The KPZ fixed point

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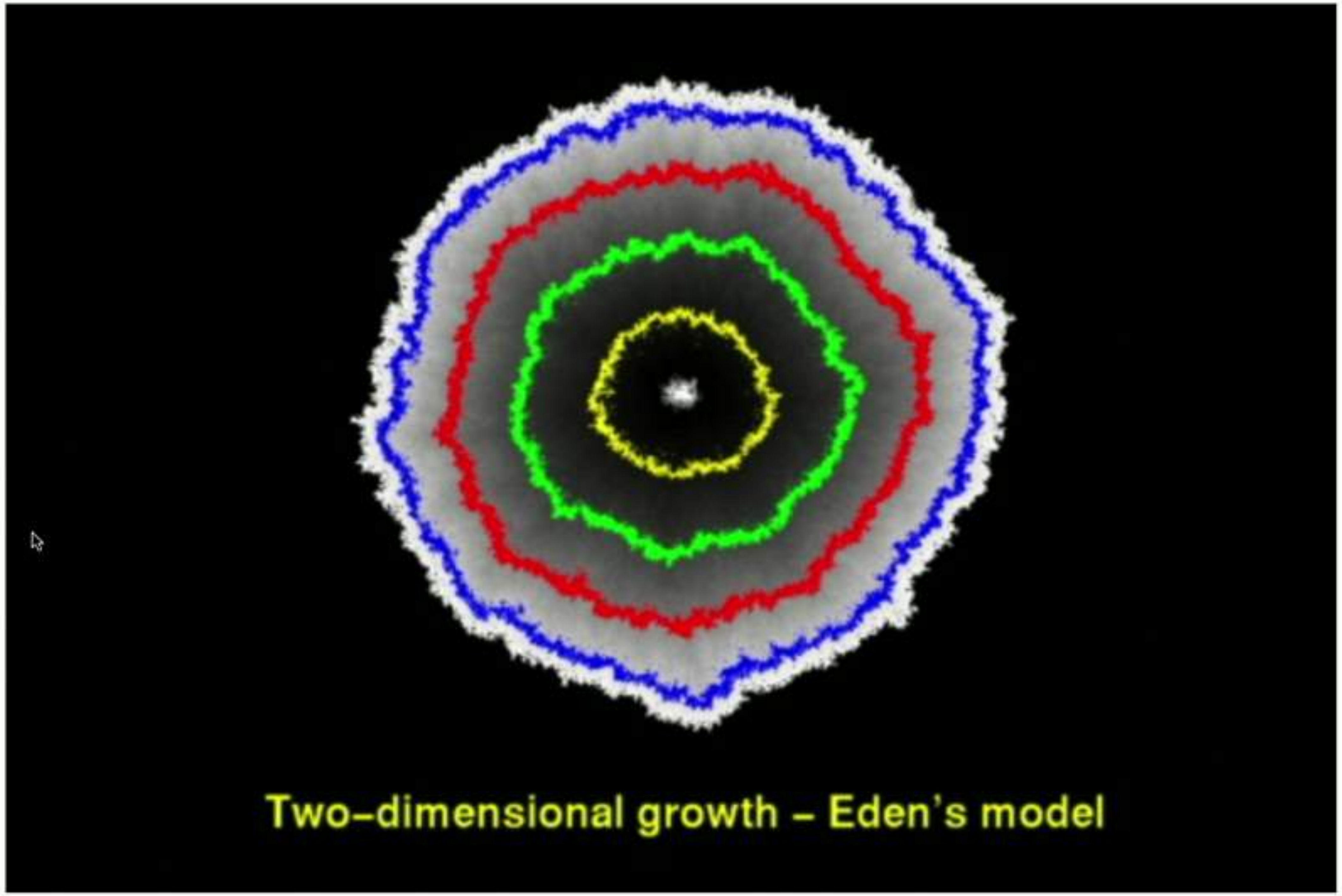
joint work with

Konstantin Matetski (Columbia University) and Daniel Remenik (Universidad de Chile)

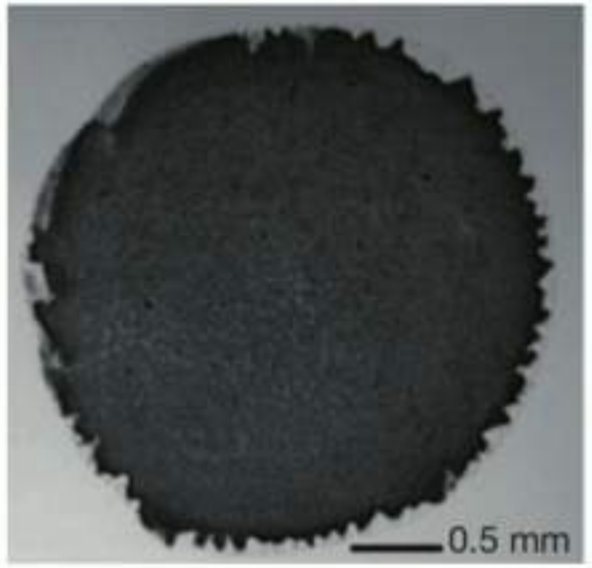
Pacific Northwest Probability Seminar 20 Oct 2018



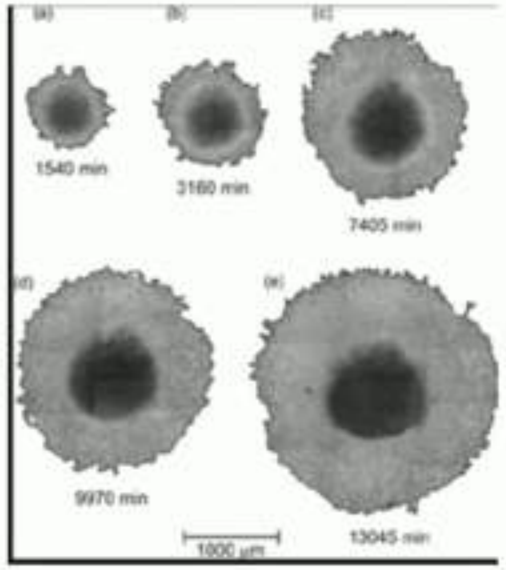
$$t(\text{some shape}) + t^{\chi}\text{Fluctuations}$$

