The Thermality of Quantum Approximate Markov Chains
with implications to the Locality of Edge States and Entanglement Spectrum

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Motivation

When many-body systems are described by local (short-range) Hamiltonians, states have special correlation properties.

Area law for gapped ground states: restricts entanglement (rigorously proven for 1D systems [Hastings, 07])

Area law for Gibbs (thermal) states: restricts correlations (proven for any dim. [Wolf, et al., 07])

Efficient descriptions of many-body states (MPS, PEPS, MPO,...)

A useful consequence of area laws: small “conditional mutual information (CMI)” on certain regions (Applications: [Kim, ‘12,’13], [Swingle & Kim, 14], [Kastryano & Brandao, ‘16] ...)

Q. How to characterize?
Motivation

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A useful consequence of area laws: small “conditional mutual information (CMI)” on certain regions

This talk:

1. Characterizing states with small CMI in terms of Gibbs states (cf. previous talk by Kastoryano)

2. An application to “entanglement spectrum” of 2D gapped systems

Efficient descriptions of many-body states (MPS, PEPS, MPO,...)

Q. How to characterize?
Outline of this talk

Part I: A characterization of approximate Markov chains
- Area law for Gibbs States
- Quantum Markov Chains & Approximate Quantum Markov Chains
- Equivalence to Gibbs states of short-range Hamiltonians

Part II: An application to entanglement spectrum in 2D systems
- Topological Entanglement Entropy and Entanglement Spectrum
- Previous Results on Entanglement Spectrum
- Locality of Entanglement Hamiltonian and Spectrum
Part I:  
A characterization of approximate Markov chains
Area law for Gibbs states

Hamiltonian

\[ H = \sum_i h_{i,i+1}, \quad \|h_i\| \leq J. \]

WLOG: nearest-neighbor

Gibbs state

\[ \rho = \frac{1}{Z} e^{-\beta H}, \quad Z = \text{tr} e^{-\beta H}. \]

[Wolf, et al., '07]

\[ I(A: B)_\rho := S(A)_\rho + S(B)_\rho - S(AB)_\rho \leq 2\beta J |\partial A| \]

\[ \triangleright S(A)_\rho := -\text{tr} \rho_A \log_2 \rho_A \]
Conditional Mutual Information of Gibbs States

The conditional mutual information:

\[ I(A: C | B) \rho := I(A: BC) \rho - I(A: B) \rho \geq 0 \]

- Monotonicity of MI: \( I(A: BC) \rho \geq I(A: B) \rho \)

\[ \rightarrow I(A: B_1) \rho \leq I(A: B_1 B_2) \rho \leq \cdots \leq I(A: B_1 \ldots B_m) \rho \leq 2\beta J|\partial A| \]

small for large \( m \)!
Quantum Markov Chain (for three systems)

If $I(A: C|B)_\rho = 0$, quantum state $\rho_{ABC}$ is called a Quantum Markov Chain $A - B - C$.

1. There exists a CPTP-map $\Lambda_{B\rightarrow BC} : B \rightarrow BC$ s.t.

$$\rho_{ABC} = \text{id}_A \otimes \Lambda_{B\rightarrow BC}(\rho_{AB})$$

2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.

$$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0 \quad (\rho_{ABC} > 0)$$

[Hayden, et al., 03], [Brown & Poulin, ‘12]
Longer Chains

The quantum Markov chain \( \rho_A \) on the chain \( A_1 A_2 \ldots A_n \) is a (quantum) Markov chain if

\[
I(A_1 \ldots A_{i-1} : A_{i+1} \ldots A_n | A_i)_{\rho} = 0
\]

for arbitrary \( i \in [n] \).

