Cutoff for product replacement on finite groups

Alex Zhai

Stanford University

Joint work with

Yuval Peres
Microsoft Research

Ryokichi Tanaka
Tohoku University

Product replacement walk

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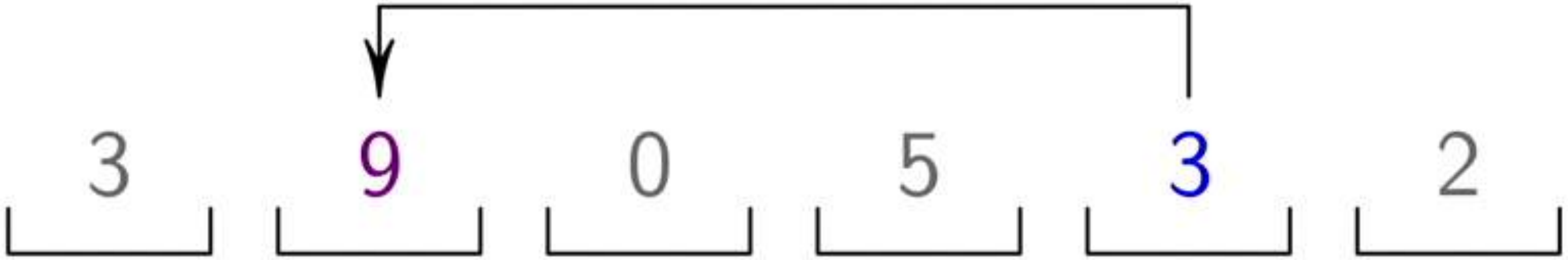

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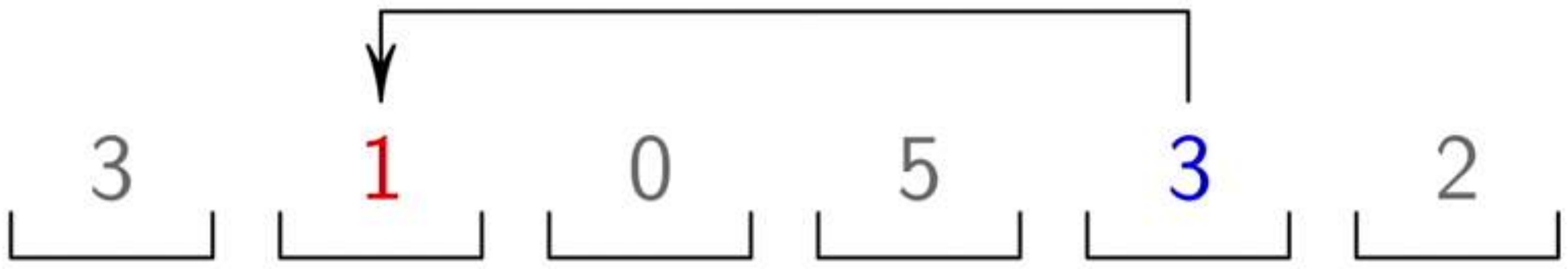

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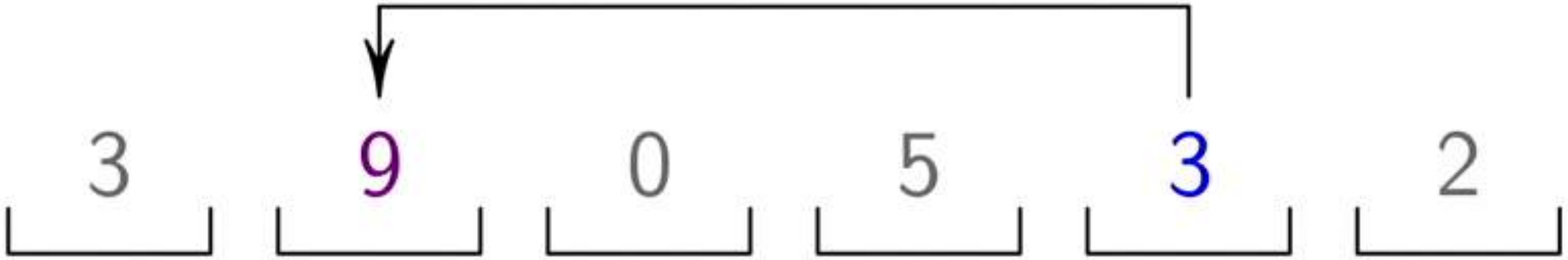

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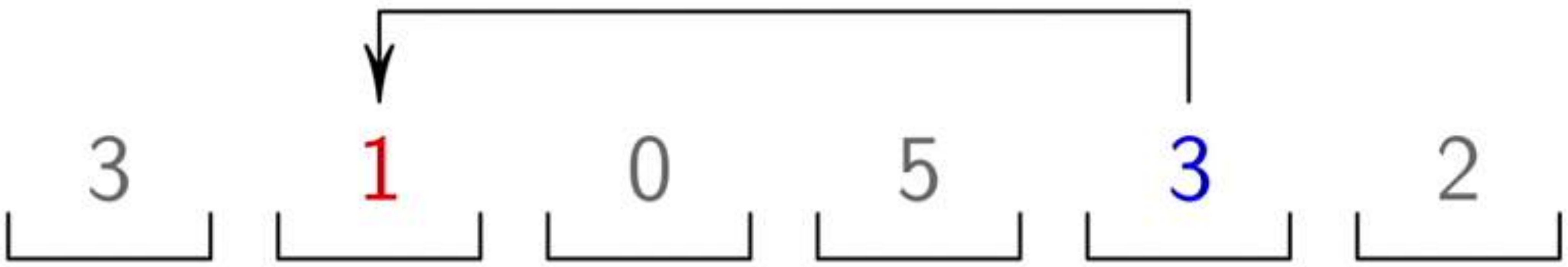

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- The graph is not connected, e.g. the subgroup generated by the elements remains invariant throughout the process.
- We focus on "generating n-tuples", i.e. n-tuples that generate the whole group.
- As long as n is big enough (say $n \gg \log|G|$), any two generating n-tuples can reach each other by product replacement steps. Also, the vast majority of configurations are generating n-tuples.