$1 + 1$ d: Fluctuation exponent $\chi = 1/3$

Burning paper



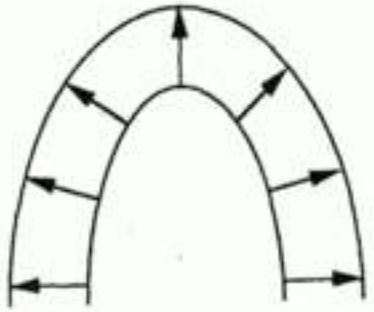
Coffee Stains



Tumour growth

Kardar-Parisi-Zhang (KPZ) Equation 86

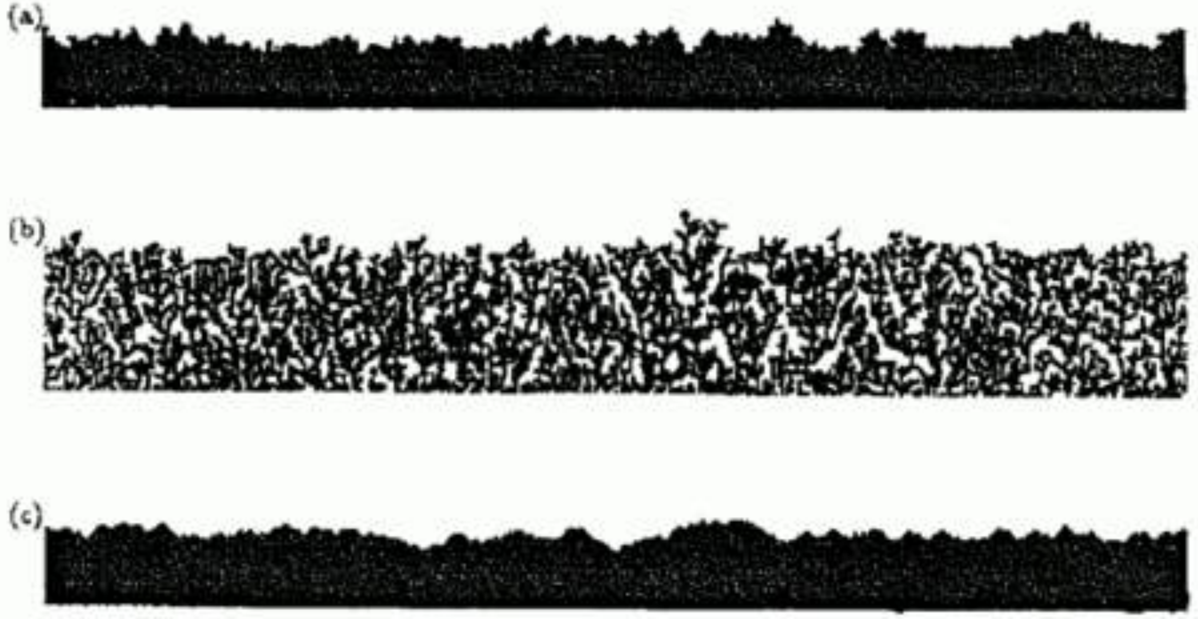
$$\partial_t h = \underbrace{(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\partial_x^2 h}_{\text{relaxation}} + \underbrace{\xi}_{\substack{\text{space-time}\\ \text{white noise}}} \qquad \underbrace{h(t,x)}_{\substack{\text{height at time}\\ \text{t position x}}}$$



$$F(\partial_x h) \sim F(0) + F'(0)\partial_x h + \tfrac{1}{2}F''(0)(\partial_x h)^2 + \cdots$$

$$\partial_x h = u \rightsquigarrow \text{stoch Burgers eqn}$$
$$\partial_t u = 2u\partial_x u + \partial_x^2 u + \partial_x \xi$$

T. Halpin-Healy, Y.-C. Zhang / Physics Reports 254 (1995) 215–414



Eden growth

Ballistic aggregration

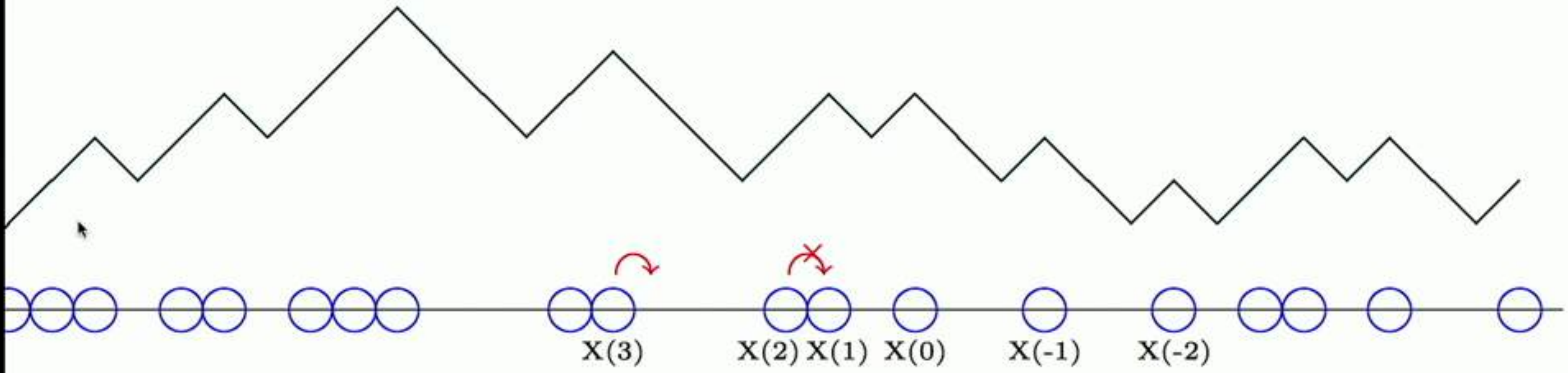
ASEP

2.2. Snapshots illustrating surface and bulk properties of three distinct stochastic growth models: (a) Eden cluster, (b) stic deposit, and (c) RSOS solid. All belong to the KPZ universality class [HH93].

A special discretization of KPZ equation (TASEP)

$h(x+1) = h(x) \pm 1$, $x \in \mathbb{Z}$

local max $\mapsto$ local min at rate 1



$$-2\mathbf{1}_{\wedge} = \frac{1}{2}\left[(\nabla^{-}h)(\nabla^{+}h) - 1 + \Delta h\right]$$

symmetric random walk invariant (except for height shift)