*We can generalize the concept of Markov chains to general graphs as Markov networks.*
Hammersley-Clifford Theorem (1D)

[Hammersley&Clifford, ’71]:
Random variables $X_1, X_2, ..., X_n$ forms a (positive) Markov chain if, and only if, the distribution can be written as

$$p_{X_1X_2...X_n}(x_1, x_2, ..., x_n) = \frac{1}{Z} \exp \left( - \sum_i h_i(x_i, x_{i+1}) \right)$$

$X_1 \ X_2 \ \ h_i(x_i, x_{i+1}) \ \ ... \ X_n$

* also holds for Markov networks

Positive Markov chains

$\uparrow$

Gibbs distributions of 1D short-range Hamiltonians
Quantum Hammersley-Clifford Theorem (1D)

[Leifer & Poulin, ’08], [Brown & Poulin, ’12]:

A quantum state $\rho_{A_1 \ldots A_n} > 0$ on a chain forms a Markov chain if, and only if, the state can be written as

$$
\rho_{A_1 \ldots A_n} = \frac{1}{Z} \exp \left( - \sum_i h_{A_i A_{i+1}} \right), \quad [h_{A_i A_{i+1}}, h_{A_j A_{j+1}}] = 0
$$

Positive quantum Markov chains

2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.

$$
\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0
$$

Gibbs states of 1D commuting short-range Hamiltonians

* also holds for Markov networks
Quantum Hammersley-Clifford Theorem (1D)

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A quantum state $\rho_{A_1...A_n} > 0$ on a chain forms a Markov chain if, and only if, the state can be written as

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$A_1A_2...A_n$

* also holds for Markov networks

2. There exists a Hamiltonian $H_{ABC} = H_{AB} + H_{BC}$ s.t.

$$\rho_{ABC} = e^{-H_{ABC}}, [H_{AB}, H_{BC}] = 0$$
Properties of Approximate Markov Chains

How about states having small but non-zero CMI?

Naïve guess: all properties of Markov chains approximately hold for approximate Markov chains

Classical:

\[
I(X:Z|Y)_{p} = \min_{q: \text{Markov}} S(p_{XYZ} \| q_{XYZ})
\]

\[
I(X:Z|Y)_{p} \leq \varepsilon \iff p_{XYZ} \approx_{\varepsilon} q_{XYZ}
\]

However...

Quantum:

\[
I(A:C|B)_{\rho} \neq \min_{\sigma: \text{Markov}} S(\rho_{ABC} \| \sigma_{ABC}) \quad [\text{Ibinson, et al., '06}]
\]

\exists \text{ property of Markov chains which is invalid for approximate Markov chains}
Local Recoverability of States with Small CMI

Some properties still approximately hold for approximate Markov chains

[Fawzi & Renner, ‘15]:
There exists a CPTP-map $\Lambda_{B\rightarrow BC}$ s.t.
$$I(A: C|B)_\rho \geq -2\log_2 F(\rho_{ABC}, \Lambda_{B\rightarrow BC}(\rho_{AB}))$$

$$I(A: C|B)_\rho \approx 0 \iff$$

1. There exists a CPTP-map $\Lambda_{B\rightarrow BC}: B \rightarrow BC$ s.t.
$$\rho_{ABC} \approx \text{id}_A \otimes \Lambda_{B\rightarrow BC}(\rho_{AB})$$

*The converse part can be shown by using the Alicki-Fannes inequality.*
Question

Q. How about the quantum Hammersley-Clifford theorem for approximate Markov chains?

Quantum approximate Markov chains

Gibbs states of 1D short-range Hamiltonians
Approximate Quantum HC Theorem (1D)

Result 1.
For any \( \epsilon \)-approximate Markov chain \( \rho_{A_1A_2...A_n} \), there exists a Hamiltonian \( H_A = \sum h_{A_iA_{i+1}} \) s.t.,

\[
S(\rho_A || e^{-H_A}) \leq n\epsilon.
\]

Application to gapped systems (next part)

Any approximate Markov chain can be approximated by local Gibbs states

\( \rho_A \) is an \( \epsilon \)-approximate Markov chain if

\[
I(A_1 ... A_{i-1} : A_{i+1} ... A_n | A_i)_{\rho} \leq \epsilon
\]

for arbitrary \( i \in [n] \).
Approximate Quantum HC Theorem (1D)

Result 2. For any Gibbs state $\rho$ of a short-range Hamiltonian $H$ at temperature $T$,

$$I(A: C | B)_\rho \leq c e^{-q(T)\sqrt{l}}$$

for $q(T) = e^{-c' T^{-1}}, c \geq 0, c' > 0$ and any partition $ABC$ as in the diagram.

