathcal{E}, \mathcal{D}} \frac{1}{m} \sum_{k=1}^{m} \text{Tr} \mathcal{M}(|k\langle k|)|k\rangle\langle k|, \text{ with } \mathcal{M} = \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}. \]
General codes

- $p_s(N, m) = \sup_{\mathcal{E}, \mathcal{D}} \frac{1}{m} \sum_{k=1}^{m} \text{Tr} \mathcal{M}(|k\langle k|) |k\rangle\langle k|$, with $\mathcal{M} = \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}$.

- **No-signalling code** $\Pi$ is bipartite channel $\Pi : \mathcal{L}(A_i) \otimes \mathcal{L}(B_i) \rightarrow \mathcal{L}(A_o) \otimes \mathcal{L}(B_o)$ with NS constraints (Leung & Matthews’16; Duan & Winter’16), i.e., $A$ and $B$ cannot use the channel to communicate classical information.

- Also see causal operations (Beckman, Gottesman, Nielsen, Preskill’01; Eggeling, Schlingemann, Werner’02, Piani, Horodecki et al.’06).
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General codes

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- A **hierarchy** of codes by adding constraints on \( \Pi \), e.g., Positive-partial-transpose preserving (PPT) constraint (Rains’01; Leung & Matthews’16).
Optimal success probability

Optimal success probability of $\Omega$ codes ($\Omega = \text{NS}$ or $\text{NS} \cap \text{PPT}$ in this talk)

$$p_{s,\Omega}(\mathcal{N}, m) = \sup_{\Pi \in \Omega} \frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|], \quad \mathcal{M} \text{ given by } \mathcal{N}, \Pi.$$
Result 1: Optimal success probability for NS/PPT codes

Theorem

For any $\mathcal{N}$, the optimal success probability to transmit $m$ messages assisted by $\text{NS}\cap\text{PPT}$ codes is given by the following SDP:

$$p_{s,\text{NS}\cap\text{PPT}}(\mathcal{N}, m) = \max \text{Tr} \ J_{\mathcal{N}} F_{AB}$$

s.t. $0 \leq F_{AB} \leq \rho_A \otimes 1_B$, $\text{Tr} \rho_A = 1$,

$$\text{Tr}_A F_{AB} = 1_B / m,$$

$$0 \leq F_{AB}^T \leq \rho_A \otimes 1_B \ (\text{PPT}),$$

where $J_{\mathcal{N}}$ is the Choi-Jamiołkowski matrix of $\mathcal{N}$.

When assisted by NS codes, one can remove PPT constraint to obtain

$$p_{s,\text{NS}}(\mathcal{N}, m) = \max \text{Tr} \ J_{\mathcal{N}} F_{AB} \ s.t. \ 0 \leq F_{AB} \leq \rho_A \otimes 1_B$, $\text{Tr} \rho_A = 1$,

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Sketch of proof

- Target:

\[ p_{s,\Omega}(\mathcal{N}, m) = \sup_{\Pi \in \Omega} \frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|], \quad (1) \]
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- Recall \( J_{\mathcal{M}} = \sum_{ij} |i\rangle\langle j|_{A_i} \otimes \mathcal{M}(|i\rangle\langle j|_{A_i}) \) and let \( V = \sum_{k=1}^{m} |kk\rangle\langle kk| \)

Xin Wang (UTS:QSI) | SDP strong converse bounds for quantum channel capacities | QIP 17, Microsoft Research, Seattle
Sketch of proof

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  Key: \[ \frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|] = \frac{1}{m} \text{Tr}[J_{\mathcal{M}} V_{A_iB_o}]. \quad (2) \]
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\[
p_{s,\Omega}(\mathcal{N}, m) = \sup_{\Pi \in \Omega} \frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(|k\rangle\langle k|) |k\rangle\langle k|],
\]

(1)

- **Recall** \( J_\mathcal{M} = \sum_{ij} |i\rangle\langle j|_{A_i} \otimes \mathcal{M}(|i\rangle\langle j|_{A_i}) \) and let \( V = \sum_{k=1}^{m} |kk\rangle\langle kk| \)

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\frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(|k\rangle\langle k|) |k\rangle\langle k|] = \frac{1}{m} \text{Tr}[J_\mathcal{M} V_{A_i B_o}].
\]

(2)