Lab mouse of non-equilibrium stat mech since the late 60's

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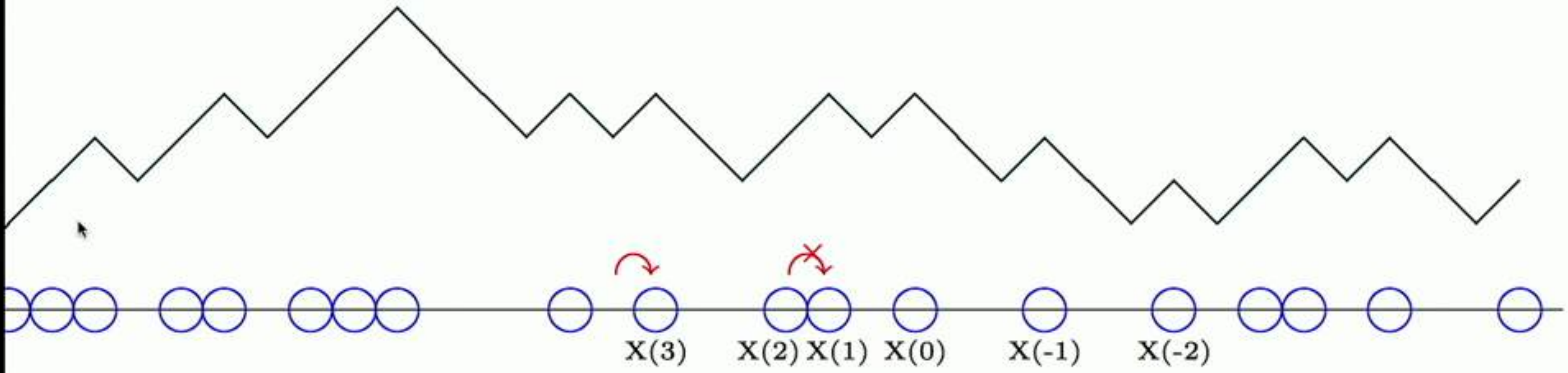

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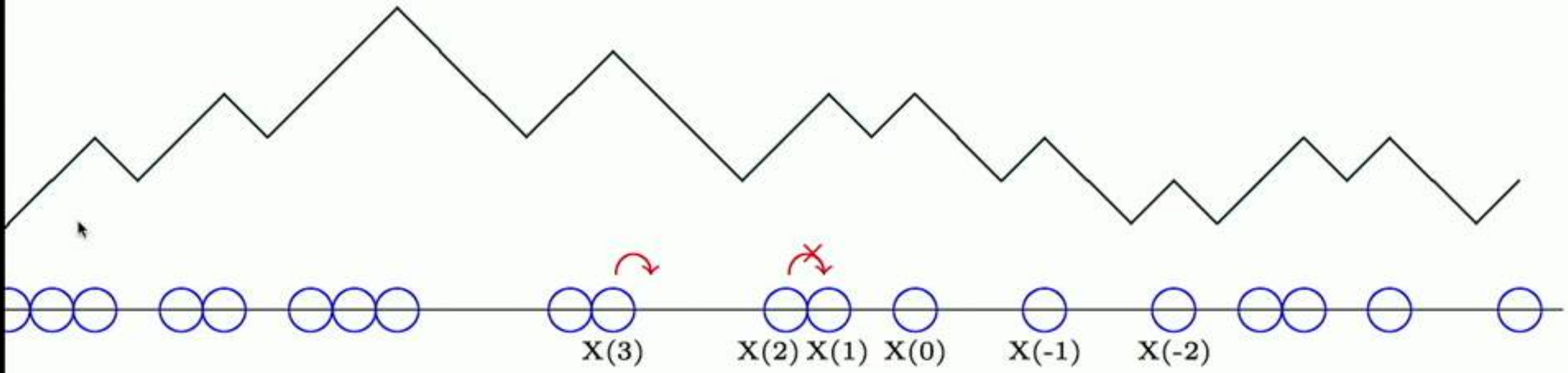

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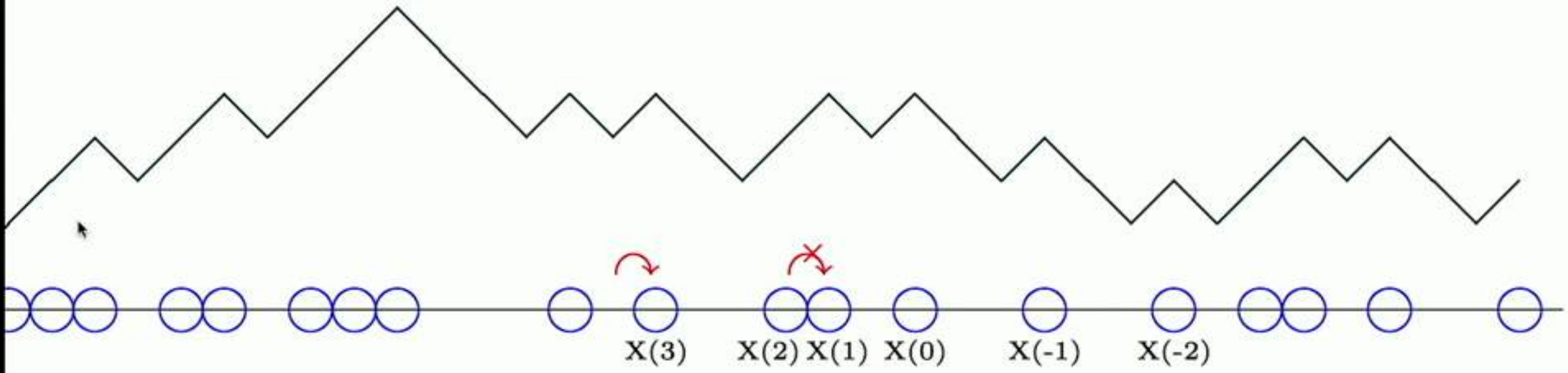

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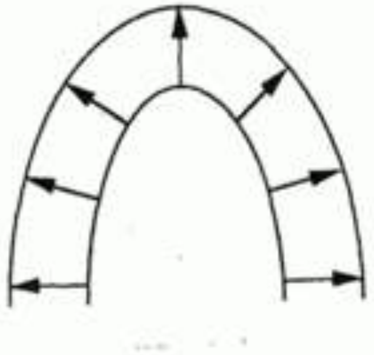