Application to Gibbs state preparation (see previous talk)

All 1D Gibbs states of short-range Hamiltonians are approximate Markov chains (Strengthen the area law of 1D Gibbs states)
Approximate Quantum HC Theorem (1D)

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Quantum approximate Markov chains

Application to Gibbs state preparation (see previous talk)

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PartII: An application to entanglement spectrum in 2D systems
Area Law in 2D Gapped Systems

- Ground states of 2D gapped local Hamiltonians typically obey area law:

\[ S(A)_\rho = \alpha |\partial A| - n_{\partial A} \gamma + o(1) \]

\( \gamma \): topological entanglement entropy

[Ref: Kitaev & Preskill, ‘06], [Levin & Wen ‘06]

\((\gamma > 0 \leftrightarrow \text{the g.s. is in a topologically ordered phase (??)})\)

A strong type of area law (rest of this talk)

\[ S(A)_\rho = \alpha |\partial A| - n_{\partial A} \gamma + e^{-|\partial A|/\xi} \]

For any \(ABC\) with no holes,

\[ I(A: C | B)_\rho \leq e^{-cl} \]

\( \rho_{ABC} \) is an approximate Markov chain
Entanglement Hamiltonian and Spectrum

- Other tools to study gapped g.s.

\[ \rho_A =: e^{-H_A} \quad \text{entanglement Hamiltonian} \]

\[ \lambda(H_A): \text{entanglement spectrum} \]

- Correspondence to edge theory in FQHE [Li & Haldane, ‘08] also has been studied in other systems [Ali, et al., ‘09, Lauchli & Bergholtz, ‘10, ...]

- Previous observations in the PEPS formalism [Cirac et al., ‘11], [Schuch, et al., ‘13], [Cirac, et al., ‘16]

\[ \rho_l = V \sigma_b^2 V^\dagger \quad V: \text{isometry} \]

\[ H_b = \begin{cases} 
\text{short-range} \\
\text{(in trivial phase)}
\end{cases} \]

\[ \text{short-range + global interactions} \]

\[ \text{(in topologically ordered phases)} \]
Entanglement Hamiltonian and Spectrum

• Other tools to study gapped g.s.

\[ \rho_A = e^{-H_A} \]

Entanglement Hamiltonian

\( \lambda(H_A) \): entanglement spectrum

(logarithm of the Schmidt coefficients)

This talk: connection to the topological entanglement entropy
also has been studied in other systems [Ali, et al., ’09, Lauchli & Bergholtz, ’10,...]

• Previous observations in the PEPS formalism
[Cirac et al., ’11], [Schuch, et al., ’13], [Cirac, et al., ’16]

\[ \rho_l = V \sigma_b^2 V^\dagger \]

V: isometry

\[ H_b = \begin{cases} 
\text{short-range} \\
\text{(in trivial phase)} \\
\text{short-range + global interactions} \\
\text{(in topologically ordered phases)}
\end{cases} \]

Q. How general this observation in PEPS?
Locality of Entanglement Spectrum \((\gamma = 0)\)

Suppose \(|\psi_{YXY'}\rangle\) satisfies the area law and \(\gamma = 0\) (trivial phase).

\[ \rightarrow \rho_{X_1\ldots X_m} \text{ is an approx. Markov chain} \]

\[ \rightarrow \rho_{X_1\ldots X_m} \approx \frac{1}{Z} \exp(-\sum h_{X_iX_{i+1}}) \]