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  - Higher-dimensional versions?

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- Recall for $\epsilon \in (0,1)$ that the **mixing time** $t_{\mathrm{mix}}(\epsilon)$ is the earliest time t such that
$$\|\mathbf{P}(\sigma_t \in \cdot) - \pi\|_{\mathrm{TV}} \leq \epsilon,$$
where π is the stationary distribution.
- **Cutoff phenomenon**: refers to when the TV-distance from stationary rapidly drops from near 1 to near 0 within a short timescale.

Cutoff for product replacement

Theorem (Peres-Tanaka-Z.)

Let G be a fixed finite group, and consider the product replacement walk $(\sigma_t)_{t\geq 0}$. Then, for any fixed $\epsilon > 0$,

$$\left(\frac{3}{2} - o(1)\right) n \log n \leq t_{\mathrm{mix}}(1-\epsilon) \leq t_{\mathrm{mix}}(\epsilon) \leq \left(\frac{3}{2} + o(1)\right) n \log n$$

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- Extends work of Ben-Hamou and Peres who showed this for $G = \mathbb{Z}/2$.
- Diaconis and Saloff-Coste proved $t_{\mathrm{mix}} = O(n^2 \log n)$ and conjectured $t_{\mathrm{mix}} = O(n \log n)$.

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For simplicity, we'll focus on the case $G = \mathbb{Z}/q$ for a prime q. Also, we'll focus on the upper bound for mixing.

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\mathbf{P}(b=0) &\approx 1-\frac{k}{n} && \implies \text{\# non-zero stays same} \\
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- Can analyze k as a birth-or-death process, reaches at least $\frac{n}{3}$ w.h.p. in about $n \log n$ steps.

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- Think of $N_a(\sigma_t)$ as a function of a and t. Our equation approximates the differential equation (after appropriate rescaling)

$$\frac{\partial}{\partial t} N = -N + N * \left(\frac{N + N^-}{2}\right),$$

where $N^-(a,t) = N(-a,t)$.

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- After about $\frac{1}{2} n \log n$ steps, the x_k are about $O(1/\sqrt{n})$, which corresponds to $N_a = \frac{n}{|G|} + O(\sqrt{n})$.

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- Suffices to couple our process σ_t to σ'_t (with probability $1 - \epsilon$) within $\frac{3}{2} n \log n + O(n)$ steps.

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- By similar argument to before, after $\frac{3}{2} n \log n$ steps we'll have something like

$$N_{a,b} \approx \frac{n}{|G|^2} + O(\sqrt{n})$$

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  - This can be done (approximately) because b and b' are nearly uniformly distributed over G.
- If done carefully, can ensure that D has $\Omega(1)$ probability of either increasing or decreasing, and in expectation, it decreases. Simple random walk started at $O(\sqrt{n})$ is likely to hit 0 within $O(n)$ steps.

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- Thank you!

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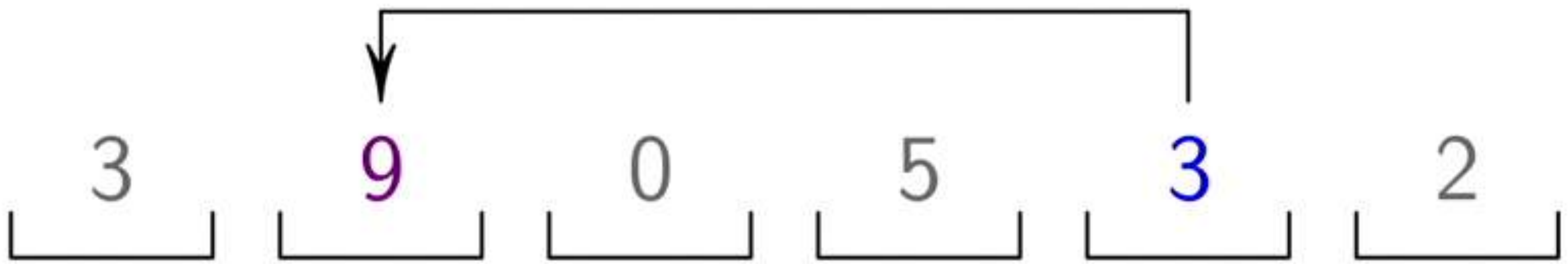
as $n \to \infty$.

Product replacement walk

- Let G be a finite group, and consider an n-tuple of group elements (call this a **configuration**). For example, $G = \mathbb{Z}/11$ and $n = 6$.
- We consider the **product replacement walk**: randomly pick two of the elements a and b, and replace a with $a + b$ or $a - b$.

Product replacement walk



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