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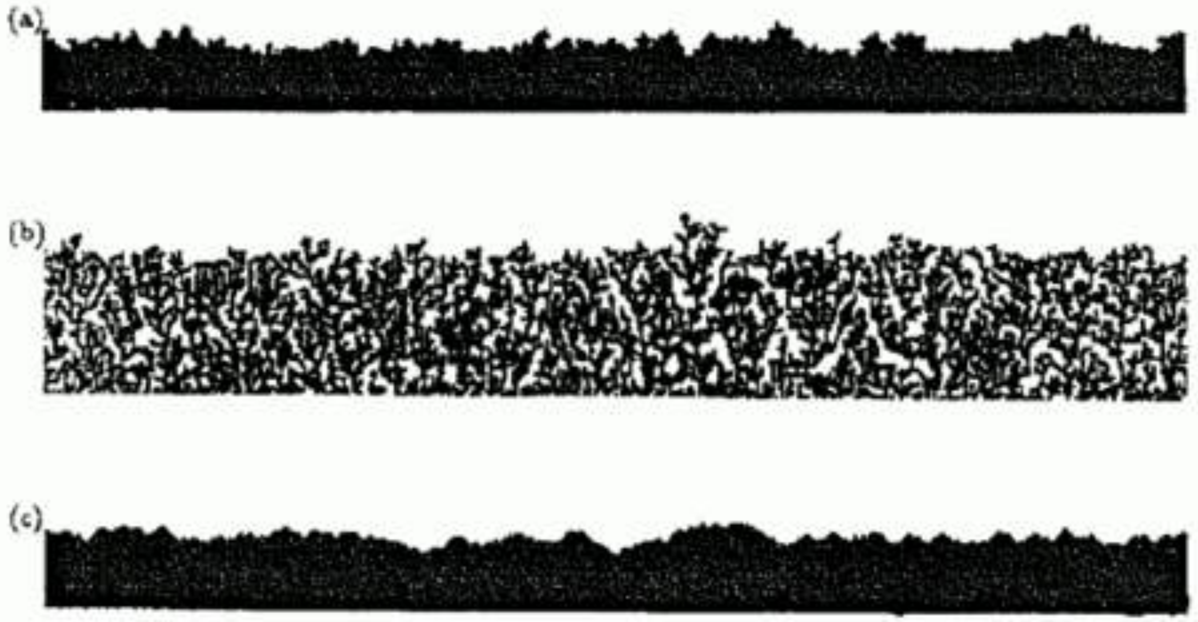

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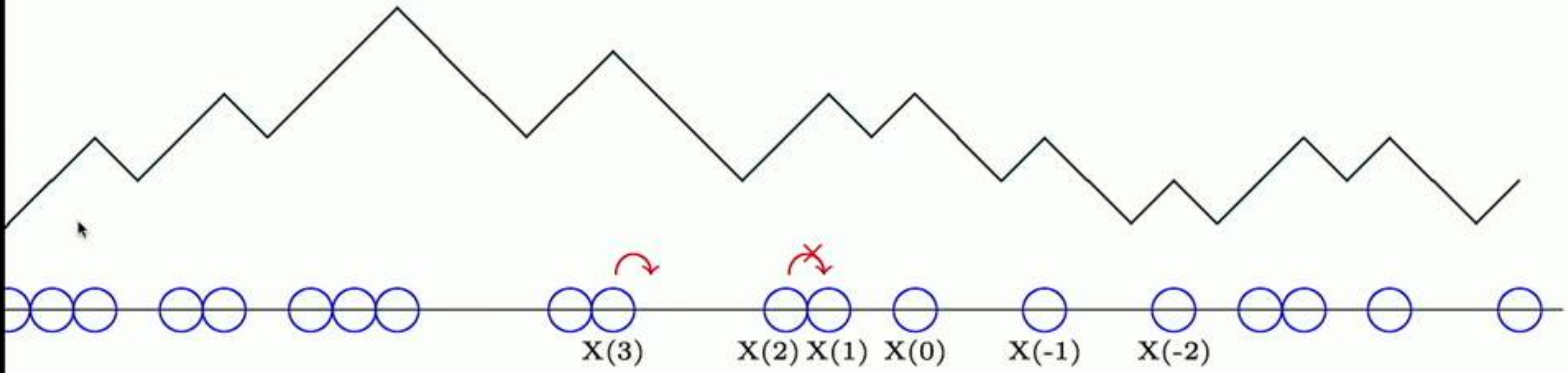

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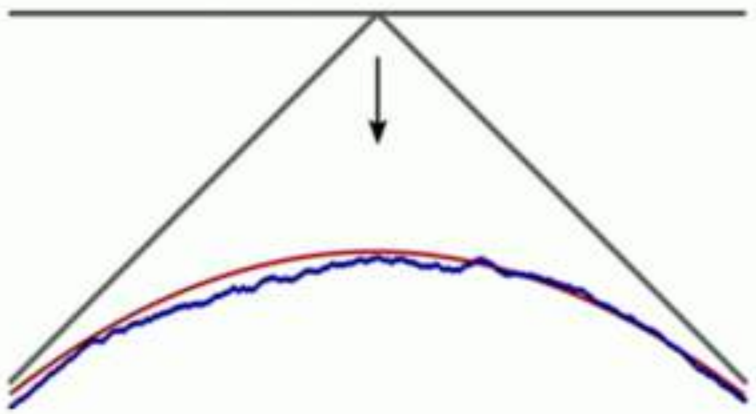
Asymptotic fluctuations depend on initial data

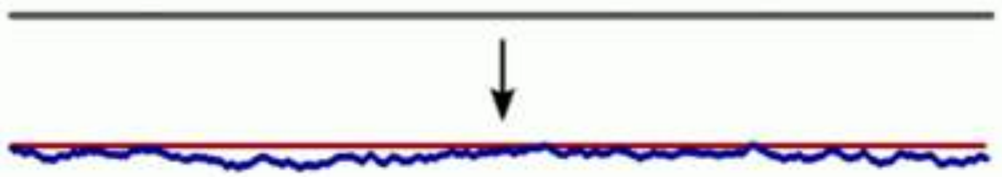
In TASEP special initial data could be computed 00's Johansson, Spohn, Borodin, Sasamoto,...

Corner



$$h(t,x) \sim c_1 t - c_2 \frac{x^2}{t} + c_3 t^{1/3} F_{\mathrm{GUE}}$$

Flat



$$h(t,x) \sim c_3 t + c_4 t^{1/3} F_{\mathrm{GOE}}$$

$F_{\mathrm{GUE}}/F_{\mathrm{GOE}}$ are the rescaled top eigenvalues of a matrix from the Gaussian Unitary/Orthogonal Ensembles

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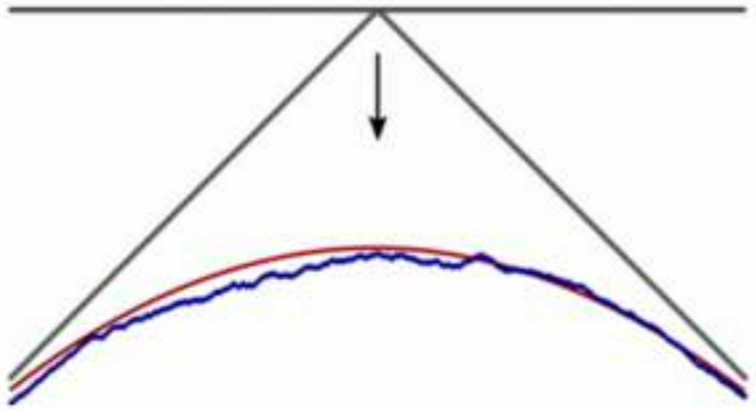