- \(|\psi_{YXY'}\rangle\) is pure \(\rightarrow \lambda(\rho_{YY'}) = \lambda(\rho_{X_1\ldots X_m})\)
- \(I(Y:Y')_\rho = I(Y:Y'|X)_\psi \approx 0 \rightarrow \rho_{YY'} \approx \rho_Y \otimes \rho_{Y'} = \rho_Y \otimes^2\)

\[ H_Y^{(2)} := \log \rho_Y \otimes I + I \otimes \log \rho_Y \]

\[ \Rightarrow \left\| \lambda \left( H_Y^{(2)} \right) - \lambda\left( \sum h_{X_iX_{i+1}} \right) \right\|_1 \leq e^{-cl} \]

for some \(c > 0\).
How about the case of $\gamma > 0$?

Result 3.
Under our assumption, for some $c > 0$ and sufficiently large $l$,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X || e^{-H_X}) + e^{-cl} \geq 0 \ (l \gg 1)$$

$$\mathcal{H}_2 := \{ H = \sum h_{X_iX_{i+1}}, \|h_{X_iX_{i+1}}\| \leq \mathcal{O}(|X|) \}$$

$\gamma > 0 \rightarrow -\log \rho_X$ is non-local

Note: EH is local after tracing out $X_i$.

$$\text{tr}_{X_1} e^{-H_X} = \exp(-h_{X_2X_3} \cdots - h_{X_{m-1}X_m})$$

Conjecture (no rigorous proof): The non-local part is dominated by $m$-body interactions
Non-Locality of Entanglement Spectrum ($\gamma > 0$)

Result 3.
Under our assumption, for some $c > 0$ and sufficiently large $l$,

$$2\gamma = \min_{H_X \in \mathcal{H}_2} S(\rho_X \| e^{-H_X}) + e^{-cl}$$

$$\mathcal{H}_2 := \{ H = \sum h_{X_iX_{i+1}}, \| h_{X_iX_{i+1}} \| \leq \mathcal{O}(|X|) \}$$

$$\downarrow$$

$$\| \lambda(H_{Y}^{(2)}) - \lambda(H_X) \|_1 \leq e^{-cl}$$

for a non-local $H_X$. 
Difference to The Previous Results

Assumption: PEPS formalism (fixed-point) [Cirac et al., ‘11], [Schuch, et al., ‘13], [Cirac, et al., ‘16]

\[ \lambda(-\log \rho_l) = \lambda(H_b) \]

\[ H_b = \begin{cases} 
\text{short-range (in trivial phase)} \\
\text{short-range + global interactions (in topologically ordered phases)}
\end{cases} \]

Assumption: Strong type of area law (+ reflection symmetry) this talk

\[ \left\| \lambda \left( H_Y^{(2)} \right) - \lambda(H_X) \right\|_1 \leq e^{-cl} \]

\[ H_X = \begin{cases} 
\text{short-range } (\gamma = 0) \\
\text{short-range + global interactions } (\gamma > 0)
\end{cases} \]
Take-home massages:
Part I: Quantum approximate Markov chains are Gibbs states of 1D short-range Hamiltonians.

Part II: The locality of the entanglement spectrum of gapped g.s. on a cylinder is related to the TEE.

Open problems:
Part I: Better bounds on CMI of 1D Gibbs states?
Generalization of the equivalence to Markov networks?
(→ application for Gibbs state preparation)

Part II: Weaker assumptions?
Do we really need double of the ES?
Consequences of the (non-)locality of ES?
THANK YOU!
Idea of the proof