- **Moreover,** \( J_\mathcal{M} \) can be represented by \( J_\mathcal{N} \) and \( J_\Pi \) (Leung & Matthews’16; based on Chiribella, D’Ariano, Perinotti’08)

\[
J_\mathcal{M} = \text{Tr}_{A_o B_i} (J_\mathcal{N}^T \otimes 1_{A_i B_o}) J_\Pi.
\]

(3)
Sketch of proof

- Target: 
  \[ p_{s,\Omega}(\mathcal{N}, m) = \sup_{\Pi \in \Omega} \frac{1}{m} \sum_{k=1}^{m} \text{Tr}[\mathcal{M}(\langle k|k\rangle)|k\rangle\langle k|], \]  
  \hspace{1cm} (1) 

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  \hspace{1cm} (3) 

- Combining Eqs. (1), (2), (3), we have

  \[ p_{s,\Omega}(\mathcal{N}, m) = \max_{\Pi \in \Omega} \text{Tr}[ (J_{\mathcal{N}}^T \otimes 1_{A_i B_o}) J_{\Pi}(1_{A_o B_i} \otimes V_{A_i B_o}) ]/m, \]
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  \[ J_{\mathcal{M}} = \text{Tr}_{A_oB_i}(J_{\mathcal{N}}^T \otimes 1_{A_iB_o}) J_{\Pi}. \quad (3) \]

- **Combining Eqs. (1), (2), (3), we have**
  \[ p_{s,\Omega}(\mathcal{N}, m) = \max_{\Pi \in \Omega} \frac{\text{Tr}[(J_{\mathcal{N}}^T \otimes 1_{A_iB_o}) J_{\Pi}(1_{A_oB_i} \otimes V_{A_iB_o})]}{m}, \]

- Imposing the NS and PPT constraints of \( \Pi \) to obtain the SDP.
- Exploit the **permutation invariance** of \( V_{A_iB_o} \) to simplify SDP.
Example: assess the performance of AD channel

- For amplitude damping channel $\mathcal{N}_\gamma^{AD}(\rho) = \sum_{i=0}^{1} E_i \rho E_i^\dagger$ with $E_0 = |0\rangle\langle 0| + \sqrt{1 - \gamma} |1\rangle\langle 1|$ and $E_1 = \sqrt{\gamma} |0\rangle\langle 1|$, if we use the channel 3 times, the optimal success probability to transmit 1 bit is given as follows:
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- if we use the channel 3 times, the optimal success probabilty to transmit 1 bit is given as follows:

![Graph showing success probability vs. $\gamma$ from 0 to 1]
Result 2: One-shot capacities

- One-shot $\epsilon$-error capacity assisted with $\Omega$-codes:

$$C^{(1)}_{\Omega}(N, \epsilon) := \sup \{\log \lambda : 1 - p_{s,\Omega}(N, \lambda) \leq \epsilon\}.$$
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**Theorem**

For given channel $\mathcal{N}$ and error threshold $\epsilon$,

$$C^{(1)}_{\text{NS} \cap \text{PPT}}(\mathcal{N}, \epsilon) = -\log \min \eta \text{ s.t.} \quad 0 \leq F_{AB} \leq \rho_A \otimes 1_B, \ Tr \rho_A = 1,$$

$$Tr_A F_{AB} = \eta 1_B, \ Tr J_N F_{AB} \geq 1 - \epsilon,$$

$$0 \leq F_{AB}^{TB} \leq \rho_A \otimes 1_B \ (\text{PPT}).$$

To obtain $C^{(1)}_{\text{NS}}(\mathcal{N}, \epsilon)$, one only needs to remove the PPT constraint:

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For given channel $\mathcal{N}$ and error threshold $\epsilon$,

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C_{\text{NS} \cap \text{PPT}}^{(1)}(\mathcal{N}, \epsilon) = -\log \min \eta \quad \text{s.t.} \quad 0 \leq F_{AB} \leq \rho_A \otimes 1_B, \quad \text{Tr} \rho_A = 1,
\]
\[
\text{Tr}_A F_{AB} = \eta 1_B, \quad \text{Tr} J_{\mathcal{N}} F_{AB} \geq 1 - \epsilon,
\]
\[
0 \leq F_{AB}^{T_B} \leq \rho_A \otimes 1_B \quad \text{(PPT)},
\]