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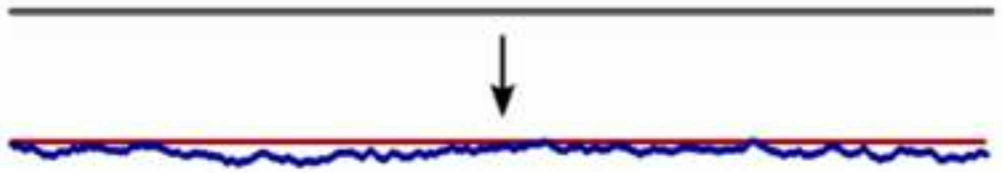

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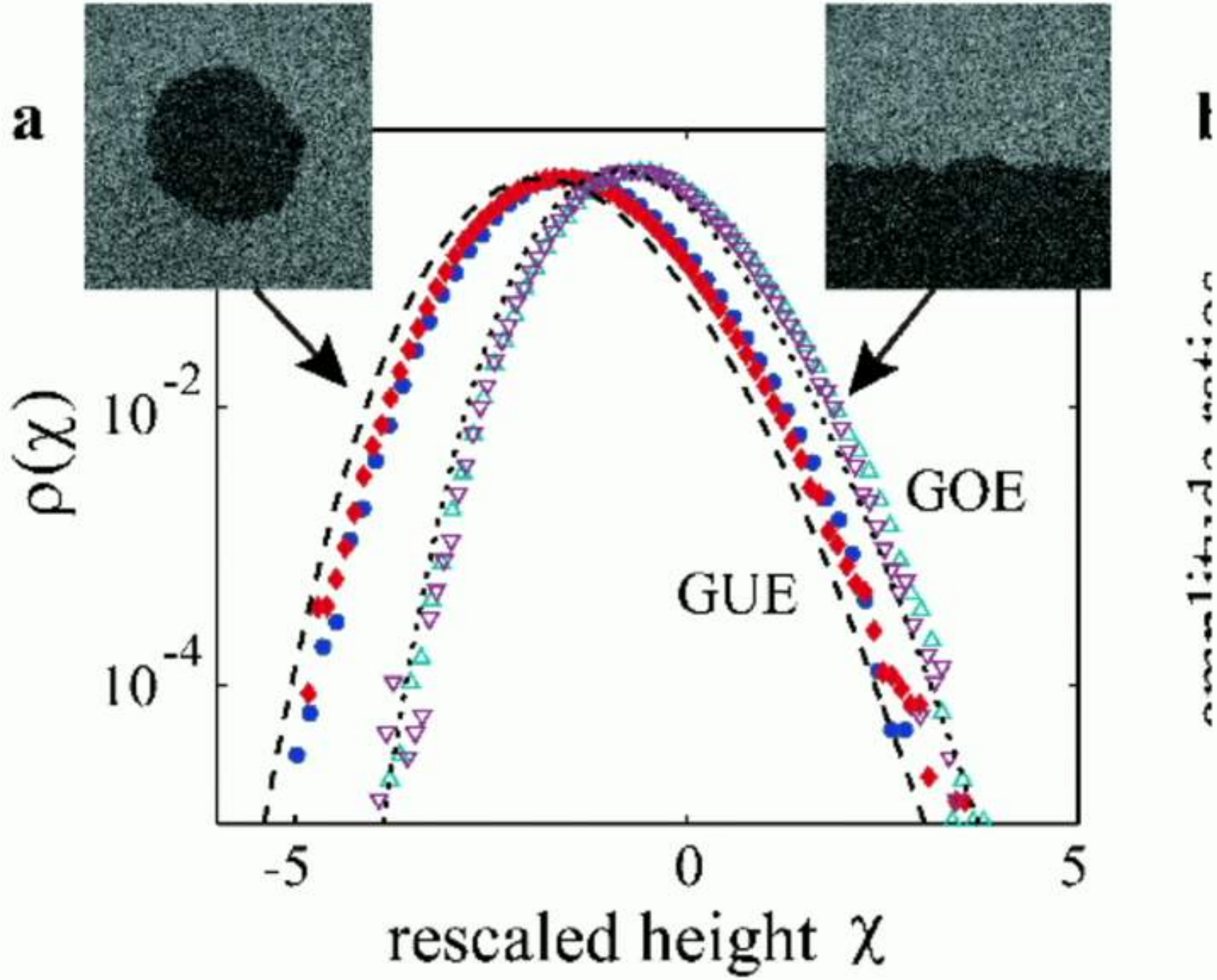
a
ρ(χ)
10^-2
10^-4
GOE
GUE
-5
0
5
rescaled height χ

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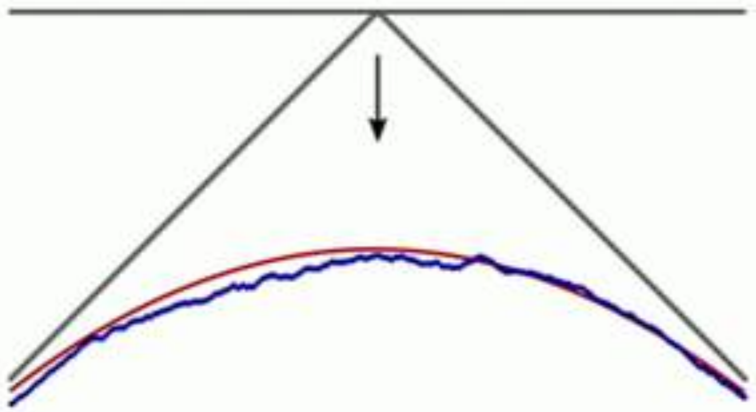
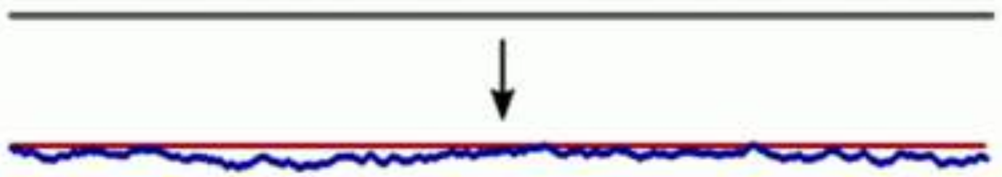


$$h(t,x) \sim c_1 t - c_2 \frac{x^2}{t} + c_3 t^{1/3} \mathcal{A}_2(t^{-2/3} x)$$