**Result 1.**
For any $\varepsilon$–approximate Markov chain $\rho_{A_1A_2\ldots A_n}$, there exists a Hamiltonian $H_A = \sum h_{A_iA_{i+1}}$ s.t.,

$$S(\rho_A || e^{-HA}) \leq n\varepsilon.$$

- **The maximum entropy principle** [Jaynes, ‘57]
  The maximum entropy state $\sigma_A$ satisfying

  $$\sigma_{A_iA_{i+1}} = \rho_{A_iA_{i+1}}, \forall i$$

  has the form

  $$\sigma_{A_iA_{i+1}} = e^{-\sum h_{A_iA_{i+1}}}.$$

- **A result from information geometry** [Knauf & Weis, ‘10]
  $$\inf_{H_A = \sum h_{A_iA_{i+1}}} S(\rho_A || e^{-HA}) = S(A)_\rho - S(A)_\sigma$$

  Small by the assumption + SSA
Idea of the proof

**Result 2.**
For any Gibbs state $\rho$ of a short-range Hamiltonian $H$ at temperature $T$,

$$I(A: C | B)_\rho \leq ce^{-q(T)\sqrt{l}}$$

for $q(T) = e^{-c'T^{-1}}$, $c \geq 0$, $c' > 0$ and any partition $ABC$ as in the below.

Explicitly construct a recovery map $\Lambda_{B \rightarrow BC}$ s.t.

$$\|\rho_{ABC} - \Lambda_{B \rightarrow BC}(\rho_{AB})\|_1 \leq c'e^{-q'\sqrt{l}}$$

- **Quantum belief propagation equation** [Hastings, ‘07][Kim, ‘11]

For 1D Hamiltonian with short-range $H$, $\exists O_I$ s.t.

$$\|e^{-\beta(H+V)} - O_I e^{-\beta H} O_I^\dagger\| \leq e^{-q''l}$$
Idea of the proof

From the quantum belief propagation equation, there exists $X_B$ s.t.

$$
\rho_{ABC} \approx \kappa_{B\rightarrow BC}(\rho_{AB}) = X_B \left( \text{tr}_{B^R} \left[ X_B^{-1} \rho_{AB} (X_B^{-1})^\dagger \right] \otimes \rho_{B^R C} \right) X_B^\dagger
$$

**Note:** Probably $\kappa_{B\rightarrow BC}$ is not a quantum operation.
Repeat-until-success method

We normalize $\kappa_{B\rightarrow BC}$ and define a CPTD-map $\tilde{\Lambda}_{B\rightarrow BC}$.
→ Succeed to recover with a constant probability $p$ (in 1D systems).

We normalize $\kappa_{B\rightarrow BC}$ and define a CPTD-map $\tilde{\Lambda}_{B\rightarrow BC}$.

Choose $N \sim l$ ($|B| = O(l^2)$).
We can construct a CPTP-map $\Lambda_{B\rightarrow BC}$ satisfying

$$
\|\rho_{ABC} - \text{id}_A \otimes \Lambda_{B\rightarrow BC} (\rho_{AB})\|_1 \leq e^{-O(l)}.
$$
Idea of the proof

**Result 3.**
Under our assumption, for some $c > 0$ and sufficiently large $l$,

$$2\gamma = \min_{H_x \in \mathcal{H}_2} S(\rho_x || e^{-H x}) + e^{-cl}$$

$$\mathcal{H}_2 := \{ H = \sum h_{X_i X_{i+1}}, \| h_{X_i X_{i+1}} \| \leq O(|X|) \}$$

By assumption, $I(X_1: X_3 X_{m-1} | X_2 X_m) \rho \approx 0$.

$\rightarrow \exists$ recovery map $\Lambda_{2m \rightarrow 12m}: X_2 X_m \rightarrow X_2 X_m X_1$

$$\sigma_X := \Lambda_{2m \rightarrow 12m}(\rho_{X_2 \ldots X_m})$$

**Facts:** $\sigma_{X_i X_{i+1}} \approx \rho_{X_i X_{i+1}}$

$\rightarrow \sigma_X \approx \arg\min_{H_x \in \mathcal{H}_2} S(\rho_x || e^{-H x}), \quad S(\rho_x || \sigma_X) \approx 2\gamma.$