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\]
\[
\text{Tr}_A F_{AB} = \eta 1_B, \quad \text{Tr} J_{\mathcal{N}} F_{AB} \geq 1 - \epsilon.
\]

- Study zero-error capacity by setting $\epsilon = 0$, e.g., $C_{\text{NS}}^{(1)}(\mathcal{N}, 0)$ recovers the one-shot NS assisted zero-error capacity in (Duan & Winter’16).
Comparison with previous converse bounds

- Converse for classical channel (Polyanskiy, Poor, Verdú 2010) and classical-quantum channel (Wang & Renner 2010).
- (Matthews & Wehner 2014) shows SDP converse bounds

\[
C_E^{(1)}(\mathcal{N}, \epsilon) \leq \max_{\rho_A} \min_{\sigma_B} D^\epsilon_H((id_A' \otimes \mathcal{N})(\rho_{A'}A)||\rho_{A'} \otimes \sigma_B),
\]

\[
C^{(1)}(\mathcal{N}, \epsilon) \leq \max_{\rho_A} \min_{\sigma_B} D^\epsilon_{H,PPT}((id_A' \otimes \mathcal{N})(\rho_{A'}A)||\rho_{A'} \otimes \sigma_B),
\]

where \(D^\epsilon_H\) and \(D^\epsilon_{H,PPT}\) are hypothesis testing relative entropies.

- (Datta & Hsieh’13) gives converse for \(C_E^{(1)}(\mathcal{N}, \epsilon)\) (hard to compute).
Comparsion with previous converse bounds

- Converse for classical channel (Polyanskiy, Poor, Verdú 2010) and classical-quantum channel (Wang & Renner 2010).
- (Matthews & Wehner 2014) shows SDP converse bounds

\[
C_E^{(1)}(\mathcal{N}, \epsilon) \leq \max_{\rho_A} \min_{\sigma_B} D_H^\epsilon((id_{A'} \otimes \mathcal{N})(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B),
\]

\[
C^{(1)}(\mathcal{N}, \epsilon) \leq \max_{\rho_A} \min_{\sigma_B} D_{H,PPT}^\epsilon((id_{A'} \otimes \mathcal{N})(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B),
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where \( D_H^\epsilon \) and \( D_{H,PPT}^\epsilon \) are hypothesis testing relative entropies.

- (Datta & Hsieh’13) gives converse for \( C_E^{(1)}(\mathcal{N}, \epsilon) \) (hard to compute).
- One-shot \( \epsilon \)-error capacities can provide better efficiently computable converse bounds:

\[
C_E^{(1)}(\mathcal{N}, \epsilon) \leq C_{NS}^{(1)}(\mathcal{N}, \epsilon) \leq \max_{\rho_A} \min_{\sigma_B} D_H^\epsilon((id_{A'} \otimes \mathcal{N})(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B),
\]

\[
C^{(1)}(\mathcal{N}, \epsilon) \leq C_{NS\cap PPT}^{(1)}(\mathcal{N}, \epsilon) \leq \max_{\rho_A} \min_{\sigma_B} D_{H,PPT}^\epsilon((id_{A'} \otimes \mathcal{N})(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B).
\]

The blue inequalities can be strict for amplitude damping channels.
Asymptotic communication capability
The converse part of the HSW theorem due to Holevo (1973) only establishes a **weak converse**, which states that there cannot be an error-free communication scheme if rate exceeds capacity.
Weak vs Strong Converse

- The converse part of the HSW theorem due to Holevo (1973) only establishes a **weak converse**, which states that there cannot be an error-free communication scheme if rate exceeds capacity.
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• The converse part of the HSW theorem due to Holevo (1973) only establishes a weak converse, which states that there cannot be an error-free communication scheme if rate exceeds capacity.

• A strong converse bound: $p_{\text{succ}} \to 0$ as $n$ increases if the rate exceeds this bound.

• If the capacity of a channel is also its strong converse bound, then the strong converse property holds.
Result 3: Strong converse bound for classical capacity

- Known strong converse bound: the entanglement-assisted capacity (Bennett, Shor, Smolin, Thapliyal 1999, 2002)

**Theorem (SDP strong converse bound for C)**

For any quantum channel $\mathcal{N}$,

$$C(\mathcal{N}) \leq C_\beta(\mathcal{N}) = \log \min \text{Tr} S_B$$

s.t. $-R_{AB} \leq J_{\mathcal{N}}^{TB} \leq R_{AB}$,

$$-\mathbb{1}_A \otimes S_B \leq R_{AB}^{TB} \leq \mathbb{1}_A \otimes S_B.$$

And $p_{\text{succ}} \to 0$ when the rate exceeds $C_\beta(\mathcal{N})$.