Flat



$$h(t,x) \sim c_3 t + c_4 t^{1/3} \mathcal{A}_1(t^{-2/3} x)$$

$\mathrm{Airy}_2/\mathrm{Airy}_1$ are special stochastic processes

The Airy$_2$ process $\mathcal{A}_2(x)$ 05 Prahofer-Spohn, Okounkov, Johansson

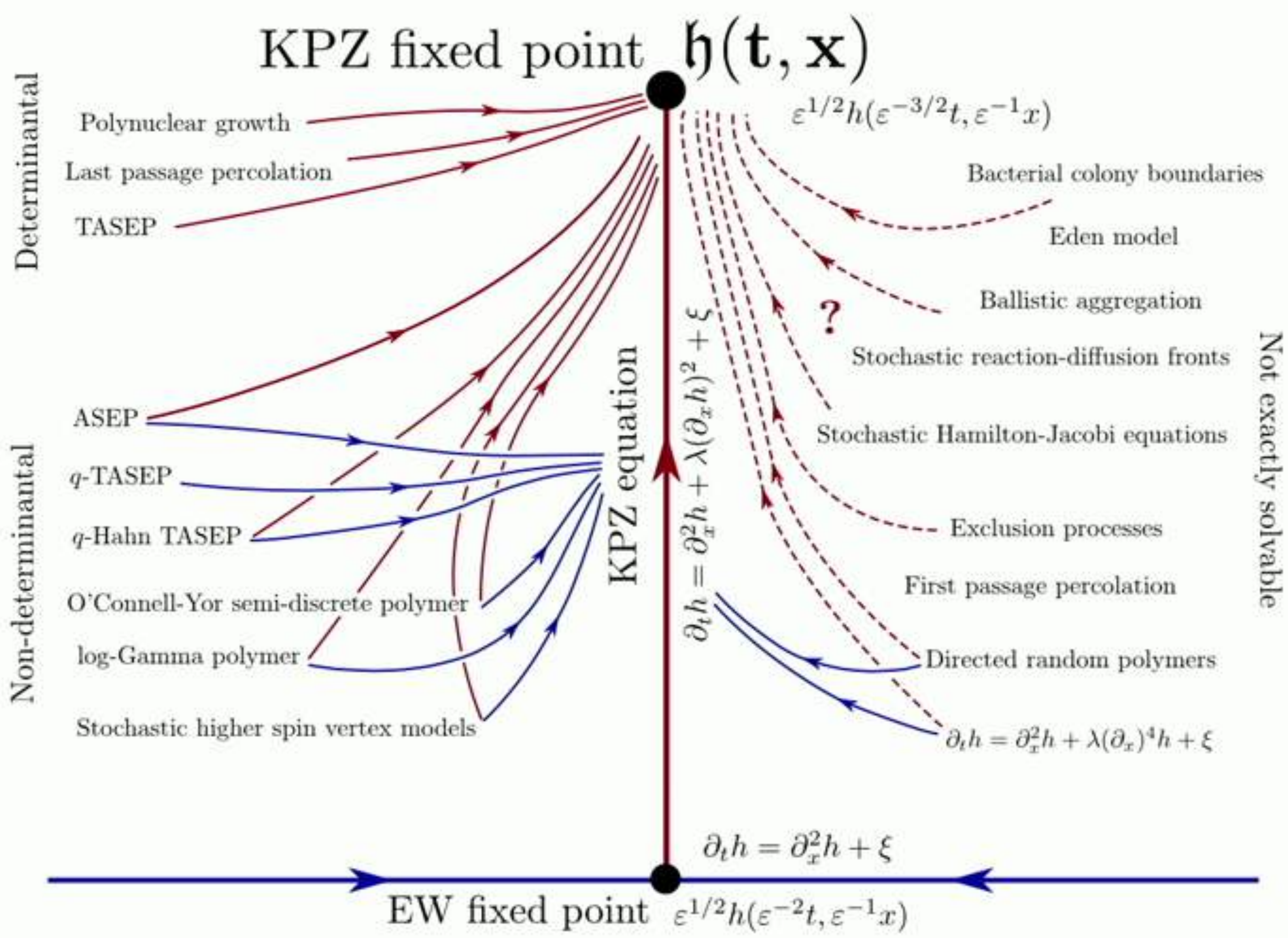
Defined via its finite dimensional distributions, given by Fredholm determinants of "extended kernels"

$x_1 < \cdots < x_n$ $\qquad\qquad$ $\chi_{\vec{g}} = \mathbf{1}_{z>g_j}$

$$\mathbb{P}(\mathcal{A}_2(x_1) \le g_1, \ldots, \mathcal{A}_2(x_n) \le g_n) = \det(I - \chi_{\vec{g}} K^{\mathrm{ext}}_{\mathrm{Ai}} \chi_{\vec{g}})_{L^2(\{x_1,\ldots,x_n\}\times\mathbb{R})}$$

$$K_2^{\mathrm{ext}}(x,z;x',z') = \begin{cases} \int_0^\infty d\lambda\, e^{-\lambda(x-x')}\,\mathrm{Ai}(z+\lambda)\,\mathrm{Ai}(z'+\lambda) & x \ge x' \\ -\int_{-\infty}^0 d\lambda\, e^{-\lambda(x-x')}\,\mathrm{Ai}(z+\lambda)\,\mathrm{Ai}(z'+\lambda) & x < x' \end{cases}$$

Conjectural KPZ universality class



KPZ fixed point was a complete mystery, both in math and physics.
All we had was a few self-similar solutions (Airy processes)

What is the fixed point?

KPZ 1:2:3 scaling $h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2}t, \epsilon^{-1}x)$ takes KPZ eqn to

$$\partial_t h_\epsilon = (\partial_x h_\epsilon)^2 + \epsilon^{1/2}\partial_x^2 h_\epsilon + \epsilon^{1/4}\xi$$

Goal to understand the limit $\mathfrak{h}$ of h_ϵ as $\epsilon \to 0$

Or $u = \partial_x h$, $\mathfrak{u} = \partial_x \mathfrak{h}$ stochastic Burgers fixed point

Weak solution of Burgers equation $\partial_t \mathfrak{u} = \partial_x \mathfrak{u}^2$, but *not* unique

$$\underset{\partial_t u = \partial_x u^2 + \epsilon \partial_x^2 u}{\text{Dissipationless limit}} \neq \underset{\partial_t u = \partial_x u^2 + \epsilon \partial_x^3 u}{\text{dispersionless limit}}$$ (Lax, Levermore, Venakidis, Deift, Zhou, ...)

KPZ fixed point *not* dissipative limit because Hopf-Lax formula $h(t,x) = \sup_y \{ h_0(y) - \frac{1}{t}(x-y)^2 \}$ does *not* preserve Brownian motion

With Matetski and Remenik (2017) we *solve* TASEP and take the 1:2:3 scaling limit to find the KPZ fixed point

The KPZ fixed point

Markov process on state space UC = upper semi-cont fns, with local Hausdorff topology

(eg. Narrow wedge $= \lim_{\epsilon\to 0} -\epsilon^{1/2}|\epsilon^{-1}\mathbf{x}|$)

Transition probabilities

$$\boxed{\mathbb{P}_{\mathfrak{h}_0}\big(\mathfrak{h}(\boldsymbol{t},\mathbf{x}_1)\le \mathbf{a}_1,\dots,\mathfrak{h}(\boldsymbol{t},\mathbf{x}_M)\le \mathbf{a}_M\big) = \det(\mathbf{I}-\mathbf{K}_{\mathfrak{h}_0,\boldsymbol{t},\mathbf{x},\mathbf{a}})}$$

Fredholm determinant of a compact (trace class) operator K is

$$\det(I+K)_{L^2(S,\mu)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\int_{S^n} \det\left[K(u_i,u_j)\right]_{i,j=1}^{n} d\mu(u_1)\cdots d\mu(u_n)$$

Stochastic integrable system: dynamics linearized by a new type of transform

Brownian scattering transform

For $\mathfrak{h}_0 \in UC$ define for $\mathbf{B}$ Brownian motion

$$\mathbf{P}^{\mathrm{hit}\,\mathfrak{h}_0}_{-L,L}(u_1,u_2) = \mathbb{P}_{\mathbf{B}(-L)=u_1,\mathbf{B}(L)=u_2}\Big(\mathbf{B} \text{ hits } \mathrm{hypo}(\mathfrak{h}_0) \text{ on } [-L,L]\Big)$$

The Brownian scattering transform of $\mathfrak{h}_0$ is

$$\boxed{\mathbf{K}^{\mathfrak{h}_0} = \lim_{L\to\infty} e^{-L\partial^2}\mathbf{P}^{\mathrm{hit}\,\mathfrak{h}_0}_{-L,L}e^{-L\partial^2}}$$

Looks terrible because of backwards heat equation, but we only ever use

$$U_{\boldsymbol{t}}\mathbf{K}^{\mathfrak{h}_0}U_{\boldsymbol{t}}^{-1} \qquad U_{\boldsymbol{t}} = e^{\frac{\boldsymbol{t}}{3}\partial^3}$$

ok because $e^{\mathbf{x}\partial^2+\frac{\boldsymbol{t}}{3}\partial^3}$ is convolution with

$$\boldsymbol{t}^{-1/3}e^{\frac{2\mathbf{x}^3}{3\boldsymbol{t}^2}-\frac{z\mathbf{x}}{\boldsymbol{t}}}\,\mathrm{Ai}(-\boldsymbol{t}^{-1/3}z+\boldsymbol{t}^{-4/3}\mathbf{x}^2)$$

Properties

Brownian scattering transform $\mathbf{K}^{\mathfrak{h}_0} = \lim_{L\to\infty} e^{-L\partial^2}\mathbf{P}^{\text{hit}\,\mathfrak{h}_0}_{-L,L} e^{-L\partial^2}$

Extend $\mathbf{K}^{\text{ext}}_{\mathfrak{h}_0}(\mathbf{x}_i,\cdot;\mathbf{x}_j,\cdot) = -e^{(\mathbf{x}_j-\mathbf{x}_i)\partial^2}\mathbb{1}_{\mathbf{x}_i<\mathbf{x}_j} + e^{-\mathbf{x}_i\partial^2}\mathbf{K}^{\mathfrak{h}_0} e^{\mathbf{x}_j\partial^2}$

1. For $\boldsymbol{t}\neq 0$ the map $\mathfrak{h}_0 \mapsto \chi_{\mathbf{a}} U_{\boldsymbol{t}} \mathbf{K}^{\text{ext}}_{\mathfrak{h}_0} U_{\boldsymbol{t}}^{-1}\chi_{\mathbf{a}}$ is continuous from UC into the trace class on $L^2(\{\mathbf{x}_1,\cdots,\mathbf{x}_M\}\times\mathbb{R})$. $\chi_{\mathbf{a}}(\mathbf{x}_i,u) = \mathbb{1}_{u>\mathbf{a}_i}$

2. Inverted by the determinant

$$\det\left(\mathbf{I} - \chi_{\mathbf{a}} U_{\boldsymbol{t}} \mathbf{K}^{\text{ext}}_{\mathfrak{h}_0} U_{\boldsymbol{t}}^{-1}\chi_{\mathbf{a}}\right)_{L^2(\{\mathbf{x}_1,\cdots,\mathbf{x}_M\}\times\mathbb{R})} \xrightarrow{\boldsymbol{t}\to 0} \prod_{i=1}^{M} \mathbb{1}_{\mathfrak{h}_0(\mathbf{x}_i)\le \mathbf{a}_i}$$

3. The determinants define Markov transition probabilities on UC, i.e. the Chapman-Kolmogorov equations

$$\int_{\mathrm{UC}} \mathbb{P}_{\mathfrak{h}_0}(\mathbf{s}, d\mathfrak{f})\mathbb{P}_{\mathfrak{f}}(\boldsymbol{t}, B) = \mathbb{P}_{\mathfrak{h}_0}(\mathbf{s}+\boldsymbol{t}, B).$$

$$\mathbb{P}_{\mathfrak{h}_0}\left(\mathfrak{h}(\boldsymbol{t},\mathbf{x}_1)\le \mathbf{a}_1,\ldots,\mathfrak{h}(\boldsymbol{t},\mathbf{x}_M)\le \mathbf{a}_M\right) = \det\left(\mathbf{I} - \chi_{\mathbf{a}} U_{\boldsymbol{t}} \mathbf{K}^{\text{ext}}_{\mathfrak{h}_0} U_{\boldsymbol{t}}^{-1}\chi_{\mathbf{a}}\right)$$

KPZ fixed pt $\mathfrak{h}(\boldsymbol{t}, \mathbf{x}; \mathfrak{h}_0)$ unique Markov process with these transition probs starting from $\mathfrak{h}_0$

- Conjecturally unique local dynamics satisfying
  1. (1:2:3 scaling invariant) $\alpha\mathfrak{h}(\alpha^{-3}\boldsymbol{t}, \alpha^{-2}\mathbf{x}; \alpha\mathfrak{h}_0(\alpha^{-2}\mathbf{x})) \overset{dist}{=} \mathfrak{h}(\boldsymbol{t}, \mathbf{x}; \mathfrak{h}_0)$
  2. (Skew time reversible) $\mathbb{P}\big(\mathfrak{h}(\boldsymbol{t}, \mathbf{x}; \mathfrak{g}) \leq -\mathfrak{f}(\mathbf{x})\big) = \mathbb{P}\big(\mathfrak{h}(\boldsymbol{t}, \mathbf{x}; \mathfrak{f}) \leq -\mathfrak{g}(\mathbf{x})\big)$
  3. (Stationarity in space) $\mathfrak{h}(\boldsymbol{t}, \mathbf{x}+\mathbf{u}; \mathfrak{h}_0(\mathbf{x}-\mathbf{u})) \overset{dist}{=} \mathfrak{h}(\boldsymbol{t}, \mathbf{x}; \mathfrak{h}_0)$;

- For TASEP (and a few related models) we know that if the rescaled height function converges to $\mathfrak{h}_0$ in UC. Then
  $\epsilon^{1/2}\Big[h(2\epsilon^{-3/2}\boldsymbol{t}, 2\epsilon^{-1}\mathbf{x}) + \epsilon^{-3/2}\boldsymbol{t}\Big] \xrightarrow[\epsilon\to 0]{} \mathfrak{h}(\boldsymbol{t}, \mathbf{x}; \mathfrak{h}_0)$ in distr. in UC.