**Properties:**

- A relaxed bound: $C(\mathcal{N}) \leq C_\beta(\mathcal{N}) \leq \log d_B \| J_{\mathcal{N}}^{TB} \|_\infty$.
- For qudit noiseless channel $I_d$, $C(I_d) = C_\beta(I_d) = \log d$.
- $C_\beta(\mathcal{N}_1 \otimes \mathcal{N}_2) = C_\beta(\mathcal{N}_1) + C_\beta(\mathcal{N}_2)$ for any $\mathcal{N}_1$ and $\mathcal{N}_2$. 
Sketch of proof

- **Subadditive** bounds on $p_s$ (Tool: duality of SDP)

  \[
  p_{s,\text{NS} \cap \text{PPT}}(\mathcal{N} \otimes n, 2^{rn}) \leq p_s^+(\mathcal{N} \otimes n, 2^{rn}) \leq p_s^+(\mathcal{N}, 2^r)^n, \tag{4}
  \]

  where

  \[
  p_s^+(\mathcal{N}, m) = \min \text{ Tr } Z_B \text{ s.t. } -R_{AB} \leq J^T_B \leq R_{AB},
  \]

  \[
  -m1_A \otimes Z_B \leq R^T_{AB} \leq m1_A \otimes Z_B. \tag{5}
  \]
Sketch of proof

- **Subadditive** bounds on $p_s$ (Tool: duality of SDP)

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p_{s,\text{NS}\cap\text{PPT}}(\mathcal{N}^\otimes n, 2^{rn}) \leq p_s^+(\mathcal{N}^\otimes n, 2^{rn}) \leq p_s^+(\mathcal{N}, 2^r)^n,
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p_s^+(\mathcal{N}, m) = \min \quad \text{Tr } Z_B \quad \text{s.t.} \quad -R_{AB} \leq J_B^T_{\mathcal{N}} \leq R_{AB},
- m1_A \otimes Z_B \leq R_{AB}^T \leq m1_A \otimes Z_B.
\]

- For any $r > C_\beta(\mathcal{N})$, one can prove that $p_s^+(\mathcal{N}, 2^r) < 1$. Thus,

\[
p_{s,\text{NS}\cap\text{PPT}}(\mathcal{N}^\otimes n, 2^{rn}) \leq p_s^+(\mathcal{N}, 2^r)^n \rightarrow 0, \quad \text{(when } n \text{ increases)}
\]
Application 1: Amplitude damping channel

For amplitude damping channel,

\[ C(\mathcal{N}_{\gamma}^{AD}) \leq C_{\beta}(\mathcal{N}_{\gamma}^{AD}) = \log(1 + \sqrt{1 - \gamma}). \]
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- **Solid line** depicts our bound.
- **Dashed line** depicts the previously best upper bound (Brandão, Eisert, Horodecki, Yang 2011).
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- Note that \( C_\beta(\mathcal{N}^{AD}_\gamma) \geq 1 \) when \( \gamma \leq 0.5 \).
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- Note that \( C_E(\mathcal{N}^{AD}_\gamma) \geq 1 \) when \( \gamma \leq 0.5 \).
- Problem: how to further improve the lower bound or upper bound?
Application 2: Strong converse property for new channels

- Previous known channels:
  - classical-quantum channels (Ogawa, Nagaoka’99; Winter’99)
  - particular covariant quantum channels (Koenig and Wehner’09)
  - entanglement-breaking, Hadamard channels (Wilde, Winter, Yang’14).
  - Optical quantum channels (Bardhan, et al.’16)

Applying the strong converse bound $C_{\beta}$, $C(N_\alpha) = C_{NS} \cap \text{PPT}(N_\alpha) = C_{\beta}(N_\alpha) = 1$.

In (W. & D.,1608.04508), $C_{E}(N_\alpha) = 2 < \log \vartheta(N)$, and $\vartheta(N)$ is the quantum Lovász number (Duan, Severini, Winter’13).