- For $\boldsymbol{t} > 0$, $\mathfrak{h}(\boldsymbol{t}, \mathbf{x})$ is locally Brownian and locally Hölder $\frac{1}{2}-$ in $\mathbf{x}$. For fixed $\mathbf{x}$, $\mathfrak{h}(\boldsymbol{t}, \mathbf{x})$ is locally Hölder $\frac{1}{3}-$ in $\boldsymbol{t} > 0$.

- Although Brownian scattering transform looks daunting, in special cases like flat, narrow wedge, it is easy to compute and recovers the known formulas for the Airy processes. Also, new closed forms for cusps, parabolas.

A determinantal formula for TASEP

Solution to the master equation by using Bethe ansatz:

Schütz'97:

For $N \geq 1$ particles one has the determinantal formula

$$\mathbb{P}(X_t = x | X_0 = y) = \det \Big[F_{i-j}(x_{N+1-i} - y_{N+1-j}, t) \Big]_{1 \leq i,j \leq N}$$

with $x, y \in \{z_N < \cdots < z_1\} \subset \mathbb{Z}^N$ and

$$F_n(x,t) = \frac{(-1)^n}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \frac{dw}{w^{x+1}} \left(\frac{w}{1-w} \right)^n e^{t(w-1)}$$

where $\Gamma_{0,1}$ is a simple loop around 0 and 1

This formula is not suitable for $N \to \infty$, since the matrix size goes to ∞

Sasamoto (2006) realized that because of special properties of F it can be written as a (signed) determinantal point process on certain wired Gelfand-Tsetlin patterns

Biorthogonalization

For one-sided i.d. $X(0) = X(-1) = X(-2) = \cdots = +\infty$:

Borodin, Ferrari, Prähofer and Sasamoto'07:

For $1 \le n_1 < n_2 < \cdots < n_M$ one has

$$\mathbb{P}_{X_0}\Big(X_t(n_j) > a_j,\ j = 1, \ldots, M\Big) = \det\Big(I - \mathbb{1}_{x_i \le a_i} K_t \mathbb{1}_{x_i \le a_i}\Big)_{\ell^2(\{n_1,\ldots,n_M\}\times\mathbb{Z})}$$

where the RHS is a Fredholm determinant,

$$K_t(n_i, x_i; n_j, x_j) = -Q^{n_j - n_i}(x_i, x_j)\mathbb{1}_{n_i < n_j} + \sum_{k=1}^{n_j} \Psi^{n_i}_{n_i-k}(x_i)\,\Phi^{n_j}_{n_j-k}(x_j)$$

$$Q(x,y) = \mathbb{1}_{x>y} \qquad \Psi^n_k(x) = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0} \frac{dw}{w^{x-X_0(n-k)+1}}\left(\frac{1-w}{w}\right)^k e^{t(w-1/2)}$$

The functions Φ^n_k are defined implicitly by

1. Biorthogonality: $\sum_{x\in\mathbb{Z}} \Psi^n_\ell(x)\Phi^n_k(x) = \mathbb{1}_{\ell=k}$
2. Φ^n_k is a polynomial of degree k

Exact formulas for Φ_k^n

The functions Φ_k^n has been computed only for special initial datum [Borodin, Ferrari, Prähofer, Sasamoto]

- Step initial data: $X_0(i) = -i$, $i \geq 1$:

$$\Phi_k^n(x) = \frac{1}{2\pi i} \oint_{\Gamma_0} dv \, \frac{(1-v)^{x+n}}{v^{k+1}} e^{t(v+1/2)}$$

- Periodic initial data: $X_0(i) = -di$, $i \geq 1$, $d \geq 2$:

$$\Phi_k^n(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} dv \, \frac{(1-dv)(2(1-v))^{x+dn-1}}{v(2^d(1-v)^{d-1}v)^k} e^{t(v+1/2)}$$

In these cases Φ_k^n are essentially the same as the Ψ_k^n

Explicit biorthogonalization

We can write

$$\Psi_k^n(x) = e^{-\frac{t}{2}\nabla^-} Q^{-k}(X_0(n-k), x)$$

where $Q^{-1}Q = I$ and $Q(x,y) = \mathbb{1}_{x>y}$

Matetski, Q, Remenik'17:

The functions Φ_k^n are given by

$$\Phi_k^n(x) = e^{\frac{t}{2}\nabla^-} h_k^n(0, x)$$

where $h_k^n(\ell, z)$ is the unique solution to the backwards heat equation

$$\begin{cases} (Q^*)^{-1} h_k^n(\ell, z) = h_k^n(\ell+1, z) & \ell < k,\ z \in \mathbb{Z} \\ h_k^n(k, z) = 1 & z \in \mathbb{Z} \\ h_k^n(\ell, X_0(n-\ell)) = 0 & \ell < k \end{cases}$$

Note! h_k^n is well defined because $\dim\ker(Q^*)^{-1} = 1$

A simple check

Recall: $\begin{cases} (Q^*)^{-1} h_k^n(\ell, z) = h_k^n(\ell+1, z) & \ell < k,\ z \in \mathbb{Z} \\ h_k^n(k, z) = 1 & z \in \mathbb{Z} \\ h_k^n(\ell, X_0(n-\ell)) = 0 & \ell < k \end{cases}$

Then we have

$$\begin{aligned}
\sum_{z\in\mathbb{Z}} \Psi_\ell^n(z)\Phi_k^n(z) &= \sum_{z,z_1,z_2\in\mathbb{Z}} e^{-\frac{t}{2}\nabla^-}(z,z_1) Q^{-\ell}(z_1, X_0(n-\ell)) h_k^n(0,z_2) e^{\frac{t}{2}\nabla^-}(z_2,z) \\
&= (Q^*)^{-\ell} h_k^n(0, X_0(n-\ell)) \\
&= h_k^n(\ell, X_0(n-\ell)) \\
&= \mathbb{1}_{k=\ell}
\end{aligned}$$

To show that Φ_k^n is a polynomial we use $Q^{-1} = I + 2\nabla^+$ recursively