In particular, $Q(N_\alpha) < 1 = P(N_\alpha) = C(N_\alpha) = \frac{1}{2}C_{E}(N_\alpha)$.
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- The channel from $A$ to $B$ is given by $\mathcal{N}_\alpha(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$ ($0 < \alpha \leq \pi/4$) with
  
  $E_0 = \sin \alpha |0\rangle\langle 1| + |1\rangle\langle 2|$, $E_1 = \cos \alpha |2\rangle\langle 1| + |1\rangle\langle 0|$. 

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In particular, $Q(\mathcal{N}_\alpha) < 1 = P(\mathcal{N}_\alpha) = C(\mathcal{N}_\alpha) = 1/2 C_E(\mathcal{N}_\alpha)$. 

Xin Wang (UTS:QSI) | SDP strong converse bounds for quantum channel capacities | QIP 17, Microsoft Research, Seattle
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  \[
  Q(\mathcal{N}_\alpha) < 1 = P(\mathcal{N}_\alpha) = C(\mathcal{N}_\alpha) = \frac{1}{2} C_E(\mathcal{N}_\alpha).
  \]
Quantum capacity

- Quantum capacity is established by (Lloyd, Shor, Devetak 97-05) & (Barnum, Nielsen, Schumacher 96-00)

\[ Q(\mathcal{N}) = \lim_{m \to \infty} \frac{1}{m} I_c(\mathcal{N}^\otimes m). \]

- Coherent information \( I_c(\mathcal{N}) := \max_{\rho} [H(\mathcal{N}(\rho)) - H(\mathcal{N}^c(\rho))] \)
- \( Q(\mathcal{N}) \) is also difficult to evaluate.
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- Known strong converse bounds:
  - Partial Transposition bound (Holevo, Werner 2001; Muller-Hermes, Reeb, Wolf 2016)
  - Rains information (Tomamichel, Wilde, Winter 2015)
  - Channel’s entanglement cost (Berta, Brandao, Christandl, Wehner 2013)
Theorem (SDP strong converse bound for $Q$)

For any quantum channel $\mathcal{N}$,

$$Q(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) = \log \max \text{Tr } J_{\mathcal{N}} R_{AB}$$

s.t. $R_{AB}, \rho_A \geq 0, \text{Tr } \rho_A = 1,$

$$-\rho_A \otimes 1_B \leq R_{AB}^{T_B} \leq \rho_A \otimes 1_B.$$ 

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This is based on the optimal fidelity of transmitting quantum information assisted with PPT codes (Leung and Matthews'16). The proof idea is similar to previous bound for classical capacity. For noiseless quantum channel $I_d$, $Q(I_d) = Q_\Gamma(I_d) = \log_2 d$. $Q_\Gamma(N \otimes M) = Q_\Gamma(M) + Q_\Gamma(N)$ (by utilizing SDP duality).
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Comparison with other bounds

- **Partial Transposition bound** (Holevo & Werner’01, Muller-Hermes, Reeb, Wolf’16)

\[ Q(\mathcal{N}) \leq Q_\Theta(\mathcal{N}) = \log_2 \| J^T_B \|_{cb}, \]

where \( \| \cdot \|_{cb} \) uses an alternative expression from (Watrous’12).
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**Improved efficiently computable bound**

For any quantum channel \( \mathcal{N} \), \( Q_\Gamma(\mathcal{N}) \leq Q_\Theta(\mathcal{N}) \).
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**Improved efficiently computable bound**

For any quantum channel \( \mathcal{N} \), \( Q_\Gamma(\mathcal{N}) \leq Q_\Theta(\mathcal{N}) \).

- Example: \( \mathcal{N}_r = \sum_i E_i \cdot E_i^\dagger \)

with \( E_0 = |0\rangle \langle 0| + \sqrt{r} |1\rangle \langle 1| \)

and \( E_1 = \sqrt{1 - r} |0\rangle \langle 1| + |1\rangle \langle 2| \).

- **Solid line**: SDP bound \( Q_\Gamma \)

- **Dashed line**: PT bound \( Q_\Theta \)
Summary and Outlook

- Non-asymptotic classical communication (NS/NS∩PPT codes)
  - Optimal success probability of communication is given by SDP
  - One-shot $\epsilon$-error capacity is given by SDP
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  - Continuous-variable quantum channels?
arXiv:1610.06381 & 1601.06888

Wei Xie

Runyao Duan
Thank you for your attention!